Van Dooren's Index Sum Theorem and Rational Matrices with Prescribed Structural Data

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joint work with **L. M. Anguas** (UC3M, Spain),
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What is this theorem? Where can we find it?

INT. J. CONTROL, 1979, VOL. 30, NO. 2, 235–243

Properties of the system matrix of a generalized state-space system†

G. VERGHESE‡, P. VAN DOOREN§ and T. KAILATH‡

For an irreducible polynomial system matrix $P(s)=\begin{bmatrix} T(s) & -U(s) \\ V(s) & W(s) \end{bmatrix}$, Rosenbrock (1970, p. 111) has shown that the *polar* structure of the associated transfer function $R(s)=V(s)T^{-1}(s)U(s)$ at any finite frequency is isomorphic to the zero structure of $T(s)$ at that frequency, while the *zero* structure of $R(s)$ at any finite frequency is isomorphic to that of $P(s)$ at the same frequency. In this paper we obtain the appropriate extensions for the structure *at infinite frequencies* in the particular case of systems for which $T(s)=sE-A$ (with E possibly singular), $U(s)=B$, $V(s)=C$, and $W(s)=D$, under a strengthened irreducibility condition. We term such systems *generalized state-space systems*, and note that *any* rational $R(s)$ may be realized in this form. We also demonstrate in this case that a minimal basis (in the sense of Forney (1975) for the left or right null space of $P(s)$ directly generates one with the same minimal indices for the corresponding null space of $R(s)$, and vice versa. These results also enable us to identify the pole-zero excess of $R(s)$ as being equal to the sum of the minimal indices of its null spaces. Connections with Kronecker's theory of matrix pencils are made.

Why do I call this result "Van Dooren's Index Sum Theorem"?

The following theorem†, whose proof we merely outline for lack of space, demonstrates an important consequence of the preceding two theorems.

Theorem 3

Let $\delta_p(R)$ and $\delta_z(R)$ denote the total number of poles and zeros (finite *and* infinite) respectively of an arbitrary rational matrix $R(s)$, and let $\alpha(R)$ denote the sum of the minimal indices of the left and right null spaces of $R(s)$. Then

$$\delta_p(R) = \delta_z(R) + \alpha(R) \tag{21}$$

...

† First obtained, in a slightly different way, by Van Dooren, in earlier unpublished research.

G. Verghese, P. Van Dooren, T. Kailath, Int. J. Control, 1979, Vol 30, page 241.

Paul Van Dooren's Index Sum Theorem is also in his PhD Thesis

Proposition 5.10

The polar and zero degree of a rational matrix and the minimal orders of its left and right null spaces satisfy the equality

$$\boxed{\delta_p(R) = \delta_z(R) + \hat{\varepsilon}(R) + \hat{\eta}(R)}$$

Proof

From the above remarks and theorem 3.8 it follows that ($\lambda E-A$ being regular) :

$$\delta_p(R) = \delta_z(\lambda E-A) = \delta_p(\lambda E-A)$$

$$\delta_z(R) + \hat{\varepsilon}(R) + \hat{\eta}(R) = \delta_z(\lambda\hat{E}-\hat{A}) + \hat{\varepsilon}(\lambda\hat{E}-\hat{A}) + \hat{\eta}(\lambda\hat{E}-\hat{A}) = \delta_p(\lambda\hat{E}-\hat{A})$$

Since $\delta_p(\lambda E-A)$ = rank E and $\delta_p(\lambda\hat{E}-\hat{A})$ = rank $\hat{E}$ (see theorem 3.8), we have that $\delta_p(\lambda E-A) = \delta_p(\lambda\hat{E}-\hat{A})$, which completes the proof.

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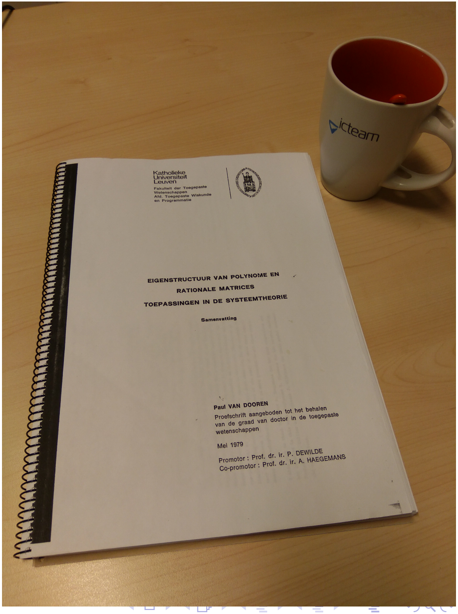

Van Dooren's Index Sum Theorem laid dormant until 1991,

- at least up to my limited knowledge, though it is studied in Kailath's book, Linear Systems, 1980.
- In 1991 the index sum theorem appears again but **only for matrix polynomials** and written in such a form that **nobody established a connection between Van Dooren's general result** for arbitrary rational matrices and the "new theorem" by
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March 6, 1991

Abstract In this paper a result on integer invariants of polynomial matrices is derived: the sum of the minimal indices and the elementary exponents equals the rank times the degree. This result is used to generalize a numerically reliable algorithm for column reduction of polynomial matrices.

Keywords: Kronecker indices, elementary divisors, column reduction

1 Introduction

Many results in the polynomial approach to systems theory depend on or take a nice form if the specific polynomials are in row or column reduced form: results concerning minimal state space representations, coprime factorizations etc, etc.

In Beelen, van den Hurk, Praagman [BHP] a numerically reliable method was derived to compute a column reduced polynomial matrix, unimodularly equivalent to a given polynomial matrix of full column rank. The proof given in [BHP] for the correctness of the algorithm hinges strongly on the assumption that the original matrix has full column rank. It turns out, however, that the algorithm still leads to correct results if this condition is not satisfied. In this paper I will present some results on integer invariants of polynomial matrices, which make a proof possible in the more general case.

2 Preliminaries

Let me start by recalling some definitions:

Definition 1 *Let $P \in \mathbb{R}^{m\times n}[s]$. Then $d(P)$, the degree of P is defined as the maximum of the degrees of its entries, and $d_j(P)$, the $j-th$ column degree of P as the maximum of the degrees in the $j-th$ column. $\delta(P)$ is the array of integers obtained by arranging the column degrees of P in non-decreasing order.*

ECC 91 *European Control Conference*, Grenoble, France, July 2-5 1991

Definition 2 *Let $P \in \mathbb{R}^{m\times m}[s]$. Then P is unimodular if $\det(P) \in \mathbb{R}\backslash\{0\}$.*

Let $\Delta^P(s) = diag(s^{-d_1(P)} \dots s^{-d_n(P)})$, then $P\Delta^P$ is a proper rational matrix.

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Definition 4 *Let M be a submodule of $\mathbb{R}^n[s]$. Then $Q \in \mathbb{R}^{n\times r}[s]$ is called a basis of M if rank $Q = r$, and $M = Im\ Q$. If, moreover, Q is column reduced, then Q is called a minimal basis of M.*

Note that if $Q(s)$ has full column rank for all $s \in \mathbb{C}$, then M is a direct summand of $\mathbb{R}^n[s]$, so in that case Q is a minimal polynomial basis in the sense of Forney [F], or Beelen [B].

For each polynomial matrix P having full column rank there exists a unimodular matrix U, such that PU is column reduced (see Wolovich [W], Kailath [K] or [F]). The proof, given in these references, is constructive and does imply:

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Unfortunately, the proof mentioned above, has awkward numerical properties, as was pointed out by Van Dooren [VD]. The numerically more satisfying method in [BHP] is based on the following theorem:

Theorem 1 *Let $P \in \mathbb{R}^{m\times n}[s]$ have full column rank, and let $(U^t\ R^t)^t$ be a minimal basis for $Ker(s^bP - I)$. Then U is unimodular and if b exceeds $(n-1)d$ then $s^{-b}R = PU$ is column reduced.*

Theorem 3 *Let $P \in \mathbb{R}^{m\times n}[s]$ be a polynomial matrix of rank r and degree d. Then the sum of its structure indices equals rd.*

Proof. It can be deduced immediately from the Kronecker normal form displayed above that the theorem holds for matrix polynomials of degree *1*. The rank of L^P equals $m(d-1)+r$, hence its number of left minimal indices (and that of P) is $m-r$.
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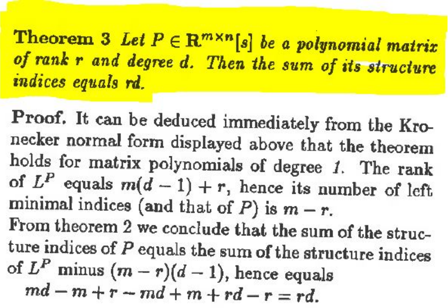

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V. N. Kublanovskaya

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$$\gamma_p[R] = \gamma_z[R] + \varepsilon[R] + \eta[R] \qquad (1.1.21)$$

for a rational $m \times n$ matrix $R(\lambda)$ of rank ρ;

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$$\beta_c[D] + \beta_\infty[D] + \varepsilon[D] + \eta[D] = \rho s \qquad (1.1.22)$$

for a polynomial $m \times n$ matrix $D(\lambda)$ of rank ρ and degree s.

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The following very long paper in LAA (2014)

Linear Algebra and its Applications 459 (2014) 264–333

Contents lists available at ScienceDirect

Linear Algebra and its Applications

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Spectral equivalence of matrix polynomials and the Index Sum Theorem

Fernando De Terán [a,1], Froilán M. Dopico [b,*,1], D. Steven Mackey [c,2]

[a] *Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. Universidad 30, 28911 Leganés, Spain*

[b] *Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM and Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. Universidad 30, 28911 Leganés, Spain*

[c] *Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, USA*

Theorem 6.5 *(Index Sum Theorem for Matrix Polynomials). Suppose $P(\lambda)$ is an arbitrary $m \times n$ matrix polynomial over an arbitrary field. Then*

$$\delta_{\text{fin}}(P) + \delta_\infty(P) + \mu(P) = \text{grade}(P) \cdot \text{rank}(P). \tag{6.4}$$

The connection was finally established in 2015 (2014) in...

SIAM J. MATRIX ANAL. APPL.
Vol. 36, No. 1, pp. 302–328

MATRIX POLYNOMIALS WITH COMPLETELY PRESCRIBED EIGENSTRUCTURE*

FERNANDO DE TERÁN†, FROILÁN M. DOPICO†, AND PAUL VAN DOOREN‡

THEOREM 3.1 (index sum theorem). *Let $P(\lambda)$ be an $m \times n$ matrix polynomial of degree d and rank r having the following eigenstructure:*

- *r invariant polynomials $p_j(\lambda)$ of degrees δ_j, for $j = 1, \ldots, r$,*
- *r infinite partial multiplicities $\gamma_1, \ldots, \gamma_r$,*
- *$n - r$ right minimal indices $\varepsilon_1, \ldots, \varepsilon_{n-r}$, and*
- *$m - r$ left minimal indices $\eta_1, \ldots, \eta_{m-r}$,*

where some of the degrees, partial multiplicities, or indices can be zero, and/or one or both of the lists of minimal indices can be empty. Then

$$\sum_{j=1}^{r} \delta_j + \sum_{j=1}^{r} \gamma_j + \sum_{j=1}^{n-r} \varepsilon_j + \sum_{j=1}^{m-r} \eta_j = dr. \tag{3.1}$$

Remark 3.2. A very interesting remark pointed out by an anonymous referee is that the index sum theorem for matrix polynomials can be obtained as an easy corollary of a more general result valid for arbitrary rational matrices, which is much older than reference [28]. This result is [36, Theorem 3], which can also be found in [18, Theorem 6.5-11]. Using the notion of structural indices at α introduced in

[28] is Praagman's 1991 paper; [36] Verghese, Van Dooren, Kailath's 1979 paper; [18] Kailath's 1980 book.

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In the rest of the talk:

1. We will explain why Van Dooren's Index Sum Theorem for Rational Matrices (1979) implies "easily" the Index Sum Theorem for Polynomial Matrices (1991).
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3. We will explain why the Index Sum Theorem for Polynomial Matrices (1991) implies "easily" Van Dooren's Index Sum Theorem for Rational Matrices (1979).
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Outline

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1. **Basic concepts on rational matrices**
2. From Van Dooren's Rational to Polynomial Index Sum Theorem
3. From Polynomial to Van Dooren's Rational Index Sum Theorem
4. The rational inverse structural data problem

- **Any rational matrix $R(\lambda) \in \mathbb{C}(\lambda)^{m \times n}$ can be uniquely expressed as**

$$R(\lambda) = P(\lambda) + R_{sp}(\lambda), \quad \text{where}$$

  - $P(\lambda)$ is a polynomial matrix (polynomial part), and
  - the rational matrix $R_{sp}(\lambda)$ is **strictly proper** (strictly proper part), i.e., $\lim_{\lambda \to \infty} R_{sp}(\lambda) = 0$.

Definition

The **Smith-McMillan form** of a rational matrix $R(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ is the following **diagonal matrix** obtained under unimodular transformations $U(\lambda)$ and $V(\lambda)$:

$$U(\lambda)R(\lambda)V(\lambda) = \left[\begin{array}{ccc|c} \frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)} & & & \\ & \ddots & & 0_{r\times(n-r)} \\ & & \frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)} & \\ \hline & 0_{(m-r)\times r} & & 0_{(m-r)\times(n-r)} \end{array}\right].$$

- $\varepsilon_1(\lambda), \cdots, \varepsilon_r(\lambda), \psi_1(\lambda), \cdots, \psi_r(\lambda)$ are monic polynomials,
- the fractions $\dfrac{\varepsilon_i(\lambda)}{\psi_i(\lambda)}$ are irreducible (invariant fractions),
- $\varepsilon_j(\lambda)$ divides $\varepsilon_{j+1}(\lambda)$ and $\psi_{j+1}(\lambda)$ divides $\psi_j(\lambda)$, for $j = 1, \ldots, r-1$,
- $r = \operatorname{rank} G(\lambda)$.

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Definition (finite zeros and finite poles)

Given the **Smith-McMillan form** of a rational matrix $R(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$:

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The **finite zeros** of $R(\lambda)$ are the roots of the numerators and the **finite poles** are the roots of the denominators.

Remark

Given any $c \in \mathbb{C}$, one can write for each $i = 1, \dots, r$,

$$\frac{\varepsilon_i(\lambda)}{\psi_i(\lambda)} = (\lambda - c)^{\sigma_i(c)} \frac{\widetilde{\varepsilon}_i(\lambda)}{\widetilde{\psi}_i(\lambda)}, \qquad \text{with } \widetilde{\varepsilon}_i(c) \neq 0,\ \widetilde{\psi}_i(c) \neq 0.$$

Definition (Structural indices at $c \in \mathbb{C}$)

The structural indices of $R(\lambda)$ at c are

$$S(R, c) = (\sigma_1(c) \leq \sigma_2(c) \leq \dots \leq \sigma_r(c)).$$

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Definition

The structural indices of $R(\lambda)$ at ∞ are identical to the structural indices of $R(1/\lambda)$ at $\lambda = 0$.

Proposition: The smallest structural index at infinity (Amparan, Marcaida & Zaballa, ELA, 2015)

The smallest structural index of $R(\lambda)$ at infinity is

1. $-$degree (polynomial part of $R(\lambda)$), if this poly part is nonzero,
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Total numbers of poles and zeros

Theorem (Van Dooren's Index Sum Theorem for Rational Matrices)

Let $\delta_p(R)$ and $\delta_z(R)$ denote the **total number of poles and zeros** *(finite and infinite) respectively of an arbitrary rational matrix $R(\lambda)$, and let $\alpha(R)$ denote the sum of its left and right minimal indices. Then*

$$\delta_p(R) = \delta_z(R) + \alpha(R) \,.$$

Definition (total numbers of poles and zeros)

- The total number of poles of $R(\lambda)$ is minus the sum of all negative structural indices of $R(\lambda)$ (including those at ∞).
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Minimal bases and minimal indices of a singular Rational Matrix

- They characterize the structure of the rational null spaces:

$$\begin{aligned}
\mathcal{N}_\ell(R) &:= \left\{ y(\lambda)^T \in \mathbb{C}(\lambda)^{1\times m} : y(\lambda)^T R(\lambda) \equiv 0^T \right\}, \\
\mathcal{N}_r(R) &:= \left\{ x(\lambda) \in \mathbb{C}(\lambda)^{n\times 1} : R(\lambda) x(\lambda) \equiv 0 \right\}.
\end{aligned}$$

Example: minimal indices

Consider the rational matrix

$$R(\lambda) = \begin{bmatrix} \frac{\lambda}{\lambda-1} & & & & & \\ & \frac{1}{\lambda-1} & & & & \\ & & (\lambda-1)^2 & & & \\ & & & 1 & \lambda^2 & \\ & & & & 1 & \lambda^7 \end{bmatrix} \in \mathbb{C}(\lambda)^{5\times 6}$$

$$\text{rank}(R) = 5 \implies \dim \mathcal{N}_\ell(R) = 0 \quad \text{and} \quad \dim \mathcal{N}_r(R) = 1$$

- $\{[0,0,0,\lambda^9,-\lambda^7,1]^T\}$ right minimal basis of $R(\lambda)$, which has only one right minimal index equal to 9.
- Sum of all minimal indices of $R(\lambda)$ is $\alpha(R) = 9$.

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- Sum of all minimal indices of $R(\lambda)$ is $\alpha(R) = 9$.

- Thus, **a polynomial matrix $P(\lambda)$ of degree d has always at least one pole of order d at infinity when seen as a rational matrix**, i.e.,
- if rank$(P) = r$, then the structural indices at infinity are

$$S(P, \infty) = (-d \leq s_2 \leq \cdots \leq s_r),$$

which are the structural indices at 0 of $P(1/\lambda)$.

- **But**, in the "community of polynomial matrices", the structure at infinity is usually defined in a different way as follows.

- Thus, **a polynomial matrix $P(\lambda)$ of degree d has always at least one pole of order d at infinity when seen as a rational matrix**, i.e.,
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Definition (Reversal polynomial)

Let $P(\lambda) = P_d\lambda^d + P_{d-1}\lambda^{d-1} + \cdots + P_0$ be a polynomial matrix of **degree** d. The **reversal** of $P(\lambda)$ is

$$\mathrm{rev}P(\lambda) := \lambda^d P\left(\frac{1}{\lambda}\right) = P_d + P_{d-1}\,\lambda + \cdots + P_0\,\lambda^d\,.$$

Definition (Eigenvalues at ∞ of a polynomial matrix)

$P(\lambda)$ **has an eigenvalue at ∞ if 0 is an eigenvalue of** $\mathrm{rev}P(\lambda)$ and the partial multiplicity sequence of $P(\lambda)$ at ∞ is the same as that of 0 in $\mathrm{rev}P(\lambda)$.

Proposition

If $P(\lambda)$ is a polynomial matrix of degree d with structural indices at ∞

$$S(P,\infty) = (-d \le s_2 \le \cdots \le s_r)\,.$$

Then, the partial multiplicity sequence of $P(\lambda)$ at infinity is

$$M(P,\infty) = (0 \le s_2 + d \le \cdots \le s_r + d)\,.$$

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$P(\lambda)$ **has an eigenvalue at ∞ if 0 is an eigenvalue of $\mathrm{rev}P(\lambda)$** and the partial multiplicity sequence of $P(\lambda)$ at ∞ is the same as that of 0 in $\mathrm{rev}P(\lambda)$.

Proposition

If $P(\lambda)$ is a polynomial matrix of degree d with structural indices at ∞

$$S(P,\infty) = (-d \le s_2 \le \cdots \le s_r)\,.$$

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Definition (Reversal polynomial)

Let $P(\lambda) = P_d\lambda^d + P_{d-1}\lambda^{d-1} + \cdots + P_0$ be a polynomial matrix of **degree** d. The **reversal** of $P(\lambda)$ is

$$\mathrm{rev}P(\lambda) := \lambda^d P\left(\frac{1}{\lambda}\right) = P_d + P_{d-1}\,\lambda + \cdots + P_0\,\lambda^d .$$

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Outline

Theorem (Van Dooren's Index Sum Theorem for Rational Matrices)

Let $\delta_p(R)$ and $\delta_z(R)$ denote the total number of poles and zeros (finite and infinite) respectively of an arbitrary rational matrix $R(\lambda)$, and let $\alpha(R)$ denote the sum of its left and right minimal indices. Then

$$\delta_p(R) = \delta_z(R) + \alpha(R)\,.$$

If $R(\lambda) = P(\lambda)$ is a polynomial matrix of degree d, $r = \operatorname{rank}(P)$, and with structural indices at ∞ given by

$$S(P,\infty) = (-d \leq s_2 \leq \cdots \leq s_k < 0 \leq s_{k+1} \leq \cdots \leq s_r),$$

then, **since $P(\lambda)$ has poles only at infinity**,

$$\delta_p(P) = -\left(-d + \sum_{i=2}^{k} s_i\right) \quad \text{and} \quad \delta_z(P) = \sum_{i=k+1}^{r} s_i + \delta_z^{finite}(P)\,.$$

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- using, these three blue equalities,
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Main ideas in proof "Polynomial IST implies Van Dooren's Rational IST"

- This implication may seem surprising at a first glance since rational matrices are not a particular case of polynomial matrices.
- Given a rational matrix $R(\lambda)$, the key point is to apply the Polynomial IST to the polynomial matrix

$$P(\lambda) = \psi_1(\lambda)\, R(\lambda),$$

where $\psi_1(\lambda)$ is the first denominator in the Smith-McMillan form of $R(\lambda)$,
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One application of IST for Polynomial Matrices is "the fundamental realization theorem for polynomial matrices" (Steve Mackey's name)

Theorem (De Terán, D, Van Dooren, SIMAX, (2015))

Consider that the following data

- m, n, d, *and* $r \leq \min\{m, n\}$ *positive integers,*
- r *scalar monic polynomials such that* $p_1(\lambda)|p_2(\lambda)|\cdots|p_r(\lambda)$,
- $0 = \gamma_1 \leq \cdots \leq \gamma_r$ *integers,*
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are prescribed. *Then,* **there exists an $m \times n$ polynomial matrix**, *with rank r, with degree d, with invariant polynomials $p_1(\lambda), \ldots, p_r(\lambda)$, with partial multiplicities at infinity $\gamma_1, \cdots, \gamma_r$, and with right and left minimal indices equal to $\alpha_1, \cdots, \alpha_{n-r}$ and $\eta_1, \cdots, \eta_{m-r}$, respectively, if and only if*

$$\sum_{j=1}^{r} \text{degree}(p_j) + \sum_{j=1}^{r} \gamma_j + \sum_{j=1}^{n-r} \alpha_j + \sum_{j=1}^{m-r} \eta_j = dr,$$

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Theorem (Anguas, D, Hollister, Mackey, submitted, (2018))

Consider that the following data

- m, n, *and* $r \leq \min\{m, n\}$ *positive integers,*
- r *(monic) irreducible fractions* $\frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)}, \ldots, \frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)}$, *such that* $\varepsilon_1(\lambda) \,|\, \cdots \,|\, \varepsilon_r(\lambda)$ *and* $\psi_r(\lambda) \,|\, \cdots \,|\, \psi_1(\lambda)$,
- $\gamma_1 \leq \cdots \leq \gamma_r$ *integers (sequence of potential structural indices at infinity),*
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- **Key idea.** Get from the prescribed data satisfying Van Dooren's IST the polynomial data:
  - m, n, and $r \leq \min\{m, n\}$ positive integers,
  - r monic polys $\frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)}\psi_1(\lambda) \,|\, \cdots \,|\, \frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)}\psi_1(\lambda)$,
  - $0 \leq \gamma_2 - \gamma_1 \leq \cdots \leq \gamma_r - \gamma_1$ integers (sequence of multiplicities at ∞),
  - $0 \leq \alpha_1 \leq \cdots \leq \alpha_{n-r}$ and $0 \leq \eta_1 \leq \cdots \leq \eta_{m-r}$ integers.

  and solve the corresponding inverse polynomial problem.