Sets of matrix polynomials with bounded rank and degree and their generic eigenstructures

Froilán M. Dopico

joint work with **Fernando De Terán** (UC3M, Spain), **Andrii Dmytryshyn** (Umeå University, Sweden), and **Paul Van Dooren** (UC Louvain, Belgium)

Departamento de Matemáticas
Universidad Carlos III de Madrid, Spain

Workshop on "Matrix Theory and beyond"
Universidade de Lisboa, Portugal, December 14, 2018

- We will consider the vector space

$$\mathrm{POL}_d^{m\times n} := \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \end{array} \right\}.$$

- The Euclidean distance in $\mathrm{POL}_d^{m\times n}$ is defined as follows. Given

$$P(\lambda) = \lambda^d P_d + \cdots + \lambda P_1 + P_0 \in \mathrm{POL}_d^{m\times n}, \quad (P_i \in \mathbb{C}^{m\times n}),$$
$$Q(\lambda) = \lambda^d Q_d + \cdots + \lambda Q_1 + Q_0 \in \mathrm{POL}_d^{m\times n}, \quad (Q_i \in \mathbb{C}^{m\times n}),$$

$$\rho(P,Q) := \sqrt{\sum_{i=0}^{d} \|P_i - Q_i\|_F^2}.$$

- It makes $\mathrm{POL}_d^{m\times n}$ a metric space and we can consider closures of subsets of $\mathrm{POL}_d^{m\times n}$, as well as any other topological concept.
- The closure of any set $\mathcal{A}$ is denoted by $\overline{\mathcal{A}}$.

Setting (I): The ambient vector space and its metric

- We will consider the vector space

$$\mathrm{POL}_d^{m\times n} := \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \end{array} \right\}.$$

- The Euclidean distance in $\mathrm{POL}_d^{m\times n}$ is defined as follows. Given

$$P(\lambda) = \lambda^d P_d + \cdots + \lambda P_1 + P_0 \in \mathrm{POL}_d^{m\times n}, \quad (P_i \in \mathbb{C}^{m\times n}),$$
$$Q(\lambda) = \lambda^d Q_d + \cdots + \lambda Q_1 + Q_0 \in \mathrm{POL}_d^{m\times n}, \quad (Q_i \in \mathbb{C}^{m\times n}),$$

$$\rho(P,Q) := \sqrt{\sum_{i=0}^{d} \|P_i - Q_i\|_F^2}\,.$$

- It makes $\mathrm{POL}_d^{m\times n}$ a metric space and we can consider closures of subsets of $\mathrm{POL}_d^{m\times n}$, as well as any other topological concept.
- The closure of any set $\mathcal{A}$ is denoted by $\overline{\mathcal{A}}$.

Setting (I): The ambient vector space and its metric

- We will consider the vector space

$$\mathrm{POL}_d^{m\times n} := \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \end{array} \right\}.$$

- The Euclidean distance in $\mathrm{POL}_d^{m\times n}$ is defined as follows. Given

$$P(\lambda) = \lambda^d P_d + \cdots + \lambda P_1 + P_0 \in \mathrm{POL}_d^{m\times n}, \quad (P_i \in \mathbb{C}^{m\times n}),$$
$$Q(\lambda) = \lambda^d Q_d + \cdots + \lambda Q_1 + Q_0 \in \mathrm{POL}_d^{m\times n}, \quad (Q_i \in \mathbb{C}^{m\times n}),$$

$$\rho(P,Q) := \sqrt{\sum_{i=0}^{d} \|P_i - Q_i\|_F^2}.$$

- It makes $\mathrm{POL}_d^{m\times n}$ a metric space and we can consider closures of subsets of $\mathrm{POL}_d^{m\times n}$, as well as any other topological concept.

- The closure of any set $\mathcal{A}$ is denoted by $\overline{\mathcal{A}}$.

Setting (I): The ambient vector space and its metric

- We will consider the vector space

$$\mathrm{POL}_d^{m\times n} := \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \end{array} \right\}.$$

- The Euclidean distance in $\mathrm{POL}_d^{m\times n}$ is defined as follows. Given

$$P(\lambda) = \lambda^d P_d + \cdots + \lambda P_1 + P_0 \in \mathrm{POL}_d^{m\times n}, \quad (P_i \in \mathbb{C}^{m\times n}),$$
$$Q(\lambda) = \lambda^d Q_d + \cdots + \lambda Q_1 + Q_0 \in \mathrm{POL}_d^{m\times n}, \quad (Q_i \in \mathbb{C}^{m\times n}),$$

$$\rho(P,Q) := \sqrt{\sum_{i=0}^{d} \|P_i - Q_i\|_F^2}.$$

- It makes $\mathrm{POL}_d^{m\times n}$ a metric space and we can consider closures of subsets of $\mathrm{POL}_d^{m\times n}$, as well as any other topological concept.

- The closure of any set $\mathcal{A}$ is denoted by $\overline{\mathcal{A}}$.

Setting (I): The ambient vector space and its metric

- We will consider the vector space

$$\mathrm{POL}_d^{m\times n} := \left\{ \begin{array}{c} m\times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \end{array} \right\}.$$

- The Euclidean distance in $\mathrm{POL}_d^{m\times n}$ is defined as follows. Given

$$P(\lambda) = \lambda^d P_d + \cdots + \lambda P_1 + P_0 \in \mathrm{POL}_d^{m\times n}, \quad (P_i \in \mathbb{C}^{m\times n}),$$
$$Q(\lambda) = \lambda^d Q_d + \cdots + \lambda Q_1 + Q_0 \in \mathrm{POL}_d^{m\times n}, \quad (Q_i \in \mathbb{C}^{m\times n}),$$

$$\rho(P,Q) := \sqrt{\sum_{i=0}^{d} \|P_i - Q_i\|_F^2}\,.$$

- It makes $\mathrm{POL}_d^{m\times n}$ a metric space and we can consider closures of subsets of $\mathrm{POL}_d^{m\times n}$, as well as any other topological concept.
- The closure of any set $\mathcal{A}$ is denoted by $\overline{\mathcal{A}}$.

Setting (II): The subsets of $\mathrm{POL}_d^{m\times n}$ studied in this talk

- Our main goal is to describe the sets

$$\mathrm{POL}_{d,r}^{m\times n} := \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \\ \text{and (normal) rank at most } r \end{array} \right\} \subseteq \mathrm{POL}_d^{m\times n},$$

- where r is a fixed positive integer such that
  - $r \leq \min\{m,n\}$, if $m \neq n$,
  - $r \leq (n-1)$, if $m = n$.
- This means that we consider sets of singular polynomials.
- The set $\mathrm{POL}_{d,r}^{m\times n}$ contains matrix polynomials with many different properties, but **generically** (most of the times) the matrix polynomials of $\mathrm{POL}_{d,r}^{m\times n}$ have just a few possible eigenstructures.
- In this talk, **generically** means that "all the matrix polynomials in an open dense subset of $\mathrm{POL}_{d,r}^{m\times n}$ have just a few possible eigenstructures",
- where we consider in $\mathrm{POL}_{d,r}^{m\times n}$ the subspace topology corresponding to the Euclidean topology in $\mathrm{POL}_d^{m\times n}$.

Setting (II): The subsets of $\mathrm{POL}_d^{m\times n}$ studied in this talk

- Our main goal is to describe the sets

$$\mathrm{POL}_{d,r}^{m\times n} := \left\{ \begin{array}{c} m\times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \\ \text{and (normal) rank at most } r \end{array} \right\} \subseteq \mathrm{POL}_d^{m\times n},$$

- where r is a fixed positive integer such that
  - $r \leq \min\{m,n\}$, if $m \neq n$,
  - $r \leq (n-1)$, if $m = n$.
- This means that we consider sets of singular polynomials.
- The set $\mathrm{POL}_{d,r}^{m\times n}$ contains matrix polynomials with many different properties, but **generically** (most of the times) the matrix polynomials of $\mathrm{POL}_{d,r}^{m\times n}$ have just a few possible eigenstructures.
- In this talk, **generically** means that "all the matrix polynomials in an open dense subset of $\mathrm{POL}_{d,r}^{m\times n}$ have just a few possible eigenstructures",
- where we consider in $\mathrm{POL}_{d,r}^{m\times n}$ the subspace topology corresponding to the Euclidean topology in $\mathrm{POL}_d^{m\times n}$.

Setting (II): The subsets of $\mathrm{POL}_d^{m\times n}$ studied in this talk

- Our main goal is to describe the sets

$$\mathrm{POL}_{d,r}^{m\times n} := \left\{ \begin{array}{c} m\times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \\ \text{and (normal) rank at most } r \end{array} \right\} \subseteq \mathrm{POL}_d^{m\times n},$$

- where r is a fixed positive integer such that
  - $r \leq \min\{m,n\}$, if $m \neq n$,
  - $r \leq (n-1)$, if $m = n$.
- This means that we consider sets of singular polynomials.
- The set $\mathrm{POL}_{d,r}^{m\times n}$ contains matrix polynomials with many different properties, but **generically** (most of the times) the matrix polynomials of $\mathrm{POL}_{d,r}^{m\times n}$ have just a few possible eigenstructures.
- In this talk, **generically** means that "all the matrix polynomials in an open dense subset of $\mathrm{POL}_{d,r}^{m\times n}$ have just a few possible eigenstructures",
- where we consider in $\mathrm{POL}_{d,r}^{m\times n}$ the subspace topology corresponding to the Euclidean topology in $\mathrm{POL}_d^{m\times n}$.

Setting (II): The subsets of $\mathrm{POL}_d^{m\times n}$ studied in this talk

- Our main goal is to describe the sets

$$\mathrm{POL}_{d,r}^{m\times n} := \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \\ \text{and (normal) rank at most } r \end{array} \right\} \subseteq \mathrm{POL}_d^{m\times n},$$

- where r is a fixed positive integer such that
  - $r \leq \min\{m,n\}$, if $m \neq n$,
  - $r \leq (n-1)$, if $m = n$.
- This means that we consider sets of singular polynomials.
- The set $\mathrm{POL}_{d,r}^{m\times n}$ contains matrix polynomials with many different properties, but **generically** (most of the times) the matrix polynomials of $\mathrm{POL}_{d,r}^{m\times n}$ have just a few possible eigenstructures.
- In this talk, **generically** means that "all the matrix polynomials in an open dense subset of $\mathrm{POL}_{d,r}^{m\times n}$ have just a few possible eigenstructures",
- where we consider in $\mathrm{POL}_{d,r}^{m\times n}$ the subspace topology corresponding to the Euclidean topology in $\mathrm{POL}_d^{m\times n}$.

- Our main goal is to describe the sets

$$\mathrm{POL}_{d,r}^{m\times n} := \left\{ \begin{array}{c} m\times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \\ \text{and (normal) rank at most } r \end{array} \right\} \subseteq \mathrm{POL}_d^{m\times n},$$

- where r is a fixed positive integer such that
  - $r \leq \min\{m,n\}$, if $m \neq n$,
  - $r \leq (n-1)$, if $m = n$.
- This means that we consider sets of singular polynomials.
- The set $\mathrm{POL}_{d,r}^{m\times n}$ contains matrix polynomials with many different properties, but **generically** (most of the times) the matrix polynomials of $\mathrm{POL}_{d,r}^{m\times n}$ have just a few possible eigenstructures.
- In this talk, **generically** means that "all the matrix polynomials in an open dense subset of $\mathrm{POL}_{d,r}^{m\times n}$ have just a few possible eigenstructures",
- where we consider in $\mathrm{POL}_{d,r}^{m\times n}$ the subspace topology corresponding to the Euclidean topology in $\mathrm{POL}_d^{m\times n}$.

Setting (II): The subsets of $\mathrm{POL}_d^{m\times n}$ studied in this talk

- Our main goal is to describe the sets

$$\mathrm{POL}_{d,r}^{m\times n} := \left\{ \begin{array}{c} m\times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \\ \text{and (normal) rank at most } r \end{array} \right\} \subseteq \mathrm{POL}_d^{m\times n},$$

- where r is a fixed positive integer such that
  - $r \leq \min\{m,n\}$, if $m \neq n$,
  - $r \leq (n-1)$, if $m = n$.
- This means that we consider sets of singular polynomials.
- The set $\mathrm{POL}_{d,r}^{m\times n}$ contains matrix polynomials with many different properties, but **generically** (most of the times) the matrix polynomials of $\mathrm{POL}_{d,r}^{m\times n}$ have just a few possible eigenstructures.
- In this talk, **generically** means that "all the matrix polynomials in an open dense subset of $\mathrm{POL}_{d,r}^{m\times n}$ have just a few possible eigenstructures",
- where we consider in $\mathrm{POL}_{d,r}^{m\times n}$ the subspace topology corresponding to the Euclidean topology in $\mathrm{POL}_d^{m\times n}$.

Setting (III): Main "informal" results of this talk

Generically a matrix polynomial in $\mathrm{POL}_{d,r}^{m\times n}$

- does not have eigenvalues,
- that is, its eigenstructure has only minimal indices,
- its left minimal indices differ at most by one (they try to be as equal as possible),
- and, its right minimal indices also differ at most by one (they also try to be as equal as possible).
- We refer to this property as "the minimal indices are generically almost homogeneous".

Of course, these properties do not tell the whole story since the generic possible values of the minimal indices have to be determined.

- **Structured matrix polynomials** (in this talk, symmetric and skew-symmetric) may have different properties but the fact that "the minimal indices are generically almost homogeneous" seems to be universal.

Setting (III): Main "informal" results of this talk

Generically a matrix polynomial in $\mathrm{POL}_{d,r}^{m\times n}$

- does not have eigenvalues,
- that is, its eigenstructure has only minimal indices,
- its left minimal indices differ at most by one (they try to be as equal as possible),
- and, its right minimal indices also differ at most by one (they also try to be as equal as possible).
- We refer to this property as "the minimal indices are generically almost homogeneous".

Of course, these properties do not tell the whole story since the generic possible values of the minimal indices have to be determined.

- **Structured matrix polynomials** (in this talk, symmetric and skew-symmetric) may have different properties but the fact that "the minimal indices are generically almost homogeneous" seems to be universal.

Setting (III): Main "informal" results of this talk

Generically a matrix polynomial in $\mathrm{POL}_{d,r}^{m\times n}$

- does not have eigenvalues,
- that is, its eigenstructure has only minimal indices,
- its left minimal indices differ at most by one (they try to be as equal as possible),
- and, its right minimal indices also differ at most by one (they also try to be as equal as possible).
- We refer to this property as "the minimal indices are generically almost homogeneous".

Of course, these properties do not tell the whole story since the generic possible values of the minimal indices have to be determined.

- **Structured matrix polynomials** (in this talk, symmetric and skew-symmetric) may have different properties but the fact that "the minimal indices are generically almost homogeneous" seems to be universal.

Generically a matrix polynomial in $\mathrm{POL}_{d,r}^{m\times n}$

- does not have eigenvalues,
- that is, its eigenstructure has only minimal indices,
- its left minimal indices differ at most by one (they try to be as equal as possible),
- and, its right minimal indices also differ at most by one (they also try to be as equal as possible).
- We refer to this property as "the minimal indices are generically almost homogeneous".

Of course, these properties do not tell the whole story since the generic possible values of the minimal indices have to be determined.

- **Structured matrix polynomials** (in this talk, symmetric and skew-symmetric) may have different properties but the fact that "the minimal indices are generically almost homogeneous" seems to be universal.

Generically a matrix polynomial in $\mathrm{POL}_{d,r}^{m\times n}$

- does not have eigenvalues,
- that is, its eigenstructure has only minimal indices,
- its left minimal indices differ at most by one (they try to be as equal as possible),
- and, its right minimal indices also differ at most by one (they also try to be as equal as possible).
- We refer to this property as "the minimal indices are generically almost homogeneous".

Of course, these properties do not tell the whole story since the generic possible values of the minimal indices have to be determined.

- **Structured matrix polynomials** (in this talk, symmetric and skew-symmetric) may have different properties but the fact that "the minimal indices are generically almost homogeneous" seems to be universal.

Generically a matrix polynomial in $\mathrm{POL}_{d,r}^{m\times n}$

- does not have eigenvalues,
- that is, its eigenstructure has only minimal indices,
- its left minimal indices differ at most by one (they try to be as equal as possible),
- and, its right minimal indices also differ at most by one (they also try to be as equal as possible).
- We refer to this property as "the minimal indices are generically almost homogeneous".

Of course, these properties do not tell the whole story since the generic possible values of the minimal indices have to be determined.

- **Structured matrix polynomials** (in this talk, symmetric and skew-symmetric) may have different properties but the fact that "the minimal indices are generically almost homogeneous" seems to be universal.

Generically a matrix polynomial in $\mathrm{POL}_{d,r}^{m\times n}$

- does not have eigenvalues,
- that is, its eigenstructure has only minimal indices,
- its left minimal indices differ at most by one (they try to be as equal as possible),
- and, its right minimal indices also differ at most by one (they also try to be as equal as possible).
- We refer to this property as "the minimal indices are generically almost homogeneous".

Of course, these properties do not tell the whole story since the generic possible values of the minimal indices have to be determined.

- **Structured matrix polynomials** (in this talk, symmetric and skew-symmetric) may have different properties but the fact that "the minimal indices are generically almost homogeneous" seems to be universal.

Setting (III): Main "informal" results of this talk

Generically a matrix polynomial in $\mathrm{POL}_{d,r}^{m\times n}$

- does not have eigenvalues,
- that is, its eigenstructure has only minimal indices,
- its left minimal indices differ at most by one (they try to be as equal as possible),
- and, its right minimal indices also differ at most by one (they also try to be as equal as possible).
- We refer to this property as "the minimal indices are generically almost homogeneous".

Of course, these properties do not tell the whole story since the generic possible values of the minimal indices have to be determined.

- **Structured matrix polynomials** (in this talk, symmetric and skew-symmetric) may have different properties but the fact that "the minimal indices are generically almost homogeneous" seems to be universal.

Setting (IV): Motivation for the problems considered in this talk

- Many papers have studied (many very recently) the effect of "low rank" perturbations on the eigenstructure of **matrices and (regular) pencils**
- from many different perspectives: generic, nongeneric, for several classes of structured matrices and pencils, etc.
- Some names involved in this research area are: Baragaña, Batzke, De Terán, Dodig, D., Gernandt, Hörmander, Mehl, Mehrmann, Melin, Moro, Ran, Rodman, Roca, Trunk, Savchenko, Silva, Stošić, Wojtylak, Zaballa, ...
- However, there are essentially no papers on the effect of "low rank" perturbations on the eigenstructure of **matrix polynomials of given degree** (for instance, quadratic polynomials),
- which is a more difficult problem.
- We think that such difficulty is related to the fact that for any fixed (low) rank the structure of the set of matrix polynomials that have (at most) that given rank and a certain bounded degree is not well understood.
- The results in this talk are a first step in this direction.

Setting (IV): Motivation for the problems considered in this talk

- Many papers have studied (many very recently) the effect of "low rank" perturbations on the eigenstructure of **matrices and (regular) pencils**
- from many different perspectives: generic, nongeneric, for several classes of structured matrices and pencils, etc.
- Some names involved in this research area are: Baragaña, Batzke, De Terán, Dodig, D., Gernandt, Hörmander, Mehl, Mehrmann, Melin, Moro, Ran, Rodman, Roca, Trunk, Savchenko, Silva, Stošić, Wojtylak, Zaballa, ...
- However, there are essentially no papers on the effect of "low rank" perturbations on the eigenstructure of **matrix polynomials of given degree** (for instance, quadratic polynomials),
- which is a more difficult problem.
- We think that such difficulty is related to the fact that for any fixed (low) rank the structure of the set of matrix polynomials that have (at most) that given rank and a certain bounded degree is not well understood.
- The results in this talk are a first step in this direction.

Setting (IV): Motivation for the problems considered in this talk

- Many papers have studied (many very recently) the effect of "low rank" perturbations on the eigenstructure of **matrices and (regular) pencils**
- from many different perspectives: generic, nongeneric, for several classes of structured matrices and pencils, etc.
- Some names involved in this research area are: Baragaña, Batzke, De Terán, Dodig, D., Gernandt, Hörmander, Mehl, Mehrmann, Melin, Moro, Ran, Rodman, Roca, Trunk, Savchenko, Silva, Stošić, Wojtylak, Zaballa, ...
- However, there are essentially no papers on the effect of "low rank" perturbations on the eigenstructure of **matrix polynomials of given degree** (for instance, quadratic polynomials),
- which is a more difficult problem.
- We think that such difficulty is related to the fact that for any fixed (low) rank the structure of the set of matrix polynomials that have (at most) that given rank and a certain bounded degree is not well understood.
- The results in this talk are a first step in this direction.

Setting (IV): Motivation for the problems considered in this talk

- Many papers have studied (many very recently) the effect of "low rank" perturbations on the eigenstructure of **matrices and (regular) pencils**
- from many different perspectives: generic, nongeneric, for several classes of structured matrices and pencils, etc.
- Some names involved in this research area are: Baragaña, Batzke, De Terán, Dodig, D., Gernandt, Hörmander, Mehl, Mehrmann, Melin, Moro, Ran, Rodman, Roca, Trunk, Savchenko, Silva, Stošić, Wojtylak, Zaballa, ...
- However, there are essentially no papers on the effect of "low rank" perturbations on the eigenstructure of **matrix polynomials of given degree** (for instance, quadratic polynomials),
- which is a more difficult problem.
- We think that such difficulty is related to the fact that for any fixed (low) rank the structure of the set of matrix polynomials that have (at most) that given rank and a certain bounded degree is not well understood.
- The results in this talk are a first step in this direction.

Setting (IV): Motivation for the problems considered in this talk

- Many papers have studied (many very recently) the effect of "low rank" perturbations on the eigenstructure of **matrices and (regular) pencils**
- from many different perspectives: generic, nongeneric, for several classes of structured matrices and pencils, etc.
- Some names involved in this research area are: Baragaña, Batzke, De Terán, Dodig, D., Gernandt, Hörmander, Mehl, Mehrmann, Melin, Moro, Ran, Rodman, Roca, Trunk, Savchenko, Silva, Stošić, Wojtylak, Zaballa, ...
- However, there are essentially no papers on the effect of "low rank" perturbations on the eigenstructure of **matrix polynomials of given degree** (for instance, quadratic polynomials),
- which is a more difficult problem.
- We think that such difficulty is related to the fact that for any fixed (low) rank the structure of the set of matrix polynomials that have (at most) that given rank and a certain bounded degree is not well understood.
- The results in this talk are a first step in this direction.

Setting (IV): Motivation for the problems considered in this talk

- Many papers have studied (many very recently) the effect of “low rank” perturbations on the eigenstructure of **matrices and (regular) pencils**
- from many different perspectives: generic, nongeneric, for several classes of structured matrices and pencils, etc.
- Some names involved in this research area are: Baragaña, Batzke, De Terán, Dodig, D., Gernandt, Hörmander, Mehl, Mehrmann, Melin, Moro, Ran, Rodman, Roca, Trunk, Savchenko, Silva, Stošić, Wojtylak, Zaballa, ...
- However, there are essentially no papers on the effect of “low rank” perturbations on the eigenstructure of **matrix polynomials of given degree** (for instance, quadratic polynomials),
- which is a more difficult problem.
- We think that such difficulty is related to the fact that for any fixed (low) rank the structure of the set of matrix polynomials that have (at most) that given rank and a certain bounded degree is not well understood.
- The results in this talk are a first step in this direction.

Setting (IV): Motivation for the problems considered in this talk

- Many papers have studied (many very recently) the effect of "low rank" perturbations on the eigenstructure of **matrices and (regular) pencils**
- from many different perspectives: generic, nongeneric, for several classes of structured matrices and pencils, etc.
- Some names involved in this research area are: Baragaña, Batzke, De Terán, Dodig, D., Gernandt, Hörmander, Mehl, Mehrmann, Melin, Moro, Ran, Rodman, Roca, Trunk, Savchenko, Silva, Stošić, Wojtylak, Zaballa, ...
- However, there are essentially no papers on the effect of "low rank" perturbations on the eigenstructure of **matrix polynomials of given degree** (for instance, quadratic polynomials),
- which is a more difficult problem.
- We think that such difficulty is related to the fact that for any fixed (low) rank the structure of the set of matrix polynomials that have (at most) that given rank and a certain bounded degree is not well understood.
- The results in this talk are a first step in this direction.

Outline

- **All the $m \times n$ pencils with the same complete eigenstructure form an orbit** under strict equivalence:

$$\mathrm{O}(\lambda A + B) := \{P(\lambda A + B)Q \mid \det P \cdot \det Q \neq 0\}.$$

- The complete eigenstructure of a pencil is determined by its **Kronecker canonical form (KCF)** under strict equivalence, which is a direct sum of four types of canonical matrix pencils:
- the regular $k \times k$ Jordan blocks for **finite and infinite eigenvalues**

$$\mathcal{J}_k(\mu) := \begin{bmatrix} \lambda - \mu & 1 & & \\ & \lambda - \mu & \ddots & \\ & & \ddots & 1 \\ & & & \lambda - \mu \end{bmatrix}, \quad \mathcal{J}_k(\infty) := \begin{bmatrix} 1 & \lambda & & \\ & 1 & \ddots & \\ & & \ddots & \lambda \\ & & & 1 \end{bmatrix} \quad k = 1, 2, 3, \ldots$$

- the singular $k \times (k+1)$ and $(k+1) \times k$ blocks for right and left **minimal indices** of value k

$$\mathcal{L}_k := \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \end{bmatrix}, \quad \mathcal{L}_k^T, \quad k = 0, 1, 2, \ldots$$

- **All the $m \times n$ pencils with the same complete eigenstructure form an orbit** under strict equivalence:

$$\mathrm{O}(\lambda A + B) := \{P(\lambda A + B)Q \mid \det P \cdot \det Q \neq 0\}.$$

- The complete eigenstructure of a pencil is determined by its **Kronecker canonical form (KCF)** under strict equivalence, which is a direct sum of four types of canonical matrix pencils:

- the regular $k \times k$ Jordan blocks for **finite and infinite eigenvalues**

$$\mathcal{J}_k(\mu) := \begin{bmatrix} \lambda-\mu & 1 & & \\ & \lambda-\mu & \ddots & \\ & & \ddots & 1 \\ & & & \lambda-\mu \end{bmatrix}, \quad \mathcal{J}_k(\infty) := \begin{bmatrix} 1 & \lambda & & \\ & 1 & \ddots & \\ & & \ddots & \lambda \\ & & & 1 \end{bmatrix} \quad k = 1, 2, 3, \ldots$$

- the singular $k \times (k+1)$ and $(k+1) \times k$ blocks for right and left **minimal indices** of value k

$$\mathcal{L}_k := \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \end{bmatrix}, \quad \mathcal{L}_k^T, \quad k = 0, 1, 2, \ldots$$

- **All the $m \times n$ pencils with the same complete eigenstructure form an orbit** under strict equivalence:

$$\mathrm{O}(\lambda A + B) := \{P(\lambda A + B)Q \ | \ \det P \cdot \det Q \neq 0\}.$$

- The complete eigenstructure of a pencil is determined by its **Kronecker canonical form (KCF)** under strict equivalence, which is a direct sum of four types of canonical matrix pencils:
- the regular $k \times k$ Jordan blocks for **finite and infinite eigenvalues**

$$\mathcal{J}_k(\mu) := \begin{bmatrix} \lambda-\mu & 1 & & \\ & \lambda-\mu & \ddots & \\ & & \ddots & 1 \\ & & & \lambda-\mu \end{bmatrix}, \quad \mathcal{J}_k(\infty) := \begin{bmatrix} 1 & \lambda & & \\ & 1 & \ddots & \\ & & \ddots & \lambda \\ & & & 1 \end{bmatrix} \quad k = 1, 2, 3, \ldots$$

- the singular $k \times (k+1)$ and $(k+1) \times k$ blocks for right and left **minimal indices** of value k

$$\mathcal{L}_k := \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \end{bmatrix}, \quad \mathcal{L}_k^T, \quad k = 0, 1, 2, \ldots$$

- **All the $m \times n$ pencils with the same complete eigenstructure form an orbit** under strict equivalence:

$$\mathrm{O}(\lambda A + B) := \{ P(\lambda A + B)Q \mid \det P \cdot \det Q \neq 0 \} .$$

- The complete eigenstructure of a pencil is determined by its **Kronecker canonical form (KCF)** under strict equivalence, which is a direct sum of four types of canonical matrix pencils:

- the regular $k \times k$ Jordan blocks for **finite and infinite eigenvalues**

$$\mathcal{J}_k(\mu) := \begin{bmatrix} \lambda-\mu & 1 & & \\ & \lambda-\mu & \ddots & \\ & & \ddots & 1 \\ & & & \lambda-\mu \end{bmatrix}, \quad \mathcal{J}_k(\infty) := \begin{bmatrix} 1 & \lambda & & \\ & 1 & \ddots & \\ & & \ddots & \lambda \\ & & & 1 \end{bmatrix} \quad k = 1, 2, 3, \ldots$$

- the singular $k \times (k+1)$ and $(k+1) \times k$ blocks for right and left **minimal indices** of value k

$$\mathcal{L}_k := \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \end{bmatrix}, \quad \mathcal{L}_k^T, \quad k = 0, 1, 2, \ldots$$

Theorem (De Terán and D., SIMAX, 2008)

Let m, n, and r be integers such that $m, n \geq 2$ and $1 \leq r \leq \min\{m, n\} - 1$. Then

$$\mathrm{POL}_{1,r}^{m\times n} = \left\{ \begin{array}{c} m \times n \text{ complex matrix pencils} \\ \text{with rank at most } r \end{array} \right\} = \bigcup_{0\leq a\leq r} \overline{\mathrm{O}}(\mathcal{K}_a),$$

where the $m \times n$ complex matrix pencils $\mathcal{K}_a, a = 0, 1, \ldots, r$, have rank r and the KCF

$$\mathcal{K}_a = \operatorname{diag}\left(\underbrace{\overbrace{\underbrace{\mathcal{L}_{\alpha+1}, \ldots, \mathcal{L}_{\alpha+1}}_{s}, \underbrace{\mathcal{L}_{\alpha}, \ldots, \mathcal{L}_{\alpha}}_{n-r-s}}^{\text{right minimal indices}}}_{\text{rank}=a}, \underbrace{\overbrace{\underbrace{\mathcal{L}_{\beta+1}^T, \ldots, \mathcal{L}_{\beta+1}^T}_{t}, \underbrace{\mathcal{L}_{\beta}^T, \ldots, \mathcal{L}_{\beta}^T}_{m-r-t}}^{\text{left minimal indices}}}_{\text{rank}=r-a} \right)$$

with $\alpha = \lfloor a/(n-r) \rfloor$ and $s = a \mod (n-r)$,
$\beta = \lfloor (r-a)/(m-r) \rfloor$ and $t = (r-a) \mod (m-r)$.

Moreover, $\overline{\mathrm{O}}(\mathcal{K}_a) \cap \mathrm{O}(\mathcal{K}_{a'}) = \varnothing$ whenever $a \neq a'$ (but $\overline{\mathrm{O}}(\mathcal{K}_a) \cap \overline{\mathrm{O}}(\mathcal{K}_{a'}) \neq \varnothing$).

F. De Terán and F.M. Dopico, A note on generic Kronecker orbits of matrix pencils with fixed rank, SIAM J. Matrix Anal. Appl., 30 (2008) 491–496

The set of matrix pencils with rank at most r

Theorem (De Terán and D., SIMAX, 2008)

Let m, n, and r be integers such that $m, n \geq 2$ and $1 \leq r \leq \min\{m, n\} - 1$. Then

$$\mathrm{POL}_{1,r}^{m\times n} = \left\{ \begin{array}{c} m \times n \text{ complex matrix pencils} \\ \text{with rank at most } r \end{array} \right\} = \bigcup_{0 \leq a \leq r} \overline{\mathrm{O}}(\mathcal{K}_a),$$

where the $m \times n$ complex matrix pencils $\mathcal{K}_a, a = 0, 1, \ldots, r$, have rank r and the KCF

$$\mathcal{K}_a = \mathrm{diag}\left(\underbrace{\overbrace{\underbrace{\mathcal{L}_{\alpha+1}, \ldots, \mathcal{L}_{\alpha+1}}_{s}, \underbrace{\mathcal{L}_{\alpha}, \ldots, \mathcal{L}_{\alpha}}_{n-r-s}}^{\text{right minimal indices}}}_{\text{rank}=a}, \underbrace{\overbrace{\underbrace{\mathcal{L}_{\beta+1}^T, \ldots, \mathcal{L}_{\beta+1}^T}_{t}, \underbrace{\mathcal{L}_{\beta}^T, \ldots, \mathcal{L}_{\beta}^T}_{m-r-t}}^{\text{left minimal indices}}}_{\text{rank}=r-a} \right)$$

with $\alpha = \lfloor a/(n-r) \rfloor$ and $s = a \mod (n-r)$,
$\beta = \lfloor (r-a)/(m-r) \rfloor$ and $t = (r-a) \mod (m-r)$.

Moreover, $\overline{\mathrm{O}}(\mathcal{K}_a) \cap \mathrm{O}(\mathcal{K}_{a'}) = \varnothing$ whenever $a \neq a'$ (but $\overline{\mathrm{O}}(\mathcal{K}_a) \cap \overline{\mathrm{O}}(\mathcal{K}_{a'}) \neq \varnothing$).

F. De Terán and F.M. Dopico, A note on generic Kronecker orbits of matrix pencils with fixed rank, SIAM J. Matrix Anal. Appl., 30 (2008) 491–496

The set of matrix pencils with rank at most r

Theorem (De Terán and D., SIMAX, 2008)

Let m, n, and r be integers such that $m, n \geq 2$ and $1 \leq r \leq \min\{m, n\} - 1$. Then

$$\mathrm{POL}_{1,r}^{m\times n} = \left\{ \begin{array}{c} m \times n \text{ complex matrix pencils} \\ \text{with rank at most } r \end{array} \right\} = \bigcup_{0\leq a\leq r} \overline{\mathrm{O}}(\mathcal{K}_a),$$

where the $m \times n$ complex matrix pencils $\mathcal{K}_a, a = 0, 1, \ldots, r$, have rank r and the KCF

$$\mathcal{K}_a = \mathrm{diag}\left(\underbrace{\overbrace{\underbrace{\mathcal{L}_{\alpha+1}, \ldots, \mathcal{L}_{\alpha+1}}_{s}, \underbrace{\mathcal{L}_{\alpha}, \ldots, \mathcal{L}_{\alpha}}_{n-r-s}}^{\text{right minimal indices}}}_{\text{rank}=a}, \underbrace{\overbrace{\underbrace{\mathcal{L}_{\beta+1}^T, \ldots, \mathcal{L}_{\beta+1}^T}_{t}, \underbrace{\mathcal{L}_{\beta}^T, \ldots, \mathcal{L}_{\beta}^T}_{m-r-t}}^{\text{left minimal indices}}}_{\text{rank}=r-a} \right)$$

with $\alpha = \lfloor a/(n-r) \rfloor$ and $s = a \mod (n-r)$,
$\beta = \lfloor (r-a)/(m-r) \rfloor$ and $t = (r-a) \mod (m-r)$.

Moreover, $\overline{\mathrm{O}}(\mathcal{K}_a) \bigcap \mathrm{O}(\mathcal{K}_{a'}) = \varnothing$ whenever $a \neq a'$ (but $\overline{\mathrm{O}}(\mathcal{K}_a) \bigcap \overline{\mathrm{O}}(\mathcal{K}_{a'}) \neq \varnothing$).

F. De Terán and F.M. Dopico, A note on generic Kronecker orbits of matrix pencils with fixed rank, SIAM J. Matrix Anal. Appl., 30 (2008) 491–496

The set of matrix pencils with rank at most r

Theorem (De Terán and D., SIMAX, 2008)

Let m, n, and r be integers such that $m, n \geq 2$ and $1 \leq r \leq \min\{m, n\} - 1$. Then

$$\mathrm{POL}_{1,r}^{m\times n} = \left\{ \begin{array}{c} m \times n \text{ complex matrix pencils} \\ \text{with rank at most } r \end{array} \right\} = \bigcup_{0\leq a\leq r} \overline{\mathrm{O}}(\mathcal{K}_a),$$

where the $m \times n$ complex matrix pencils $\mathcal{K}_a, a = 0, 1, \ldots, r$, have rank r and the KCF

$$\mathcal{K}_a = \mathrm{diag}\left(\underbrace{\overbrace{\underbrace{\mathcal{L}_{\alpha+1}, \ldots, \mathcal{L}_{\alpha+1}}_{s}, \underbrace{\mathcal{L}_{\alpha}, \ldots, \mathcal{L}_{\alpha}}_{n-r-s}}^{\text{right minimal indices}}}_{\text{rank}=a}, \underbrace{\overbrace{\underbrace{\mathcal{L}_{\beta+1}^T, \ldots, \mathcal{L}_{\beta+1}^T}_{t}, \underbrace{\mathcal{L}_{\beta}^T, \ldots, \mathcal{L}_{\beta}^T}_{m-r-t}}^{\text{left minimal indices}}}_{\text{rank}=r-a} \right)$$

with $\alpha = \lfloor a/(n-r) \rfloor$ and $s = a \mod (n-r)$,
$\beta = \lfloor (r-a)/(m-r) \rfloor$ and $t = (r-a) \mod (m-r)$.

Moreover, $\overline{\mathrm{O}}(\mathcal{K}_a) \cap \mathrm{O}(\mathcal{K}_{a'}) = \emptyset$ whenever $a \neq a'$ (but $\overline{\mathrm{O}}(\mathcal{K}_a) \cap \overline{\mathrm{O}}(\mathcal{K}_{a'}) \neq \emptyset$).

F. De Terán and F.M. Dopico, A note on generic Kronecker orbits of matrix pencils with fixed rank, SIAM J. Matrix Anal. Appl., 30 (2008) 491–496

Theorem (De Terán and D., SIMAX, 2008)

Let m, n, and r be integers such that $m, n \geq 2$ and $1 \leq r \leq \min\{m, n\} - 1$. Then

$$\mathrm{POL}_{1,r}^{m\times n} = \left\{ \begin{array}{c} m \times n \text{ complex matrix pencils} \\ \text{with rank at most } r \end{array} \right\} = \overline{\bigcup_{0\leq a\leq r} \mathrm{O}(\mathcal{K}_a)},$$

where the $m \times n$ complex matrix pencils $\mathcal{K}_a, a = 0, 1, \ldots, r$ have rank r and the KCF

$$\mathcal{K}_a = \operatorname{diag}\left(\overbrace{\underbrace{\mathcal{L}_{\alpha+1}, \ldots, \mathcal{L}_{\alpha+1}}_{s}, \underbrace{\mathcal{L}_{\alpha}, \ldots, \mathcal{L}_{\alpha}}_{n-r-s}}^{\text{right minimal indices}}, \overbrace{\underbrace{\mathcal{L}_{\beta+1}^T, \ldots, \mathcal{L}_{\beta+1}^T}_{t}, \underbrace{\mathcal{L}_{\beta}^T, \ldots, \mathcal{L}_{\beta}^T}_{m-r-t}}^{\text{left minimal indices}} \right)$$

with $\alpha = \lfloor a/(n-r) \rfloor$ and $s = a \mod (n-r)$,
$\beta = \lfloor (r-a)/(m-r) \rfloor$ and $t = (r-a) \mod (m-r)$.

Moreover, $\overline{\mathrm{O}}(\mathcal{K}_a) \bigcap \mathrm{O}(\mathcal{K}_{a'}) = \varnothing$ whenever $a \neq a'$ (but $\overline{\mathrm{O}}(\mathcal{K}_a) \bigcap \overline{\mathrm{O}}(\mathcal{K}_{a'}) \neq \varnothing$).

$\bigcup_{0\leq a\leq r} \mathrm{O}(\mathcal{K}_a)$ is an open dense subset of $\mathrm{POL}_{1,r}^{m\times n}$ (in the topology of $\mathrm{POL}_{1,r}^{m\times n}$). So,
generically, the $m \times n$ pencils with rank at most r have only $r+1$ possible KCFs given by $\mathcal{K}_a$ for $a = 0, 1, \ldots, r$.

Outline

$$P(\lambda) = \lambda^d P_d + \cdots + \lambda P_1 + P_0, \qquad P_i \in \mathbb{C}^{m\times n}$$

- **Essentially the same as in pencils** but definitions more complicated since there is NOT KCF.
- **Finite and infinite eigenvalues and their elementary divisors** defined with Smith Form under unimodular equivalence of $P(\lambda)$ and rev$P(\lambda)$:

$$U(\lambda)P(\lambda)V(\lambda) = \text{diag}\,(g_1(\lambda),\ldots,g_r(\lambda)) \oplus 0_{(m-r)\times(n-r)}, \qquad g_j(\lambda) \mid g_{j+1}(\lambda).$$

  Invariant polynomials: $g_j(\lambda) = (\lambda-\alpha_1)^{\delta_{j1}} \cdot (\lambda-\alpha_2)^{\delta_{j2}} \cdot \ldots \cdot (\lambda-\alpha_{l_j})^{\delta_{jl_j}}$.
  Elementary divisors: $(\lambda-\alpha_k)^{\delta_{jk}}$.

- **Left and right minimal indices** defined through the minimal bases of left and right rational null spaces of $P(\lambda)$:

$$\mathcal{N}_{\text{left}}(P) := \{y(\lambda)^T \in \mathbb{C}(\lambda)^{1\times m} : y(\lambda)^T P(\lambda) = 0_{1\times n}\},$$
$$\mathcal{N}_{\text{right}}(P) := \{x(\lambda) \in \mathbb{C}(\lambda)^{n\times 1} : P(\lambda)x(\lambda) = 0_{m\times 1}\}.$$

- **The definition of orbit does not involve a group action**

$$\mathrm{O}(P) = \left\{ \begin{array}{c} \text{matrix polynomials of the same size, degree,} \\ \text{and with the same complete eigenstructure as } P(\lambda) \end{array} \right\}$$

Complete eigenstructure of matrix polynomials

$$P(\lambda) = \lambda^d P_d + \cdots + \lambda P_1 + P_0, \qquad P_i \in \mathbb{C}^{m\times n}$$

- **Essentially the same as in pencils** but definitions more complicated since there is NOT KCF.
- **Finite and infinite eigenvalues and their elementary divisors** defined with Smith Form under unimodular equivalence of $P(\lambda)$ and rev$P(\lambda)$:

$$U(\lambda)P(\lambda)V(\lambda) = \text{diag}\left(g_1(\lambda),\ldots,g_r(\lambda)\right) \oplus 0_{(m-r)\times(n-r)}, \qquad g_j(\lambda) \mid g_{j+1}(\lambda).$$

Invariant polynomials: $g_j(\lambda) = (\lambda-\alpha_1)^{\delta_{j1}} \cdot (\lambda-\alpha_2)^{\delta_{j2}} \cdot \ldots \cdot (\lambda-\alpha_{l_j})^{\delta_{jl_j}}$.
Elementary divisors: $(\lambda-\alpha_k)^{\delta_{jk}}$.

- **Left and right minimal indices** defined through the minimal bases of left and right rational null spaces of $P(\lambda)$:

$$\mathcal{N}_{\text{left}}(P) := \{y(\lambda)^T \in \mathbb{C}(\lambda)^{1\times m} : y(\lambda)^T P(\lambda) = 0_{1\times n}\},$$
$$\mathcal{N}_{\text{right}}(P) := \{x(\lambda) \in \mathbb{C}(\lambda)^{n\times 1} : P(\lambda)x(\lambda) = 0_{m\times 1}\}.$$

- **The definition of orbit does not involve a group action**

$$\mathrm{O}(P) = \left\{ \begin{array}{c} \text{matrix polynomials of the same size, degree,} \\ \text{and with the same complete eigenstructure as } P(\lambda) \end{array} \right\}$$

Complete eigenstructure of matrix polynomials

$$P(\lambda) = \lambda^d P_d + \cdots + \lambda P_1 + P_0, \qquad P_i \in \mathbb{C}^{m\times n}$$

- **Essentially the same as in pencils** but definitions more complicated since there is NOT KCF.
- **Finite and infinite eigenvalues and their elementary divisors** defined with Smith Form under unimodular equivalence of $P(\lambda)$ and rev$P(\lambda)$:

$$U(\lambda)P(\lambda)V(\lambda) = \text{diag}\,(g_1(\lambda), \ldots, g_r(\lambda)) \oplus 0_{(m-r)\times(n-r)}, \quad g_j(\lambda) \mid g_{j+1}(\lambda).$$

Invariant polynomials: $g_j(\lambda) = (\lambda - \alpha_1)^{\delta_{j1}} \cdot (\lambda - \alpha_2)^{\delta_{j2}} \cdot \ldots \cdot (\lambda - \alpha_{l_j})^{\delta_{jl_j}}$.
Elementary divisors: $(\lambda - \alpha_k)^{\delta_{jk}}$.

- **Left and right minimal indices** defined through the minimal bases of left and right rational null spaces of $P(\lambda)$:

$$\mathcal{N}_{\text{left}}(P) := \{y(\lambda)^T \in \mathbb{C}(\lambda)^{1\times m} : y(\lambda)^T P(\lambda) = 0_{1\times n}\},$$

$$\mathcal{N}_{\text{right}}(P) := \{x(\lambda) \in \mathbb{C}(\lambda)^{n\times 1} : P(\lambda)x(\lambda) = 0_{m\times 1}\}.$$

- **The definition of orbit does not involve a group action**

$$\mathrm{O}(P) = \left\{ \begin{array}{c} \text{matrix polynomials of the same size, degree,} \\ \text{and with the same complete eigenstructure as } P(\lambda) \end{array} \right\}$$

$$P(\lambda) = \lambda^d P_d + \cdots + \lambda P_1 + P_0, \qquad P_i \in \mathbb{C}^{m\times n}$$

- **Essentially the same as in pencils** but definitions more complicated since there is NOT KCF.
- **Finite and infinite eigenvalues and their elementary divisors** defined with Smith Form under unimodular equivalence of $P(\lambda)$ and rev$P(\lambda)$:

$$U(\lambda)P(\lambda)V(\lambda) = \text{diag}\,(g_1(\lambda),\ldots,g_r(\lambda)) \oplus 0_{(m-r)\times(n-r)}, \quad g_j(\lambda) \mid g_{j+1}(\lambda).$$

  Invariant polynomials: $g_j(\lambda) = (\lambda-\alpha_1)^{\delta_{j1}} \cdot (\lambda-\alpha_2)^{\delta_{j2}} \cdot \ldots \cdot (\lambda-\alpha_{l_j})^{\delta_{jl_j}}$.
  Elementary divisors: $(\lambda-\alpha_k)^{\delta_{jk}}$.

- **Left and right minimal indices** defined through the minimal bases of left and right rational null spaces of $P(\lambda)$:

$$\mathcal{N}_{\text{left}}(P) := \{y(\lambda)^T \in \mathbb{C}(\lambda)^{1\times m} : y(\lambda)^T P(\lambda) = 0_{1\times n}\},$$
$$\mathcal{N}_{\text{right}}(P) := \{x(\lambda) \in \mathbb{C}(\lambda)^{n\times 1} : P(\lambda)x(\lambda) = 0_{m\times 1}\}.$$

- **The definition of orbit does not involve a group action**

$$\mathrm{O}(P) = \left\{\begin{array}{c}\text{matrix polynomials of the same size, degree,}\\ \text{and with the same complete eigenstructure as } P(\lambda)\end{array}\right\}$$

$$P(\lambda) = \lambda^d P_d + \cdots + \lambda P_1 + P_0, \qquad P_i \in \mathbb{C}^{m\times n}$$

- **Essentially the same as in pencils** but definitions more complicated since there is NOT KCF.
- **Finite and infinite eigenvalues and their elementary divisors** defined with Smith Form under unimodular equivalence of $P(\lambda)$ and rev$P(\lambda)$:

$$U(\lambda)P(\lambda)V(\lambda) = \text{diag}\left(g_1(\lambda), \dots, g_r(\lambda)\right) \oplus 0_{(m-r)\times(n-r)}, \quad g_j(\lambda) \mid g_{j+1}(\lambda).$$

Invariant polynomials: $g_j(\lambda) = (\lambda - \alpha_1)^{\delta_{j1}} \cdot (\lambda - \alpha_2)^{\delta_{j2}} \cdot \ldots \cdot (\lambda - \alpha_{l_j})^{\delta_{jl_j}}$.
Elementary divisors: $(\lambda - \alpha_k)^{\delta_{jk}}$.

- **Left and right minimal indices** defined through the minimal bases of left and right rational null spaces of $P(\lambda)$:

$$\mathcal{N}_{\text{left}}(P) := \{y(\lambda)^T \in \mathbb{C}(\lambda)^{1\times m} : y(\lambda)^T P(\lambda) = 0_{1\times n}\},$$

$$\mathcal{N}_{\text{right}}(P) := \{x(\lambda) \in \mathbb{C}(\lambda)^{n\times 1} : P(\lambda)x(\lambda) = 0_{m\times 1}\}.$$

- **The definition of orbit does not involve a group action**

$$\mathrm{O}(P) = \left\{ \begin{array}{c} \text{matrix polynomials of the same size, degree,} \\ \text{and with the same complete eigenstructure as } P(\lambda) \end{array} \right\}$$

The set of matrix polynomials with degree at most d and rank at most r

Theorem (Dmytryshyn and D., LAA, 2017)

Let m, n, r and d be integers such that $m, n \geq 2$, $d \geq 1$ and $1 \leq r \leq \min\{m, n\} - 1$. Then

$$\mathrm{POL}^{m\times n}_{d,r} = \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d, \text{ with rank at most } r \end{array} \right\} = \bigcup_{0 \leq a \leq rd} \overline{\mathrm{O}}(K_a),$$

where the $m \times n$ complex matrix polynomial K_a, $a = 0, 1, \ldots, rd$, has

- *degree exactly d, rank exactly r, and*
- *the complete eigenstructure*

$$\mathbf{K}_a : \{\overbrace{\underbrace{\alpha+1, \ldots, \alpha+1}_{s}, \underbrace{\alpha, \ldots, \alpha}_{n-r-s}}^{\text{right minimal indices}}, \overbrace{\underbrace{\beta+1, \ldots, \beta+1}_{t}, \underbrace{\beta, \ldots, \beta}_{m-r-t}}^{\text{left minimal indices}}\},$$

where $\alpha = \lfloor a/(n-r) \rfloor$ and $s = a \mod (n-r)$,
$\beta = \lfloor (rd-a)/(m-r) \rfloor$ and $t = (rd-a) \mod (m-r)$.

Moreover, $\overline{\mathrm{O}}(K_a) \bigcap \mathrm{O}(K_{a'}) = \varnothing$ whenever $a \neq a'$ (but $\overline{\mathrm{O}}(K_a) \bigcap \overline{\mathrm{O}}(K_{a'}) \neq \varnothing$).

A. Dmytryshyn and F.M. Dopico, Generic complete eigenstructures for sets of matrix polynomials with bounded rank and degree, Linear Algebra Appl., 535 (2017) 213–230

Theorem (Dmytryshyn and D., LAA, 2017)

Let m, n, r *and* d *be integers such that* $m, n \geq 2$, $d \geq 1$ *and* $1 \leq r \leq \min\{m,n\} - 1$. *Then*

$$\mathrm{POL}_{d,r}^{m\times n} = \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d, \text{ with rank at most } r \end{array} \right\} = \bigcup_{0\leq a \leq rd} \overline{\mathrm{O}}(K_a),$$

where the $m \times n$ *complex matrix polynomial* K_a, $a = 0, 1, \ldots, rd$, *has*

- *degree exactly* d, *rank exactly* r, *and*
- *the complete eigenstructure*

$$\mathbf{K}_a : \{\overbrace{\underbrace{\alpha+1,\ldots,\alpha+1}_{s}, \underbrace{\alpha,\ldots,\alpha}_{n-r-s}}^{\text{right minimal indices}}, \overbrace{\underbrace{\beta+1,\ldots,\beta+1}_{t}, \underbrace{\beta,\ldots,\beta}_{m-r-t}}^{\text{left minimal indices}}\},$$

where $\alpha = \lfloor a/(n-r) \rfloor$ *and* $s = a \mod (n-r)$,
$\beta = \lfloor (rd-a)/(m-r) \rfloor$ *and* $t = (rd-a) \mod (m-r)$.

Moreover, $\overline{\mathrm{O}}(K_a) \bigcap \mathrm{O}(K_{a'}) = \emptyset$ *whenever* $a \neq a'$ *(but* $\overline{\mathrm{O}}(K_a) \bigcap \overline{\mathrm{O}}(K_{a'}) \neq \emptyset$*).*

A. Dmytryshyn and F.M. Dopico, Generic complete eigenstructures for sets of matrix polynomials with bounded rank and degree, Linear Algebra Appl., 535 (2017) 213–230

Theorem (Dmytryshyn and D., LAA, 2017)

Let m, n, r and d be integers such that $m, n \geq 2$, $d \geq 1$ and $1 \leq r \leq \min\{m, n\} - 1$. Then

$$\mathrm{POL}_{d,r}^{m\times n} = \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d\text{, with rank at most } r \end{array} \right\} = \bigcup_{0\leq a\leq rd} \overline{\mathrm{O}}(\mathcal{K}_a),$$

where the $m \times n$ complex matrix polynomial $\mathcal{K}_a$, $a = 0, 1, \ldots, rd$, has

- *degree exactly d, rank exactly r, and*
- *the complete eigenstructure*

$$\mathbf{K}_a : \{\overbrace{\underbrace{\alpha+1, \ldots, \alpha+1}_{s}, \underbrace{\alpha, \ldots, \alpha}_{n-r-s}}^{\text{right minimal indices}}, \overbrace{\underbrace{\beta+1, \ldots, \beta+1}_{t}, \underbrace{\beta, \ldots, \beta}_{m-r-t}}^{\text{left minimal indices}}\},$$

where $\alpha = \lfloor a/(n-r) \rfloor$ and $s = a \mod (n-r)$,
$\beta = \lfloor (rd-a)/(m-r) \rfloor$ and $t = (rd-a) \mod (m-r)$.

Moreover, $\overline{\mathrm{O}}(\mathcal{K}_a) \cap \mathrm{O}(\mathcal{K}_{a'}) = \emptyset$ whenever $a \neq a'$ (but $\overline{\mathrm{O}}(\mathcal{K}_a) \cap \overline{\mathrm{O}}(\mathcal{K}_{a'}) \neq \emptyset$).

A. Dmytryshyn and F.M. Dopico, Generic complete eigenstructures for sets of matrix polynomials with bounded rank and degree, Linear Algebra Appl., 535 (2017) 213–230

The set of matrix polynomials with degree at most d and rank at most r

Theorem (Dmytryshyn and D., LAA, 2017)

Let m, n, r and d be integers such that $m, n \geq 2$, $d \geq 1$ and $1 \leq r \leq \min\{m, n\} - 1$. Then

$$\mathrm{POL}_{d,r}^{m\times n} = \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d\text{, with rank at most } r \end{array} \right\} = \bigcup_{0\leq a\leq rd} \overline{\mathrm{O}}(K_a),$$

where the $m \times n$ complex matrix polynomial K_a, $a = 0, 1, \ldots, rd$, has

- *degree exactly d, rank exactly r, and*
- *the complete eigenstructure*

$$\mathbf{K}_a : \{\overbrace{\underbrace{\alpha+1, \ldots, \alpha+1}_{s}, \underbrace{\alpha, \ldots, \alpha}_{n-r-s}}^{\text{right minimal indices}}, \overbrace{\underbrace{\beta+1, \ldots, \beta+1}_{t}, \underbrace{\beta, \ldots, \beta}_{m-r-t}}^{\text{left minimal indices}}\},$$

where $\alpha = \lfloor a/(n-r) \rfloor$ and $s = a \mod (n-r)$,
$\beta = \lfloor (rd-a)/(m-r) \rfloor$ and $t = (rd-a) \mod (m-r)$.

Moreover, $\overline{\mathrm{O}}(K_a) \bigcap \mathrm{O}(K_{a'}) = \varnothing$ whenever $a \neq a'$ (but $\overline{\mathrm{O}}(K_a) \bigcap \overline{\mathrm{O}}(K_{a'}) \neq \varnothing$).

A. Dmytryshyn and F.M. Dopico, Generic complete eigenstructures for sets of matrix polynomials with bounded rank and degree, Linear Algebra Appl., 535 (2017) 213–230

The set of matrix polynomials with degree at most d and rank at most r

Theorem (Dmytryshyn and D., LAA, 2017)

Let m, n, r and d be integers such that $m, n \geq 2$, $d \geq 1$ and $1 \leq r \leq \min\{m, n\} - 1$. Then

$$\mathrm{POL}_{d,r}^{m\times n} = \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \text{, with rank at most } r \end{array} \right\} = \overline{\bigcup_{0\leq a\leq rd} \mathrm{O}(K_a)},$$

where the $m \times n$ complex matrix polynomial K_a, $a = 0, 1, \ldots, rd$, has degree exactly d, rank exactly r, and the complete eigenstructure

$$\mathbf{K}_a : \{\overbrace{\underbrace{\alpha+1, \ldots, \alpha+1}_{s}, \underbrace{\alpha, \ldots, \alpha}_{n-r-s}}^{\text{right minimal indices}}, \overbrace{\underbrace{\beta+1, \ldots, \beta+1}_{t}, \underbrace{\beta, \ldots, \beta}_{m-r-t}}^{\text{left minimal indices}}\},$$

where $\alpha = \lfloor a/(n-r) \rfloor$ and $s = a \mod (n-r)$,
$\beta = \lfloor (rd-a)/(m-r) \rfloor$ and $t = (rd-a) \mod (m-r)$.

Moreover, $\overline{\mathrm{O}}(K_a) \cap \mathrm{O}(K_{a'}) = \emptyset$ whenever $a \neq a'$ (but $\overline{\mathrm{O}}(K_a) \cap \overline{\mathrm{O}}(K_{a'}) \neq \emptyset$).

$\bigcup_{0\leq a\leq rd} \mathrm{O}(K_a)$ is an open dense subset of $\mathrm{POL}_{d,r}^{m\times n}$ (in the topology of $\mathrm{POL}_{d,r}^{m\times n}$). So,
generically, the $m \times n$ matrix polys with degree at most d and with rank at most r have only $rd + 1$ possible complete eigenstructures given by $\mathbf{K}_a$ for $a = 0, 1, \ldots, rd$.

Let m, n, r and d be integers such that $m, n \geq 2$, $d \geq 1$, and $1 \leq r \leq \min\{m, n\} - 1$.

Corollary

For every $M \in \mathrm{POL}_{d,r}^{m\times n}$ and every $\varepsilon > 0$ there exists $M' \in \mathrm{POL}_{d,r}^{m\times n}$ such that

1. *M' has the complete eigenstructure $\mathbf{K}_a$ for some $a \in \{0, 1, \ldots, rd\}$ and*
2. *$d(M, M') < \varepsilon$.*

Corollary

Let $a \in \{0, 1, \ldots, rd\}$. Then for every $M' \in \mathrm{POL}_{d,r}^{m\times n}$ with the complete eigenstructure $\mathbf{K}_a$, there exists $\varepsilon > 0$ such that all the matrix polynomials in

$$\mathcal{B}_r(M'; \varepsilon) := \left\{ M \, : \, M \in \mathrm{POL}_{d,r}^{m\times n} \ \text{ and } \ d(M', M) < \varepsilon \right\}$$

have complete eigenstructure $\mathbf{K}_a$.

Corollaries 1 and 2 of previous MAIN theorem: Analytical interpretation

Let m, n, r and d be integers such that $m, n \geq 2$, $d \geq 1$, and $1 \leq r \leq \min\{m, n\} - 1$.

Corollary

For every $M \in \mathrm{POL}_{d,r}^{m\times n}$ and every $\varepsilon > 0$ there exists $M' \in \mathrm{POL}_{d,r}^{m\times n}$ such that

1. *M' has the complete eigenstructure $\mathbf{K}_a$ for some $a \in \{0, 1, \ldots, rd\}$ and*
2. *$d(M, M') < \varepsilon$.*

Corollary

Let $a \in \{0, 1, \ldots, rd\}$. Then for every $M' \in \mathrm{POL}_{d,r}^{m\times n}$ with the complete eigenstructure $\mathbf{K}_a$, there exists $\varepsilon > 0$ such that all the matrix polynomials in

$$\mathcal{B}_r(M'; \varepsilon) := \left\{ M \,:\, M \in \mathrm{POL}_{d,r}^{m\times n} \text{ and } d(M', M) < \varepsilon \right\}$$

have complete eigenstructure $\mathbf{K}_a$.

Corollary 3 of MAIN theorem: the set of SQUARE singular matrix polynomials with degree at most d

Remark: an $n \times n$ matrix polynomial is singular if and only if its rank is at most $n-1$.

Corollary (The main theorem with $m = n$ and $r = n-1$)

$$\left\{ \begin{array}{c} \text{singular } n \times n \text{ complex matrix} \\ \text{polynomials of degree at most } d \end{array} \right\} = \bigcup_{0 \le a \le (n-1)d} \overline{\mathrm{O}}(K_a),$$

where the complete eigenstructure of each of the matrix polynomials $K_a, a = 0, 1, \ldots, (n-1)d$, *has*

- *no elementary divisors (no eigenvalues);*
- *only one left minimal index equal to* $(n-1)d - a$;
- *only one right minimal index equal to* a.

This corollary extends the classical result for pencils:

W. Waterhouse, The codimension of singular matrix pairs, Linear Algebra Appl., 57 (1984) 227–245

Corollary 3 of MAIN theorem: the set of SQUARE singular matrix polynomials with degree at most d

Remark: an $n \times n$ matrix polynomial is singular if and only if its rank is at most $n-1$.

Corollary (The main theorem with $m = n$ and $r = n-1$)

$$\left\{ \begin{array}{c} \textit{singular } n \times n \textit{ complex matrix} \\ \textit{polynomials of degree at most } d \end{array} \right\} = \bigcup_{0 \le a \le (n-1)d} \overline{\mathrm{O}}(K_a),$$

where the complete eigenstructure of each of the matrix polynomials $K_a, a = 0, 1, \ldots, (n-1)d$, has

- *no elementary divisors (no eigenvalues);*
- *only one left minimal index equal to $(n-1)d - a$;*
- *only one right minimal index equal to a.*

This corollary extends the classical result for pencils:

W. Waterhouse, The codimension of singular matrix pairs, Linear Algebra Appl., 57 (1984) 227–245

Corollary 3 of MAIN theorem: the set of SQUARE singular matrix polynomials with degree at most d

Remark: an $n \times n$ matrix polynomial is singular if and only if its rank is at most $n-1$.

Corollary (The main theorem with $m = n$ and $r = n-1$)

$$\left\{ \begin{array}{c} \text{singular } n \times n \text{ complex matrix} \\ \text{polynomials of degree at most } d \end{array} \right\} = \bigcup_{0 \le a \le (n-1)d} \overline{\mathrm{O}}(K_a),$$

where the complete eigenstructure of each of the matrix polynomials $K_a, a = 0, 1, \ldots, (n-1)d$, *has*

- *no elementary divisors (no eigenvalues);*
- *only one left minimal index equal to* $(n-1)d - a$;
- *only one right minimal index equal to* a.

This corollary extends the classical result for pencils:

W. Waterhouse, The codimension of singular matrix pairs, Linear Algebra Appl., 57 (1984) 227–245

Example ($m = n = 2, \quad r = 1, \quad d = 3$)

$$\left\{ \begin{array}{c} \text{singular } 2\times 2 \text{ complex matrix} \\ \text{polynomials of degree at most } 3 \end{array} \right\} = \bigcup_{0 \le a \le 3} \overline{\mathrm{O}}(K_a)$$

each of the matrix polynomials K_0, K_1, K_2, and K_3 has

- no elementary divisors;
- one left minimal index equal to $3 - a$;
- one right minimal index equal to a.

$$\mathbf{K}_0 : \{ \underbrace{3}_{left}, \underbrace{0}_{right} \} \qquad \mathbf{K}_1 : \{ \underbrace{2}_{left}, \underbrace{1}_{right} \}$$

$$\mathbf{K}_2 : \{ \underbrace{1}_{left}, \underbrace{2}_{right} \} \qquad \mathbf{K}_3 : \{ \underbrace{0}_{left}, \underbrace{3}_{right} \}$$

Comments on the proof of the main theorem (I)

- The proof is delicate.
- Of course, it relies on the corresponding result for pencils (De Terán & D., SIMAX, 2008) and uses heavily the first Frobenius companion strong linearization of matrix polynomials,
- but also several key results for matrix polynomials that have been developed recently (or, rescued and improved from "old" references). We emphasize the following ones:
- Necessary and sufficient conditions for a matrix polynomial with prescribed degree and complete eigenstructure to exist.

F. De Terán, F.M. Dopico, and P. Van Dooren, Matrix polynomials with completely prescribed eigenstructure, SIAM J. Matrix Anal. Appl., 36 (2015) 302–328.

- Relations between the minimal indices of matrix polynomials and their strong linearizations, in particular the first Frobenius companion form.

F. De Terán, F.M. Dopico, and D.S. Mackey, Fiedler companion linearizations for rectangular matrix polynomials, Linear Algebra Appl., 437 (2012) 957–991.

F. De Terán, F.M. Dopico, and D.S. Mackey, Spectral equivalence of matrix polynomials and the index sum theorem, Linear Algebra Appl., 459 (2014) 264–333.

Comments on the proof of the main theorem (I)

- The proof is delicate.
- Of course, it relies on the corresponding result for pencils (De Terán & D., SIMAX, 2008) and uses heavily the first Frobenius companion strong linearization of matrix polynomials,
- but also several key results for matrix polynomials that have been developed recently (or, rescued and improved from "old" references). We emphasize the following ones:
- Necessary and sufficient conditions for a matrix polynomial with prescribed degree and complete eigenstructure to exist.

F. De Terán, F.M. Dopico, and P. Van Dooren, Matrix polynomials with completely prescribed eigenstructure, SIAM J. Matrix Anal. Appl., 36 (2015) 302–328.

- Relations between the minimal indices of matrix polynomials and their strong linearizations, in particular the first Frobenius companion form.

F. De Terán, F.M. Dopico, and D.S. Mackey, Fiedler companion linearizations for rectangular matrix polynomials, Linear Algebra Appl., 437 (2012) 957–991.

F. De Terán, F.M. Dopico, and D.S. Mackey, Spectral equivalence of matrix polynomials and the index sum theorem, Linear Algebra Appl., 459 (2014) 264–333.

Comments on the proof of the main theorem (I)

- The proof is delicate.
- Of course, it relies on the corresponding result for pencils (De Terán & D., SIMAX, 2008) and uses heavily the first Frobenius companion strong linearization of matrix polynomials,
- but also several key results for matrix polynomials that have been developed recently (or, rescued and improved from "old" references). We emphasize the following ones:
- Necessary and sufficient conditions for a matrix polynomial with prescribed degree and complete eigenstructure to exist.

F. De Terán, F.M. Dopico, and P. Van Dooren, Matrix polynomials with completely prescribed eigenstructure, SIAM J. Matrix Anal. Appl., 36 (2015) 302–328.

- Relations between the minimal indices of matrix polynomials and their strong linearizations, in particular the first Frobenius companion form.

F. De Terán, F.M. Dopico, and D.S. Mackey, Fiedler companion linearizations for rectangular matrix polynomials, Linear Algebra Appl., 437 (2012) 957–991.

F. De Terán, F.M. Dopico, and D.S. Mackey, Spectral equivalence of matrix polynomials and the index sum theorem, Linear Algebra Appl., 459 (2014) 264–333.

Comments on the proof of the main theorem (I)

- The proof is delicate.
- Of course, it relies on the corresponding result for pencils (De Terán & D., SIMAX, 2008) and uses heavily the first Frobenius companion strong linearization of matrix polynomials,
- but also several key results for matrix polynomials that have been developed recently (or, rescued and improved from "old" references). We emphasize the following ones:
- Necessary and sufficient conditions for a matrix polynomial with prescribed degree and complete eigenstructure to exist.

F. De Terán, F.M. Dopico, and P. Van Dooren, Matrix polynomials with completely prescribed eigenstructure, SIAM J. Matrix Anal. Appl., 36 (2015) 302–328.

- Relations between the minimal indices of matrix polynomials and their strong linearizations, in particular the first Frobenius companion form.

F. De Terán, F.M. Dopico, and D.S. Mackey, Fiedler companion linearizations for rectangular matrix polynomials, Linear Algebra Appl., 437 (2012) 957–991.

F. De Terán, F.M. Dopico, and D.S. Mackey, Spectral equivalence of matrix polynomials and the index sum theorem, Linear Algebra Appl., 459 (2014) 264–333.

Comments on the proof of the main theorem (I)

- The proof is delicate.
- Of course, it relies on the corresponding result for pencils (De Terán & D., SIMAX, 2008) and uses heavily the first Frobenius companion strong linearization of matrix polynomials,
- but also several key results for matrix polynomials that have been developed recently (or, rescued and improved from "old" references). We emphasize the following ones:
- Necessary and sufficient conditions for a matrix polynomial with prescribed degree and complete eigenstructure to exist.

F. De Terán, F.M. Dopico, and P. Van Dooren, Matrix polynomials with completely prescribed eigenstructure, SIAM J. Matrix Anal. Appl., 36 (2015) 302–328.

- Relations between the minimal indices of matrix polynomials and their strong linearizations, in particular the first Frobenius companion form.

F. De Terán, F.M. Dopico, and D.S. Mackey, Fiedler companion linearizations for rectangular matrix polynomials, Linear Algebra Appl., 437 (2012) 957–991.

F. De Terán, F.M. Dopico, and D.S. Mackey, Spectral equivalence of matrix polynomials and the index sum theorem, Linear Algebra Appl., 459 (2014) 264–333.

Given

$$P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0, \qquad A_i \in \mathbb{C}^{m\times n},$$

its first Frobenius companion form is

$$\mathcal{C}_P^1 = \lambda \begin{bmatrix} A_d & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix} + \begin{bmatrix} A_{d-1} & A_{d-2} & \ldots & A_0 \\ -I_n & 0 & \ldots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -I_n & 0 \end{bmatrix}$$

- $\mathcal{C}_P^1$ has size $(m + n(d-1)) \times nd$.
- $\mathcal{C}_P^1$ and P have the same finite and infinite elementary divisors.
- The left minimal indices of $\mathcal{C}_P^1$ are equal to those of P.
- The right minimal indices of $\mathcal{C}_P^1$ are greater by $d-1$ than the right minimal indices of the polynomial P.

The first Frobenius companion form

Given

$$P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0, \qquad A_i \in \mathbb{C}^{m\times n},$$

its first Frobenius companion form is

$$\mathcal{C}_P^1 = \lambda \begin{bmatrix} A_d & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix} + \begin{bmatrix} A_{d-1} & A_{d-2} & \dots & A_0 \\ -I_n & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -I_n & 0 \end{bmatrix}$$

- $\mathcal{C}_P^1$ has size $(m + n(d-1)) \times nd$.
- $\mathcal{C}_P^1$ and P have the same finite and infinite elementary divisors.
- The left minimal indices of $\mathcal{C}_P^1$ are equal to those of P.
- The right minimal indices of $\mathcal{C}_P^1$ are greater by $d-1$ than the right minimal indices of the polynomial P.

The first Frobenius companion form

Given

$$P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0, \qquad A_i \in \mathbb{C}^{m\times n},$$

its first Frobenius companion form is

$$\mathcal{C}_P^1 = \lambda \begin{bmatrix} A_d & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix} + \begin{bmatrix} A_{d-1} & A_{d-2} & \dots & A_0 \\ -I_n & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -I_n & 0 \end{bmatrix}$$

- $\mathcal{C}_P^1$ has size $(m + n(d-1)) \times nd$.
- $\mathcal{C}_P^1$ and P have the same finite and infinite elementary divisors.
- The left minimal indices of $\mathcal{C}_P^1$ are equal to those of P.
- The right minimal indices of $\mathcal{C}_P^1$ are greater by $d-1$ than the right minimal indices of the polynomial P.

Given

$$P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0, \qquad A_i \in \mathbb{C}^{m\times n},$$

its first Frobenius companion form is

$$\mathcal{C}_P^1 = \lambda \begin{bmatrix} A_d & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix} + \begin{bmatrix} A_{d-1} & A_{d-2} & \dots & A_0 \\ -I_n & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -I_n & 0 \end{bmatrix}$$

- $\mathcal{C}_P^1$ has size $(m + n(d-1)) \times nd$.
- $\mathcal{C}_P^1$ and P have the same finite and infinite elementary divisors.
- The left minimal indices of $\mathcal{C}_P^1$ are equal to those of P.
- The right minimal indices of $\mathcal{C}_P^1$ are greater by $d-1$ than the right minimal indices of the polynomial P.

Given

$$P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0, \qquad A_i \in \mathbb{C}^{m\times n},$$

its first Frobenius companion form is

$$\mathcal{C}_P^1 = \lambda \begin{bmatrix} A_d & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix} + \begin{bmatrix} A_{d-1} & A_{d-2} & \dots & A_0 \\ -I_n & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -I_n & 0 \end{bmatrix}$$

- $\mathcal{C}_P^1$ has size $(m + n(d-1)) \times nd$.
- $\mathcal{C}_P^1$ and P have the same finite and infinite elementary divisors.
- The left minimal indices of $\mathcal{C}_P^1$ are equal to those of P.
- The right minimal indices of $\mathcal{C}_P^1$ are greater by $d-1$ than the right minimal indices of the polynomial P.

Comments on the proof of the main theorem (II)

- Correspondence between perturbations of certain strong linearizations of matrix polynomials and the matrix polynomials themselves.

P. Van Dooren and P. Dewilde, The eigenstructure of a polynomial matrix: Computational aspects, Linear Algebra Appl., 50 (1983) 545–579

S. Johansson, B. Kågström, and P. Van Dooren, Stratification of full rank polynomial matrices, Linear Algebra Appl., 439 (2013) 1062–1090

F.M. Dopico, P.W. Lawrence, J. Pérez, and P. Van Dooren, Block Kronecker linearizations of matrix polynomials and their backward errors, Numer. Math., 140 (2018) 373-426.

- In addition to all these results, it is needed a delicate translation of results in the Euclidean topology of the set of $(m+n(d-1)) \times nd$ pencils and of the subset of pencils formed by the first Frobenius companion forms of all the $m \times n$ matrix polynomials of degree at most d into the Euclidean topology of the set of $m \times n$ matrix polynomials of degree at most d.

Comments on the proof of the main theorem (II)

- Correspondence between perturbations of certain strong linearizations of matrix polynomials and the matrix polynomials themselves.

P. Van Dooren and P. Dewilde, The eigenstructure of a polynomial matrix: Computational aspects, Linear Algebra Appl., 50 (1983) 545–579

S. Johansson, B. Kågström, and P. Van Dooren, Stratification of full rank polynomial matrices, Linear Algebra Appl., 439 (2013) 1062–1090

F.M. Dopico, P.W. Lawrence, J. Pérez, and P. Van Dooren, Block Kronecker linearizations of matrix polynomials and their backward errors, Numer. Math., 140 (2018) 373-426.

- In addition to all these results, it is needed a delicate translation of results in the Euclidean topology of the set of $(m+n(d-1))\times nd$ pencils and of the subset of pencils formed by the first Frobenius companion forms of all the $m\times n$ matrix polynomials of degree at most d into the Euclidean topology of the set of $m\times n$ matrix polynomials of degree at most d.

- In this case, the set $\mathrm{POL}_d^{m\times n}$ is equal to $\mathrm{POL}_{d,m}^{m\times n}$, i.e., the set of matrix polynomials of rank at most m,
- but main result assumes (and uses) $r \leq \min\{m,n\} - 1$. Nevertheless,
- since all the matrix polynomials in $\mathrm{POL}_d^{m\times n}$ are singular, this set can be described using techniques similar to those in main result, but
- a **very important difference** appears: **there is only one generic complete eigenstructure**.

Theorem (Dmytryshyn and D., LAA, 2017)

$$\mathrm{POL}_d^{m\times n} = \overline{\mathrm{O}}(K_{rp}),$$

where K_{rp} is an $m \times n$ complex matrix polynomial of degree exactly d and rank exactly m with the complete eigenstructure

$$\mathbf{K}_{rp} : \overbrace{\{\underbrace{\alpha+1,\ldots,\alpha+1}_{s}, \underbrace{\alpha,\ldots,\alpha}_{n-m-s}\}}^{\text{right minimal indices}},$$

with $\alpha = \lfloor md/(n-m) \rfloor$ and $s = md \mod (n-m)$.

- In this case, the set $\mathrm{POL}_d^{m\times n}$ is equal to $\mathrm{POL}_{d,m}^{m\times n}$, i.e., the set of matrix polynomials of rank at most m,
- but main result assumes (and uses) $r \leq \min\{m,n\} - 1$. Nevertheless,
- since all the matrix polynomials in $\mathrm{POL}_d^{m\times n}$ are singular, this set can be described using techniques similar to those in main result, but
- a **very important difference** appears: **there is only one generic complete eigenstructure**.

Theorem (Dmytryshyn and D., LAA, 2017)

$$\mathrm{POL}_d^{m\times n} = \overline{\mathrm{O}}(K_{rp}),$$

where K_{rp} is an $m \times n$ complex matrix polynomial of degree exactly d and rank exactly m with the complete eigenstructure

$$\mathbf{K}_{rp} : \{\overbrace{\underbrace{\alpha+1,\ldots,\alpha+1}_{s},\underbrace{\alpha,\ldots,\alpha}_{n-m-s}}^{\text{right minimal indices}}\},$$

with $\alpha = \lfloor md/(n-m) \rfloor$ and $s = md \mod (n-m)$.

The set $\mathrm{POL}_d^{m\times n}$ when $m < n$

- In this case, the set $\mathrm{POL}_d^{m\times n}$ is equal to $\mathrm{POL}_{d,m}^{m\times n}$, i.e., the set of matrix polynomials of rank at most m,
- but main result assumes (and uses) $r \le \min\{m,n\} - 1$. Nevertheless,
- since all the matrix polynomials in $\mathrm{POL}_d^{m\times n}$ are singular, this set can be described using techniques similar to those in main result, but
- a **very important difference** appears: **there is only one generic complete eigenstructure**.

Theorem (Dmytryshyn and D., LAA, 2017)

$$\mathrm{POL}_d^{m\times n} = \overline{\mathrm{O}}(K_{rp}),$$

where K_{rp} is an $m\times n$ complex matrix polynomial of degree exactly d and rank exactly m with the complete eigenstructure

$$\mathbf{K}_{rp} : \{\overbrace{\underbrace{\alpha+1,\ldots,\alpha+1}_{s},\underbrace{\alpha,\ldots,\alpha}_{n-m-s}}^{\textit{right minimal indices}}\},$$

with $\alpha = \lfloor md/(n-m)\rfloor$ and $s = md \mod (n-m)$.

The set $\mathrm{POL}_d^{m\times n}$ when $m < n$

- In this case, the set $\mathrm{POL}_d^{m\times n}$ is equal to $\mathrm{POL}_{d,m}^{m\times n}$, i.e., the set of matrix polynomials of rank at most m,
- but main result assumes (and uses) $r \leq \min\{m,n\} - 1$. Nevertheless,
- since all the matrix polynomials in $\mathrm{POL}_d^{m\times n}$ are singular, this set can be described using techniques similar to those in main result, but
- a **very important difference** appears: **there is only one generic complete eigenstructure**.

Theorem (Dmytryshyn and D., LAA, 2017)

$$\mathrm{POL}_d^{m\times n} = \overline{\mathrm{O}}(K_{rp}),$$

where K_{rp} is an $m \times n$ complex matrix polynomial of degree exactly d and rank exactly m with the complete eigenstructure

$$\mathbf{K}_{rp} : \{\overbrace{\underbrace{\alpha+1,\ldots,\alpha+1}_{s},\underbrace{\alpha,\ldots,\alpha}_{n-m-s}}^{\textit{right minimal indices}}\},$$

with $\alpha = \lfloor md/(n-m)\rfloor$ and $s = md \mod (n-m)$.

The set $\mathrm{POL}_d^{m\times n}$ when $m < n$

- In this case, the set $\mathrm{POL}_d^{m\times n}$ is equal to $\mathrm{POL}_{d,m}^{m\times n}$, i.e., the set of matrix polynomials of rank at most m,
- but main result assumes (and uses) $r \leq \min\{m,n\} - 1$. Nevertheless,
- since all the matrix polynomials in $\mathrm{POL}_d^{m\times n}$ are singular, this set can be described using techniques similar to those in main result, but
- a **very important difference** appears: **there is only one generic complete eigenstructure**.

Theorem (Dmytryshyn and D., LAA, 2017)

$$\mathrm{POL}_d^{m\times n} = \overline{\mathrm{O}}(K_{rp}),$$

where K_{rp} is an $m \times n$ complex matrix polynomial of degree exactly d and rank exactly m with the complete eigenstructure

$$\mathbf{K}_{rp} : \{\overbrace{\underbrace{\alpha+1,\ldots,\alpha+1}_{s},\underbrace{\alpha,\ldots,\alpha}_{n-m-s}}^{\text{right minimal indices}}\},$$

with $\alpha = \lfloor md/(n-m) \rfloor$ and $s = md \mod (n-m)$.

Analogously,

Theorem (Dmytryshyn and D., LAA, 2017)

$$\mathrm{POL}_d^{m\times n} = \overline{\mathrm{O}}(K_{lp}),$$

where K_{lp} is an $m \times n$ complex matrix polynomial of degree exactly d and rank exactly n with the complete eigenstructure

$$\mathbf{K}_{lp} : \overbrace{\{\underbrace{\beta+1,\ldots,\beta+1}_{t},\underbrace{\beta,\ldots,\beta}_{m-n-t}\}}^{\text{left minimal indices}},$$

with $\beta = \lfloor nd/(m-n) \rfloor$ and $t = nd \mod (m-n)$.

The two previous theorems extend classical results for matrix pencils:

J. Demmel and A. Edelman, The dimension of matrices (matrix pencils) with given Jordan (Kronecker) canonical forms, Linear Algebra Appl., 230 (1995) 61–87

P. Van Dooren, The generalized eigenstructure problem: Applications in linear system theory, PhD thesis, Kath. Univ. Leuven, Leuven, Belgium, 1979.

The set $\mathrm{POL}_d^{m\times n}$ when $m > n$

Analogously,

Theorem (Dmytryshyn and D., LAA, 2017)

$$\mathrm{POL}_d^{m\times n} = \overline{\mathrm{O}}(K_{lp}),$$

where K_{lp} is an $m \times n$ complex matrix polynomial of degree exactly d and rank exactly n with the complete eigenstructure

$$\mathbf{K}_{lp} : \{\overbrace{\underbrace{\beta+1,\ldots,\beta+1}_{t},\underbrace{\beta,\ldots,\beta}_{m-n-t}}^{\textit{left minimal indices}}\},$$

with $\beta = \lfloor nd/(m-n) \rfloor$ *and* $t = nd \mod (m-n)$.

The two previous theorems extend classical results for matrix pencils:

J. Demmel and A. Edelman, The dimension of matrices (matrix pencils) with given Jordan (Kronecker) canonical forms, Linear Algebra Appl., 230 (1995) 61–87

P. Van Dooren, The generalized eigenstructure problem: Applications in linear system theory, PhD thesis, Kath. Univ. Leuven, Leuven, Belgium, 1979.

Outline

A few properties of skew-symmetric matrix polynomials

- **Definition:** $P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0$ with $A_i^T = -A_i \in \mathbb{C}^{m\times m}$.
- Skew-symmetric matrix polynomials with size $m \times m$ and degree at most d form a vector space and we can define on it the same Euclidean distance as before.
- Their **rank is always even**.
- Their invariant polynomials are paired-up and their left minimal indices are equal to the right ones.
- When the (at most) degree is odd, they can be always strongly linearized through a skew-symmetric block-tridiagonal companion form (Antoniou-Vologiannidis, ELA, 2004 and Mackey et al, LAA, 2013) that allows us to recover via a shift the minimal indices of the polynomial (Dmytryshyn, LAA, 2017).

A few properties of skew-symmetric matrix polynomials

- **Definition:** $P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0$ with $A_i^T = -A_i \in \mathbb{C}^{m\times m}$.
- Skew-symmetric matrix polynomials with size $m \times m$ and degree at most d form a vector space and we can define on it the same Euclidean distance as before.
- Their **rank is always even**.
- Their invariant polynomials are paired-up and their left minimal indices are equal to the right ones.
- When the (at most) degree is odd, they can be always strongly linearized through a skew-symmetric block-tridiagonal companion form (Antoniou-Vologiannidis, ELA, 2004 and Mackey et al, LAA, 2013) that allows us to recover via a shift the minimal indices of the polynomial (Dmytryshyn, LAA, 2017).

A few properties of skew-symmetric matrix polynomials

- **Definition:** $P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0$ with $A_i^T = -A_i \in \mathbb{C}^{m\times m}$.
- Skew-symmetric matrix polynomials with size $m \times m$ and degree at most d form a vector space and we can define on it the same Euclidean distance as before.
- Their **rank is always even**.
- Their invariant polynomials are paired-up and their left minimal indices are equal to the right ones.
- When the (at most) degree is odd, they can be always strongly linearized through a skew-symmetric block-tridiagonal companion form (Antoniou-Vologiannidis, ELA, 2004 and Mackey et al, LAA, 2013) that allows us to recover via a shift the minimal indices of the polynomial (Dmytryshyn, LAA, 2017).

A few properties of skew-symmetric matrix polynomials

- **Definition:** $P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0$ with $A_i^T = -A_i \in \mathbb{C}^{m\times m}$.
- Skew-symmetric matrix polynomials with size $m \times m$ and degree at most d form a vector space and we can define on it the same Euclidean distance as before.
- Their **rank is always even**.
- Their invariant polynomials are paired-up and their left minimal indices are equal to the right ones.
- When the (at most) degree is odd, they can be always strongly linearized through a skew-symmetric block-tridiagonal companion form (Antoniou-Vologiannidis, ELA, 2004 and Mackey et al, LAA, 2013) that allows us to recover via a shift the minimal indices of the polynomial (Dmytryshyn, LAA, 2017).

A few properties of skew-symmetric matrix polynomials

- **Definition:** $P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0$ with $A_i^T = -A_i \in \mathbb{C}^{m\times m}$.
- Skew-symmetric matrix polynomials with size $m \times m$ and degree at most d form a vector space and we can define on it the same Euclidean distance as before.
- Their **rank is always even**.
- Their invariant polynomials are paired-up and their left minimal indices are equal to the right ones.
- When the (at most) degree is odd, they can be always strongly linearized through a skew-symmetric block-tridiagonal companion form (Antoniou-Vologiannidis, ELA, 2004 and Mackey et al, LAA, 2013) that allows us to recover via a shift the minimal indices of the polynomial (Dmytryshyn, LAA, 2017).

Given

$$P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0, \qquad A_i^T = -A_i \in \mathbb{C}^{m\times m},$$

with d **odd**, the **skew-symmetric** matrix pencil

$$\mathcal{F}_P = \begin{bmatrix} \lambda A_d + A_{d-1} & -I & & & & & \\ I & \ddots & \ddots & & & & \\ & \ddots & 0 & -\lambda I & & & \\ & & \lambda I & \lambda A_3 + A_2 & -I & & \\ & & & I & 0 & -\lambda I \\ & & & & \lambda I & \lambda A_1 + A_0 \end{bmatrix}$$

satisfies the following properties

- $\mathcal{F}_P$ has size $md \times md$.
- $\mathcal{F}_P$ and P have the same finite and infinite elementary divisors.
- The left (and right) minimal indices of $\mathcal{F}_P$ are greater by $\dfrac{d-1}{2}$ than the left (and right) minimal indices of the polynomial P.

The skew-symmetric companion form

Given

$$P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0, \qquad A_i^T = -A_i \in \mathbb{C}^{m\times m},$$

with d **odd**, the **skew-symmetric** matrix pencil

$$\mathcal{F}_P = \begin{bmatrix} \lambda A_d + A_{d-1} & -I & & & & \\ I & \ddots & \ddots & & & \\ & \ddots & 0 & -\lambda I & & \\ & & \lambda I & \lambda A_3 + A_2 & -I & \\ & & & I & 0 & -\lambda I \\ & & & & \lambda I & \lambda A_1 + A_0 \end{bmatrix}$$

satisfies the following properties

- $\mathcal{F}_P$ has size $md \times md$.
- $\mathcal{F}_P$ and P have the same finite and infinite elementary divisors.
- The left (and right) minimal indices of $\mathcal{F}_P$ are greater by $\dfrac{d-1}{2}$ than the left (and right) minimal indices of the polynomial P.

The skew-symmetric companion form

Given
$$P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0, \qquad A_i^T = -A_i \in \mathbb{C}^{m\times m},$$
with d **odd**, the **skew-symmetric** matrix pencil

$$\mathcal{F}_P = \begin{bmatrix} \lambda A_d + A_{d-1} & -I & & & & \\ I & \ddots & \ddots & & & \\ & \ddots & 0 & -\lambda I & & \\ & & \lambda I & \lambda A_3 + A_2 & -I & \\ & & & I & 0 & -\lambda I \\ & & & & \lambda I & \lambda A_1 + A_0 \end{bmatrix}$$

satisfies the following properties

- $\mathcal{F}_P$ has size $md \times md$.
- $\mathcal{F}_P$ and P have the same finite and infinite elementary divisors.
- The left (and right) minimal indices of $\mathcal{F}_P$ are greater by $\dfrac{d-1}{2}$ than the left (and right) minimal indices of the polynomial P.

Given

$$P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0, \qquad A_i^T = -A_i \in \mathbb{C}^{m\times m},$$

with d **odd**, the **skew-symmetric** matrix pencil

$$\mathcal{F}_P = \begin{bmatrix} \lambda A_d + A_{d-1} & -I & & & & \\ I & \ddots & \ddots & & & \\ & \ddots & 0 & -\lambda I & & \\ & & \lambda I & \lambda A_3 + A_2 & -I & \\ & & & I & 0 & -\lambda I \\ & & & & \lambda I & \lambda A_1 + A_0 \end{bmatrix}$$

satisfies the following properties

- $\mathcal{F}_P$ has size $md \times md$.
- $\mathcal{F}_P$ and P have the same finite and infinite elementary divisors.
- The left (and right) minimal indices of $\mathcal{F}_P$ are greater by $\dfrac{d-1}{2}$ than the left (and right) minimal indices of the polynomial P.

For describing the set of skew-symm polys of bounded rank and degree

...we should take into account that:

- In contrast to the unstructured case, the description of the sets of skew-symmetric matrix pencils with bounded ranks was not previously available and we had to develop it.
- All the topological concepts refer to the metric space of skew-symmetric matrix polynomials with degree at most d (with d odd).
- Thus, in this case, the orbit of a skew-symmetric matrix polynomial P is defined as:

$$\mathrm{O}(P) = \left\{ \begin{array}{c} \text{skew-symmetric matrix polynomials} \\ \text{of the same size, degree, and} \\ \text{with the same complete eigenstructure as } P(\lambda) \end{array} \right\}$$

- Based on the result for skew-symmetric pencils and developing structured counterparts of all needed unstructured results, we have proved...

For describing the set of skew-symm polys of bounded rank and degree

...we should take into account that:

- In contrast to the unstructured case, the description of the sets of skew-symmetric matrix pencils with bounded ranks was not previously available and we had to develop it.
- All the topological concepts refer to the metric space of skew-symmetric matrix polynomials with degree at most d (with d odd).
- Thus, in this case, the orbit of a skew-symmetric matrix polynomial P is defined as:

$$\mathrm{O}(P) = \left\{ \begin{array}{c} \text{skew-symmetric matrix polynomials} \\ \text{of the same size, degree, and} \\ \text{with the same complete eigenstructure as } P(\lambda) \end{array} \right\}$$

- Based on the result for skew-symmetric pencils and developing structured counterparts of all needed unstructured results, we have proved...

...we should take into account that:

- In contrast to the unstructured case, the description of the sets of skew-symmetric matrix pencils with bounded ranks was not previously available and we had to develop it.
- All the topological concepts refer to the metric space of skew-symmetric matrix polynomials with degree at most d (with d odd).
- Thus, in this case, the orbit of a skew-symmetric matrix polynomial P is defined as:

$$\mathrm{O}(P) = \left\{ \begin{array}{c} \text{skew-symmetric matrix polynomials} \\ \text{of the same size, degree, and} \\ \text{with the same complete eigenstructure as } P(\lambda) \end{array} \right\}$$

- Based on the result for skew-symmetric pencils and developing structured counterparts of all needed unstructured results, we have proved...

...we should take into account that:

- In contrast to the unstructured case, the description of the sets of skew-symmetric matrix pencils with bounded ranks was not previously available and we had to develop it.
- All the topological concepts refer to the metric space of skew-symmetric matrix polynomials with degree at most d (with d odd).
- Thus, in this case, the orbit of a skew-symmetric matrix polynomial P is defined as:

$$\mathrm{O}(P) = \left\{ \begin{array}{c} \text{skew-symmetric matrix polynomials} \\ \text{of the same size, degree, and} \\ \text{with the same complete eigenstructure as } P(\lambda) \end{array} \right\}$$

- Based on the result for skew-symmetric pencils and developing structured counterparts of all needed unstructured results, we have proved...

Theorem (Dmytryshyn and D., LAA, 2018)

Let m, r and d be integers such that $m \geq 2$, $d \geq 1$ is odd, and $2 \leq 2r \leq (m-1)$. Then

$$\left\{ \begin{array}{c} m \times m \text{ complex skew-symmetric matrix polynomials} \\ \text{with degree at most } d \text{ and with rank at most } 2r \end{array} \right\} = \overline{\mathrm{O}}(W),$$

where the $m \times m$ complex skew-symmetric matrix polynomial W has degree exactly d, rank exactly $2r$, and the complete eigenstructure

$$\mathbf{W} : \Big\{ \overbrace{\underbrace{\beta+1, \ldots, \beta+1}_{t}, \underbrace{\beta, \ldots, \beta}_{m-2r-t}}^{\text{left minimal indices}}, \overbrace{\underbrace{\beta+1, \ldots, \beta+1}_{t}, \underbrace{\beta, \ldots, \beta}_{m-2r-t}}^{\text{right minimal indices}} \Big\}$$

with $\beta = \lfloor rd/(m-2r) \rfloor$ and $t = rd \mod (m-2r)$.

A. Dmytryshyn and F.M. Dopico, Generic skew-symmetric matrix polynomials with fixed rank and fixed odd grade, Linear Algebra Appl., 536 (2018) 1–18

Theorem (Dmytryshyn and D., LAA, 2018)

Let m, r and d be integers such that $m \geq 2$, $d \geq 1$ is odd, and $2 \leq 2r \leq (m-1)$. Then

$$\left\{ \begin{array}{c} m \times m \text{ complex skew-symmetric matrix polynomials} \\ \text{with degree at most } d \text{ and with rank at most } 2r \end{array} \right\} = \overline{\mathrm{O}}(W),$$

where the $m \times m$ complex skew-symmetric matrix polynomial W has degree exactly d, rank exactly $2r$, and the complete eigenstructure

$$\mathbf{W}: \Big\{ \overbrace{\underbrace{\beta+1, \ldots, \beta+1}_{t}, \underbrace{\beta, \ldots, \beta}_{m-2r-t}}^{\text{left minimal indices}}, \overbrace{\underbrace{\beta+1, \ldots, \beta+1}_{t}, \underbrace{\beta, \ldots, \beta}_{m-2r-t}}^{\text{right minimal indices}} \Big\}$$

with $\beta = \lfloor rd/(m-2r) \rfloor$ and $t = rd \mod (m-2r)$.

A. Dmytryshyn and F.M. Dopico, Generic skew-symmetric matrix polynomials with fixed rank and fixed odd grade, Linear Algebra Appl., 536 (2018) 1–18

The set of skew-symm polys of degree at most d and rank at most $2r$

Theorem (Dmytryshyn and D., LAA, 2018)

Let m, r and d be integers such that $m \geq 2$, $d \geq 1$ is odd, and $2 \leq 2r \leq (m-1)$. Then

$$\left\{ \begin{array}{c} m \times m \text{ complex skew-symmetric matrix polynomials} \\ \text{with degree at most } d \text{ and with rank at most } 2r \end{array} \right\} = \overline{\mathrm{O}}(W),$$

where the $m \times m$ complex skew-symmetric matrix polynomial W has degree exactly d, rank exactly $2r$, and the complete eigenstructure

$$\mathbf{W} : \left\{ \overbrace{\underbrace{\beta+1, \ldots, \beta+1}_{t}, \underbrace{\beta, \ldots, \beta}_{m-2r-t}}^{\text{left minimal indices}}, \overbrace{\underbrace{\beta+1, \ldots, \beta+1}_{t}, \underbrace{\beta, \ldots, \beta}_{m-2r-t}}^{\text{right minimal indices}} \right\}$$

with $\beta = \lfloor rd/(m-2r) \rfloor$ and $t = rd \mod (m-2r)$.

The effect of imposing structure is dramatic since in the skew-symmetric case **there is only one** generic eigenstructure compared to the $(2r)d+1$ generic eigenstructures of the unstructured case.

A few properties of symmetric matrix polynomials

- **Definition:** $P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0$ with $A_i^T = A_i \in \mathbb{C}^{n\times n}$.
- Symmetric matrix polynomials with size $n \times n$ and degree at most d form a vector space and we can define on it the same Euclidean distance as before.
- Their left minimal indices are equal to the right ones.
- Their **rank can be even or odd**, which immediately implies that in some cases generic eigenstructures must contain eigenvalues.
- When the (at most) degree is odd, they can be always strongly linearized through a symmetric block-tridiagonal companion form (Antoniou and Vologiannidis, ELA, 2004, De Terán, Dopico, Mackey, SIMAX, 2010) that allows us to recover via a shift the minimal indices of the polynomial.

A few properties of symmetric matrix polynomials

- **Definition:** $P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0$ with $A_i^T = A_i \in \mathbb{C}^{n\times n}$.
- Symmetric matrix polynomials with size $n \times n$ and degree at most d form a vector space and we can define on it the same Euclidean distance as before.
- Their left minimal indices are equal to the right ones.
- Their **rank can be even or odd**, which immediately implies that in some cases generic eigenstructures must contain eigenvalues.
- When the (at most) degree is odd, they can be always strongly linearized through a symmetric block-tridiagonal companion form (Antoniou and Vologiannidis, ELA, 2004, De Terán, Dopico, Mackey, SIMAX, 2010) that allows us to recover via a shift the minimal indices of the polynomial.

- **Definition:** $P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0$ with $A_i^T = A_i \in \mathbb{C}^{n\times n}$.
- Symmetric matrix polynomials with size $n \times n$ and degree at most d form a vector space and we can define on it the same Euclidean distance as before.
- Their left minimal indices are equal to the right ones.
- Their **rank can be even or odd**, which immediately implies that in some cases generic eigenstructures must contain eigenvalues.
- When the (at most) degree is odd, they can be always strongly linearized through a symmetric block-tridiagonal companion form (Antoniou and Vologiannidis, ELA, 2004, De Terán, Dopico, Mackey, SIMAX, 2010) that allows us to recover via a shift the minimal indices of the polynomial.

A few properties of symmetric matrix polynomials

- **Definition:** $P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0$ with $A_i^T = A_i \in \mathbb{C}^{n\times n}$.
- Symmetric matrix polynomials with size $n \times n$ and degree at most d form a vector space and we can define on it the same Euclidean distance as before.
- Their left minimal indices are equal to the right ones.
- Their **rank can be even or odd**, which immediately implies that in some cases generic eigenstructures must contain eigenvalues.
- When the (at most) degree is odd, they can be always strongly linearized through a symmetric block-tridiagonal companion form (Antoniou and Vologiannidis, ELA, 2004, De Terán, Dopico, Mackey, SIMAX, 2010) that allows us to recover via a shift the minimal indices of the polynomial.

- **Definition:** $P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0$ with $A_i^T = A_i \in \mathbb{C}^{n\times n}$.
- Symmetric matrix polynomials with size $n \times n$ and degree at most d form a vector space and we can define on it the same Euclidean distance as before.
- Their left minimal indices are equal to the right ones.
- Their **rank can be even or odd**, which immediately implies that in some cases generic eigenstructures must contain eigenvalues.
- When the (at most) degree is odd, they can be always strongly linearized through a symmetric block-tridiagonal companion form (Antoniou and Vologiannidis, ELA, 2004, De Terán, Dopico, Mackey, SIMAX, 2010) that allows us to recover via a shift the minimal indices of the polynomial.

Given

$$P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0, \qquad A_i^T = A_i \in \mathbb{C}^{n\times n},$$

with d **odd**, the **symmetric** matrix pencil

$$\mathcal{F}_P = \begin{bmatrix} \lambda A_d + A_{d-1} & -I & & & & & \\ -I & \ddots & \ddots & & & & \\ & \ddots & 0 & \lambda I & & & \\ & & \lambda I & \lambda A_3 + A_2 & -I & & \\ & & & -I & 0 & \lambda I \\ & & & & \lambda I & \lambda A_1 + A_0 \end{bmatrix}$$

satisfies the following properties

- $\mathcal{F}_P$ has size $nd \times nd$.
- $\mathcal{F}_P$ and P have the same finite and infinite elementary divisors.
- The left (and right) minimal indices of $\mathcal{F}_P$ are greater by $\dfrac{d-1}{2}$ than the left (and right) minimal indices of the polynomial P.

The symmetric companion form

Given

$$P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0, \qquad A_i^T = A_i \in \mathbb{C}^{n\times n},$$

with d **odd**, the **symmetric** matrix pencil

$$\mathcal{F}_P = \begin{bmatrix} \lambda A_d + A_{d-1} & -I & & & & & \\ -I & \ddots & \ddots & & & & \\ & \ddots & 0 & \lambda I & & & \\ & & \lambda I & \lambda A_3 + A_2 & -I & & \\ & & & -I & 0 & \lambda I \\ & & & & \lambda I & \lambda A_1 + A_0 \end{bmatrix}$$

satisfies the following properties

- $\mathcal{F}_P$ has size $nd \times nd$.
- $\mathcal{F}_P$ and P have the same finite and infinite elementary divisors.
- The left (and right) minimal indices of $\mathcal{F}_P$ are greater by $\dfrac{d-1}{2}$ than the left (and right) minimal indices of the polynomial P.

The symmetric companion form

Given

$$P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0, \qquad A_i^T = A_i \in \mathbb{C}^{n\times n},$$

with d **odd**, the **symmetric** matrix pencil

$$\mathcal{F}_P = \begin{bmatrix} \lambda A_d + A_{d-1} & -I & & & & & \\ -I & \ddots & \ddots & & & & \\ & \ddots & 0 & \lambda I & & & \\ & & \lambda I & \lambda A_3 + A_2 & -I & & \\ & & & -I & 0 & \lambda I \\ & & & & \lambda I & \lambda A_1 + A_0 \end{bmatrix}$$

satisfies the following properties

- $\mathcal{F}_P$ has size $nd \times nd$.
- $\mathcal{F}_P$ and P have the same finite and infinite elementary divisors.
- The left (and right) minimal indices of $\mathcal{F}_P$ are greater by $\dfrac{d-1}{2}$ than the left (and right) minimal indices of the polynomial P.

The symmetric companion form

Given

$$P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + A_0, \qquad A_i^T = A_i \in \mathbb{C}^{n\times n},$$

with d **odd**, the **symmetric** matrix pencil

$$\mathcal{F}_P = \begin{bmatrix} \lambda A_d + A_{d-1} & -I & & & & & \\ -I & \ddots & \ddots & & & & \\ & \ddots & 0 & \lambda I & & & \\ & & \lambda I & \lambda A_3 + A_2 & -I & & \\ & & & -I & 0 & \lambda I \\ & & & & \lambda I & \lambda A_1 + A_0 \end{bmatrix}$$

satisfies the following properties

- $\mathcal{F}_P$ has size $nd \times nd$.
- $\mathcal{F}_P$ and P have the same finite and infinite elementary divisors.
- The left (and right) minimal indices of $\mathcal{F}_P$ are greater by $\dfrac{d-1}{2}$ than the left (and right) minimal indices of the polynomial P.

For describing the set of symmetric polys of bounded rank and degree

...we should take into account that:

- The description of the sets of symmetric matrix pencils with bounded ranks was not previously available and we had to develop it.
- The solution of this problem required different techniques that the same problem for skew-symmetric, T-even, T-odd, palindromic, and anti-palindromic pencils, since in these cases there is only one generic eigenstructure, while in the symmetric case there are more than one.

F. De Terán, A. Dmytryshyn and F.M. Dopico, Generic symmetric matrix pencils with bounded rank, submitted (2018)

- In what follows, all the topological concepts refer to the metric space of symmetric matrix polynomials with degree at most d (with d odd).
- In the symmetric case, the description of the sets of symmetric polys of bounded rank and degree is presented in terms of bundles, instead of orbits.

For describing the set of symmetric polys of bounded rank and degree

...we should take into account that:

- The description of the sets of symmetric matrix pencils with bounded ranks was not previously available and we had to develop it.
- The solution of this problem required different techniques that the same problem for skew-symmetric, T-even, T-odd, palindromic, and anti-palindromic pencils, since in these cases there is only one generic eigenstructure, while in the symmetric case there are more than one.

F. De Terán, A. Dmytryshyn and F.M. Dopico, Generic symmetric matrix pencils with bounded rank, submitted (2018)

- In what follows, all the topological concepts refer to the metric space of symmetric matrix polynomials with degree at most d (with d odd).
- In the symmetric case, the description of the sets of symmetric polys of bounded rank and degree is presented in terms of bundles, instead of orbits.

For describing the set of symmetric polys of bounded rank and degree

...we should take into account that:

- The description of the sets of symmetric matrix pencils with bounded ranks was not previously available and we had to develop it.
- The solution of this problem required different techniques that the same problem for skew-symmetric, T-even, T-odd, palindromic, and anti-palindromic pencils, since in these cases there is only one generic eigenstructure, while in the symmetric case there are more than one.

F. De Terán, A. Dmytryshyn and F.M. Dopico, Generic symmetric matrix pencils with bounded rank, submitted (2018)

- In what follows, all the topological concepts refer to the metric space of symmetric matrix polynomials with degree at most d (with d odd).
- In the symmetric case, the description of the sets of symmetric polys of bounded rank and degree is presented in terms of bundles, instead of orbits.

...we should take into account that:

- The description of the sets of symmetric matrix pencils with bounded ranks was not previously available and we had to develop it.
- The solution of this problem required different techniques that the same problem for skew-symmetric, T-even, T-odd, palindromic, and anti-palindromic pencils, since in these cases there is only one generic eigenstructure, while in the symmetric case there are more than one.

F. De Terán, A. Dmytryshyn and F.M. Dopico, Generic symmetric matrix pencils with bounded rank, submitted (2018)

- In what follows, all the topological concepts refer to the metric space of symmetric matrix polynomials with degree at most d (with d odd).
- In the symmetric case, the description of the sets of symmetric polys of bounded rank and degree is presented in terms of bundles, instead of orbits.

For describing the set of symmetric polys of bounded rank and degree

...we should take into account that:

- The description of the sets of symmetric matrix pencils with bounded ranks was not previously available and we had to develop it.
- The solution of this problem required different techniques that the same problem for skew-symmetric, T-even, T-odd, palindromic, and anti-palindromic pencils, since in these cases there is only one generic eigenstructure, while in the symmetric case there are more than one.

F. De Terán, A. Dmytryshyn and F.M. Dopico, Generic symmetric matrix pencils with bounded rank, submitted (2018)

- In what follows, all the topological concepts refer to the metric space of symmetric matrix polynomials with degree at most d (with d odd).
- In the symmetric case, the description of the sets of symmetric polys of bounded rank and degree is presented in terms of bundles, instead of orbits.

Definition (Bundle of a symmetric polynomial)

The bundle of a symmetric matrix polynomial $P(\lambda)$ is defined as

$$\mathrm{B}(P) = \left\{ \begin{array}{c} \text{symmetric matrix polynomials with the same size and grade,} \\ \text{and with the same complete eigenstructure as } P(\lambda), \\ \text{except that the values of the eigenvalues are unspecified} \end{array} \right\}$$

Example

$$P(\lambda) = \begin{bmatrix} (\lambda-1)(\lambda-2) & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^2 \\ 0 & 0 & 0 & 1 \\ 0 & \lambda^2 & 1 & 0 \end{bmatrix}, \quad Q(\lambda) = \begin{bmatrix} (\lambda-6)(\lambda-7) & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^2 \\ 0 & 0 & 0 & 1 \\ 0 & \lambda^2 & 1 & 0 \end{bmatrix}$$

Complete eigenstructure

$(\lambda-1), (\lambda-2)$ elementary divisors

2 is the unique left minimal index

2 is the unique right minimal index

Complete eigenstructure

$(\lambda-6), (\lambda-7)$ elementary divisors

2 is the unique left minimal index

2 is the unique right minimal index

P and Q are in the same bundle but NOT in the same orbit.

Definition and example of bundle

Definition (Bundle of a symmetric polynomial)

The bundle of a symmetric matrix polynomial $P(\lambda)$ is defined as

$$\mathrm{B}(P) = \left\{ \begin{array}{c} \text{symmetric matrix polynomials with the same size and grade,} \\ \text{and with the same complete eigenstructure as } P(\lambda), \\ \text{except that the values of the eigenvalues are unspecified} \end{array} \right\}$$

Example

$$P(\lambda) = \begin{bmatrix} (\lambda-1)(\lambda-2) & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^2 \\ 0 & 0 & 0 & 1 \\ 0 & \lambda^2 & 1 & 0 \end{bmatrix}, \qquad Q(\lambda) = \begin{bmatrix} (\lambda-6)(\lambda-7) & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^2 \\ 0 & 0 & 0 & 1 \\ 0 & \lambda^2 & 1 & 0 \end{bmatrix}$$

Complete eigenstructure

$(\lambda-1), (\lambda-2)$ elementary divisors

2 is the unique left minimal index

2 is the unique right minimal index

Complete eigenstructure

$(\lambda-6), (\lambda-7)$ elementary divisors

2 is the unique left minimal index

2 is the unique right minimal index

P and Q are in the same bundle but NOT in the same orbit.

Definition (Bundle of a symmetric polynomial)

The bundle of a symmetric matrix polynomial $P(\lambda)$ is defined as

$$\mathrm{B}(P) = \left\{ \begin{array}{c} \text{symmetric matrix polynomials with the same size and grade,} \\ \text{and with the same complete eigenstructure as } P(\lambda), \\ \text{except that the values of the eigenvalues are unspecified} \end{array} \right\}$$

Example

$$P(\lambda) = \begin{bmatrix} (\lambda-1)(\lambda-2) & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^2 \\ 0 & 0 & 0 & 1 \\ 0 & \lambda^2 & 1 & 0 \end{bmatrix}, \qquad Q(\lambda) = \begin{bmatrix} (\lambda-6)(\lambda-7) & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^2 \\ 0 & 0 & 0 & 1 \\ 0 & \lambda^2 & 1 & 0 \end{bmatrix}$$

Complete eigenstructure

$(\lambda-1), (\lambda-2)$ elementary divisors

2 is the unique left minimal index

2 is the unique right minimal index

Complete eigenstructure

$(\lambda-6), (\lambda-7)$ elementary divisors

2 is the unique left minimal index

2 is the unique right minimal index

***P* and *Q* are in the same bundle** but NOT in the same orbit.

Theorem (De Terán, Dmytryshyn and D., in preparation, 2018)

Let n, r and d be integers such that $n \geq 2$, $d \geq 1$ is odd, and $1 \leq r \leq (n-1)$. Then

$$\left\{ \begin{array}{l} n \times n \text{ complex symmetric matrix polynomials} \\ \text{with degree at most } d \text{ and with rank at most } r \end{array} \right\} = \bigcup_{0 \leq a \leq \lfloor \frac{rd}{2} \rfloor} \overline{\mathrm{B}}(K_a),$$

where the $n \times n$ complex symmetric matrix polynomial K_a, $a = 0, 1, \ldots, \left\lfloor \frac{rd}{2} \right\rfloor$, has degree exactly d, rank exactly r, and the complete eigenstructure

$$\mathbf{K}_a : \Big\{ \overbrace{\underbrace{\alpha+1, \ldots, \alpha+1}_{s}, \underbrace{\alpha, \ldots, \alpha}_{n-r-s}}^{\text{left minimal indices}}, \overbrace{\underbrace{\alpha+1, \ldots, \alpha+1}_{s}, \underbrace{\alpha, \ldots, \alpha}_{n-r-s}}^{\text{right minimal indices}}, (\lambda - \mu_1), \ldots, (\lambda - \mu_{rd-2a}) \Big\}$$

where $\alpha = \lfloor a/(n-r) \rfloor$ and $s = a \mod (n-r)$ and $\mu_i \neq \mu_j$, if $i \neq j$.

Moreover, $\overline{\mathrm{B}}(K_a) \cap \mathrm{B}(K_{a'}) = \varnothing$ whenever $a \neq a'$ (but $\overline{\mathrm{B}}(K_a) \cap \overline{\mathrm{B}}(K_{a'}) \neq \varnothing$).

F. De Terán, A. Dmytryshyn and F.M. Dopico, Generic complete eigenstructures for sets of symmetric matrix polynomials with bounded rank and degree, in preparation.

Theorem (De Terán, Dmytryshyn and D., in preparation, 2018)

Let n, r and d be integers such that $n \geq 2$, $d \geq 1$ is odd, and $1 \leq r \leq (n-1)$. Then

$$\left\{ \begin{array}{l} n \times n \text{ complex symmetric matrix polynomials} \\ \text{with degree at most } d \text{ and with rank at most } r \end{array} \right\} = \bigcup_{0 \leq a \leq \lfloor \frac{rd}{2} \rfloor} \overline{\mathrm{B}}(K_a),$$

where the $n \times n$ complex symmetric matrix polynomial K_a, $a = 0, 1, \ldots, \lfloor \frac{rd}{2} \rfloor$, has degree exactly d, rank exactly r, and the complete eigenstructure

$$\mathbf{K}_a : \Big\{ \overbrace{\underbrace{\alpha+1, \ldots, \alpha+1}_{s}, \underbrace{\alpha, \ldots, \alpha}_{n-r-s}}^{\text{left minimal indices}}, \overbrace{\underbrace{\alpha+1, \ldots, \alpha+1}_{s}, \underbrace{\alpha, \ldots, \alpha}_{n-r-s}}^{\text{right minimal indices}}, (\lambda - \mu_1), \ldots, (\lambda - \mu_{rd-2a}) \Big\}$$

where $\alpha = \lfloor a/(n-r) \rfloor$ and $s = a \mod (n-r)$ and $\mu_i \neq \mu_j$, if $i \neq j$.

Moreover, $\overline{\mathrm{B}}(K_a) \cap \mathrm{B}(K_{a'}) = \varnothing$ whenever $a \neq a'$ (but $\overline{\mathrm{B}}(K_a) \cap \overline{\mathrm{B}}(K_{a'}) \neq \varnothing$).

F. De Terán, A. Dmytryshyn and F.M. Dopico, Generic complete eigenstructures for sets of symmetric matrix polynomials with bounded rank and degree, in preparation.

Theorem (De Terán, Dmytryshyn and D., in preparation, 2018)

Let n, r and d be integers such that $n \geq 2$, $d \geq 1$ is odd, and $1 \leq r \leq (n-1)$. Then

$$\left\{ \begin{array}{c} n \times n \text{ complex symmetric matrix polynomials} \\ \text{with degree at most } d \text{ and with rank at most } r \end{array} \right\} = \bigcup_{0 \leq a \leq \lfloor \frac{rd}{2} \rfloor} \overline{\mathrm{B}}(K_a),$$

where the $n \times n$ complex symmetric matrix polynomial K_a, $a = 0, 1, \ldots, \lfloor \frac{rd}{2} \rfloor$, has degree exactly d, rank exactly r, and the complete eigenstructure

$$\mathbf{K}_a : \Big\{ \overbrace{\underbrace{\alpha+1, \ldots, \alpha+1}_{s}, \underbrace{\alpha, \ldots, \alpha}_{n-r-s}}^{\text{left minimal indices}}, \overbrace{\underbrace{\alpha+1, \ldots, \alpha+1}_{s}, \underbrace{\alpha, \ldots, \alpha}_{n-r-s}}^{\text{right minimal indices}}, (\lambda - \mu_1), \ldots, (\lambda - \mu_{rd-2a}) \Big\}$$

where $\alpha = \lfloor a/(n-r) \rfloor$ and $s = a \mod (n-r)$ and $\mu_i \neq \mu_j$, if $i \neq j$.

Moreover, $\overline{\mathrm{B}}(K_a) \cap \mathrm{B}(K_{a'}) = \varnothing$ whenever $a \neq a'$ (but $\overline{\mathrm{B}}(K_a) \cap \overline{\mathrm{B}}(K_{a'}) \neq \varnothing$).

F. De Terán, A. Dmytryshyn and F.M. Dopico, Generic complete eigenstructures for sets of symmetric matrix polynomials with bounded rank and degree, in preparation.

Theorem (De Terán, Dmytryshyn and D., in preparation, 2018)

Let n, r and d be integers such that $n \geq 2$, $d \geq 1$ is odd, and $1 \leq r \leq (n-1)$. Then

$$\left\{ \begin{array}{c} n \times n \text{ complex symmetric matrix polynomials} \\ \text{with degree at most } d \text{ and with rank at most } r \end{array} \right\} = \bigcup_{0 \leq a \leq \lfloor \frac{rd}{2} \rfloor} \overline{\mathrm{B}}(K_a),$$

where the $n \times n$ complex symmetric matrix polynomial K_a, $a = 0, 1, \ldots, \left\lfloor \frac{rd}{2} \right\rfloor$, has degree exactly d, rank exactly r, and the complete eigenstructure

$$\mathbf{K}_a : \Big\{ \overbrace{\underbrace{\alpha+1, \ldots, \alpha+1}_{s}, \underbrace{\alpha, \ldots, \alpha}_{n-r-s}}^{\text{left minimal indices}}, \overbrace{\underbrace{\alpha+1, \ldots, \alpha+1}_{s}, \underbrace{\alpha, \ldots, \alpha}_{n-r-s}}^{\text{right minimal indices}}, (\lambda - \mu_1), \ldots, (\lambda - \mu_{rd-2a}) \Big\}$$

where $\alpha = \lfloor a/(n-r) \rfloor$ and $s = a \mod (n-r)$ and $\mu_i \neq \mu_j$, if $i \neq j$.

Moreover, $\overline{\mathrm{B}}(K_a) \cap \mathrm{B}(K_{a'}) = \varnothing$ whenever $a \neq a'$ (but $\overline{\mathrm{B}}(K_a) \cap \overline{\mathrm{B}}(K_{a'}) \neq \varnothing$).

F. De Terán, A. Dmytryshyn and F.M. Dopico, Generic complete eigenstructures for sets of symmetric matrix polynomials with bounded rank and degree, in preparation.

Theorem (De Terán, Dmytryshyn and D., in preparation, 2018)

Let n, r and d be integers such that $n \geq 2$, $d \geq 1$ is odd, and $1 \leq r \leq (n-1)$. Then

$$\left\{ \begin{array}{c} n \times n \text{ complex symmetric matrix polynomials} \\ \text{with degree at most } d \text{ and with rank at most } r \end{array} \right\} = \bigcup_{0 \leq a \leq \lfloor \frac{rd}{2} \rfloor} \overline{\mathrm{B}}(K_a),$$

where the $n \times n$ complex symmetric matrix polynomial K_a, $a = 0, 1, \ldots, \lfloor \frac{rd}{2} \rfloor$, has degree exactly d, rank exactly r, and the complete eigenstructure

$$\mathbf{K}_a : \Big\{ \overbrace{\underbrace{\alpha+1, \ldots, \alpha+1}_{s}, \underbrace{\alpha, \ldots, \alpha}_{n-r-s}}^{\text{left minimal indices}}, \overbrace{\underbrace{\alpha+1, \ldots, \alpha+1}_{s}, \underbrace{\alpha, \ldots, \alpha}_{n-r-s}}^{\text{right minimal indices}}, (\lambda - \mu_1), \ldots, (\lambda - \mu_{rd-2a}) \Big\}$$

where $\alpha = \lfloor a/(n-r) \rfloor$ and $s = a \mod (n-r)$ and $\mu_i \neq \mu_j$, if $i \neq j$.

Moreover, $\overline{\mathrm{B}}(K_a) \bigcap \mathrm{B}(K_{a'}) = \varnothing$ whenever $a \neq a'$ (but $\overline{\mathrm{B}}(K_a) \bigcap \overline{\mathrm{B}}(K_{a'}) \neq \varnothing$).

The effect of imposing the symmetric structure is very strong in two senses:

- **There are only $\lfloor \frac{rd}{2} \rfloor + 1$ generic eigenstructures** instead of $rd + 1$.
- **The generic eigenstructures include eigenvalues**.

Comments on the eigenvalues of the generic eigenstructures

The generic eigenstructures

$$\mathbf{K}_a : \Big\{ \overbrace{\underbrace{\alpha+1,\dots,\alpha+1}_{s},\underbrace{\alpha,\dots,\alpha}_{n-r-s}}^{\text{left minimal indices}}, \overbrace{\underbrace{\alpha+1,\dots,\alpha+1}_{s},\underbrace{\alpha,\dots,\alpha}_{n-r-s}}^{\text{right minimal indices}}, (\lambda-\mu_1),\dots,(\lambda-\mu_{rd-2a}) \Big\},$$

for $a = 0, 1, \dots, \left\lfloor \frac{rd}{2} \right\rfloor$, have the following codimensions

$$\operatorname{cod} \mathrm{B}(K_a) = (n-r)\left(n + \frac{d+1}{2}\right) - a(n-r-1).$$

Thus,

- If $r = n-1$, then all the codimensions are equal; but if $r < n-1$, then
- the codimension of the bundle decreases if the number of eigenvalues decreases, or, equivalently,
- the dimension of the bundle increases if the number of eigenvalues decreases.

Comments on the eigenvalues of the generic eigenstructures

The generic eigenstructures

$$\mathbf{K}_a : \Big\{ \overbrace{\underbrace{\alpha+1,\ldots,\alpha+1}_{s}, \underbrace{\alpha,\ldots,\alpha}_{n-r-s}}^{\text{left minimal indices}}, \overbrace{\underbrace{\alpha+1,\ldots,\alpha+1}_{s}, \underbrace{\alpha,\ldots,\alpha}_{n-r-s}}^{\text{right minimal indices}}, (\lambda-\mu_1),\ldots,(\lambda-\mu_{rd-2a}) \Big\},$$

for $a = 0, 1, \ldots, \left\lfloor \frac{rd}{2} \right\rfloor$, have the following codimensions

$$\operatorname{cod} \mathrm{B}(K_a) = (n-r)\left(n + \frac{d+1}{2}\right) - a(n-r-1).$$

Thus,

- If $r = n-1$, then all the codimensions are equal; but if $r < n-1$, then
- the codimension of the bundle decreases if the number of eigenvalues decreases, or, equivalently,
- the dimension of the bundle increases if the number of eigenvalues decreases.

Comments on the eigenvalues of the generic eigenstructures

The generic eigenstructures

$$\mathbf{K}_a : \Big\{ \overbrace{\underbrace{\alpha+1,\dots,\alpha+1}_{s}, \underbrace{\alpha,\dots,\alpha}_{n-r-s}}^{\text{left minimal indices}}, \overbrace{\underbrace{\alpha+1,\dots,\alpha+1}_{s}, \underbrace{\alpha,\dots,\alpha}_{n-r-s}}^{\text{right minimal indices}}, (\lambda-\mu_1),\dots,(\lambda-\mu_{rd-2a}) \Big\},$$

for $a = 0, 1, \dots, \left\lfloor \frac{rd}{2} \right\rfloor$, have the following codimensions

$$\operatorname{cod} \mathrm{B}(K_a) = (n-r)\left(n + \frac{d+1}{2}\right) - a(n-r-1).$$

Thus,

- If $r = n-1$, then all the codimensions are equal; but if $r < n-1$, then
- the codimension of the bundle decreases if the number of eigenvalues decreases, or, equivalently,
- the dimension of the bundle increases if the number of eigenvalues decreases.

Comments on the eigenvalues of the generic eigenstructures

The generic eigenstructures

$$\mathbf{K}_a : \Big\{ \overbrace{\underbrace{\alpha+1,\ldots,\alpha+1}_{s}, \underbrace{\alpha,\ldots,\alpha}_{n-r-s}}^{\text{left minimal indices}}, \overbrace{\underbrace{\alpha+1,\ldots,\alpha+1}_{s}, \underbrace{\alpha,\ldots,\alpha}_{n-r-s}}^{\text{right minimal indices}}, (\lambda-\mu_1),\ldots,(\lambda-\mu_{rd-2a}) \Big\},$$

for $a = 0, 1, \ldots, \left\lfloor \frac{rd}{2} \right\rfloor$, have the following codimensions

$$\operatorname{cod} \mathrm{B}(K_a) = (n-r)\left(n + \frac{d+1}{2}\right) - a(n-r-1).$$

Thus,

- If $r = n-1$, then all the codimensions are equal; but if $r < n-1$, then
- the codimension of the bundle decreases if the number of eigenvalues decreases, or, equivalently,
- the dimension of the bundle increases if the number of eigenvalues decreases.

Comments on the eigenvalues of the generic eigenstructures

The generic eigenstructures

$$\mathbf{K}_a : \Big\{ \overbrace{\underbrace{\alpha+1,\ldots,\alpha+1}_{s}, \underbrace{\alpha,\ldots,\alpha}_{n-r-s}}^{\text{left minimal indices}}, \overbrace{\underbrace{\alpha+1,\ldots,\alpha+1}_{s}, \underbrace{\alpha,\ldots,\alpha}_{n-r-s}}^{\text{right minimal indices}}, (\lambda-\mu_1),\ldots,(\lambda-\mu_{rd-2a}) \Big\},$$

for $a = 0, 1, \ldots, \lfloor \frac{rd}{2} \rfloor$, have the following codimensions

$$\operatorname{cod} \mathrm{B}(K_a) = (n-r)\left(n + \frac{d+1}{2}\right) - a(n-r-1).$$

Thus,

- If $r = n-1$, then all the codimensions are equal; but if $r < n-1$, then
- the codimension of the bundle decreases if the number of eigenvalues decreases, or, equivalently,
- the dimension of the bundle increases if the number of eigenvalues decreases.

Generic eigenstructures of sets of general and structured matrix polynomials with bounded rank and degree: Solved and Open problems

In table: r = rank, d = degree, # = number of generic eigenstructures.

	Pencils	**Polynomials** $d > 1$
General	De Terán and D., 2008 $\# = r + 1$	Dmytryshyn and D., 2017 $\# = rd + 1$
Skew-Symmetric	Dmytryshyn and D., 2018 $\# = 1$	Dmytryshyn and D., 2018 (d odd) $\# = 1$
T-(anti)palindromic	De Terán, 2018, $\# = 1$	open
T-even and odd	De Terán, 2018, $\# = 1$	open
Symmetric	De Terán, Dmytryshyn and D., 2018 $\lfloor r/2 \rfloor + 1$	De Terán, Dmytryshyn and D., 2018 (d odd) $\lfloor rd/2 \rfloor + 1$
Hermitian	open	open

F. De Terán, A geometric description of the sets of palindromic and alternating matrix pencils with bounded rank, SIAM J. Matrix Anal. Appl., 39 (2018) 1116–1134

Most relevant question in this setting: What happens with structured matrix polynomials of degree at most d with d even?

Generic eigenstructures of sets of general and structured matrix polynomials with bounded rank and degree: Solved and Open problems

In table: r = rank, d = degree, # = number of generic eigenstructures.

	Pencils	**Polynomials $d > 1$**
General	De Terán and D., 2008 $\# = r + 1$	Dmytryshyn and D., 2017 $\# = rd + 1$
Skew-Symmetric	Dmytryshyn and D., 2018 $\# = 1$	Dmytryshyn and D., 2018 (d odd) $\# = 1$
T-(anti)palindromic	De Terán, 2018, $\# = 1$	open
T-even and odd	De Terán, 2018, $\# = 1$	open
Symmetric	De Terán, Dmytryshyn and D., 2018 $\lfloor r/2 \rfloor + 1$	De Terán, Dmytryshyn and D., 2018 (d odd) $\lfloor rd/2 \rfloor + 1$
Hermitian	open	open

F. De Terán, A geometric description of the sets of palindromic and alternating matrix pencils with bounded rank, SIAM J. Matrix Anal. Appl., 39 (2018) 1116–1134

Most relevant question in this setting: What happens with structured matrix polynomials of degree at most d with d even?

Outline

Ideas, difficulties, and simple statement of results

- Any $m \times n$ constant matrix A of rank r is

$$A = L\,R, \quad \text{where} \begin{cases} L \text{ is } m \times r \text{ and } \operatorname{rank} L = r, \\ R \text{ is } r \times n \text{ and } \operatorname{rank} R = r. \end{cases}$$

- The idea is to get a similar description of $\mathrm{POL}_{d,r}^{m\times n}$ but the degree of the factors makes the problem not trivial: it might be cancellations of "high degrees", how to distribute degrees between the factors, etc.

- Nevertheless, generically, i.e., using closures of open dense sets, we can prove that if $P(\lambda) \in \mathrm{POL}_{d,r}^{m\times n}$

$$P(\lambda) = L(\lambda)R(\lambda),$$

  where

  1. $L(\lambda)$ is an $m \times r$ matrix polynomial, $\operatorname{rank} L(\lambda) = r$, and degrees of its columns differ at most by one,
  2. $R(\lambda)$ is an $r \times n$ matrix polynomial, $\operatorname{rank} R(\lambda) = r$, and degrees of its rows differ at most by one, and
  3. $\deg \mathsf{col}_i(L(\lambda)) + \deg \mathsf{row}_i(R(\lambda)) = d$, for $i = 1, \ldots, r$.

Ideas, difficulties, and simple statement of results

- Any $m \times n$ constant matrix A of rank r is

$$A = L\,R, \quad \text{where } \begin{cases} L \text{ is } m \times r \text{ and } \operatorname{rank} L = r, \\ R \text{ is } r \times n \text{ and } \operatorname{rank} R = r. \end{cases}$$

- The idea is to get a similar description of $\mathrm{POL}_{d,r}^{m\times n}$ but the degree of the factors makes the problem not trivial: it might be cancellations of "high degrees", how to distribute degrees between the factors, etc.

- Nevertheless, generically, i.e., using closures of open dense sets, we can prove that if $P(\lambda) \in \mathrm{POL}_{d,r}^{m\times n}$

$$P(\lambda) = L(\lambda)R(\lambda),$$

where

1. $L(\lambda)$ is an $m \times r$ matrix polynomial, $\operatorname{rank} L(\lambda) = r$, and degrees of its columns differ at most by one,
2. $R(\lambda)$ is an $r \times n$ matrix polynomial, $\operatorname{rank} R(\lambda) = r$, and degrees of its rows differ at most by one, and
3. $\deg \mathrm{col}_i(L(\lambda)) + \deg \mathrm{row}_i(R(\lambda)) = d$, for $i = 1, \ldots, r$.

- Any $m \times n$ constant matrix A of rank r is

$$A = L\,R, \quad \text{where} \begin{cases} L \text{ is } m \times r \text{ and } \operatorname{rank} L = r, \\ R \text{ is } r \times n \text{ and } \operatorname{rank} R = r. \end{cases}$$

- The idea is to get a similar description of $\mathrm{POL}_{d,r}^{m \times n}$ but the degree of the factors makes the problem not trivial: it might be cancellations of "high degrees", how to distribute degrees between the factors, etc.

- Nevertheless, generically, i.e., using closures of open dense sets, we can prove that if $P(\lambda) \in \mathrm{POL}_{d,r}^{m \times n}$

$$P(\lambda) = L(\lambda)R(\lambda),$$

  where

  1. $L(\lambda)$ is an $m \times r$ matrix polynomial, $\operatorname{rank} L(\lambda) = r$, and degrees of its columns differ at most by one,
  2. $R(\lambda)$ is an $r \times n$ matrix polynomial, $\operatorname{rank} R(\lambda) = r$, and degrees of its rows differ at most by one, and
  3. $\deg \mathrm{col}_i(L(\lambda)) + \deg \mathrm{row}_i(R(\lambda)) = d$, for $i = 1, \ldots, r$.

Theorem (Dmytryshyn, D., and Van Dooren, in progress, 2018)

$$\mathrm{POL}_{d,r}^{m\times n} = \left\{ \begin{array}{c} m\times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \text{, with rank at most } r \end{array} \right\} = \bigcup_{0\le a\le rd} \overline{\mathcal{B}_a}\,,$$

where, for $a = 0, 1, \dots, rd$,

$$\mathcal{B}_a := \left\{ L(\lambda)R(\lambda) \; : \; \begin{array}{l} L(\lambda) \in \mathbb{C}[\lambda]^{m\times r},\; R(\lambda) \in \mathbb{C}[\lambda]^{r\times n}, \\ \deg \mathit{row}_i(R) = d_R + 1, \quad \text{for } i = 1, \dots, t_R, \\ \deg \mathit{row}_i(R) = d_R, \quad \text{for } i = t_R + 1, \dots, r, \\ \deg \mathit{col}_i(L) = d - \deg(R_{i*}), \quad \text{for } i = 1, \dots, r \end{array} \right\},$$

with $d_R = \lfloor a/r \rfloor$ *and* $t_R = a \bmod r$. *Moreover,*

$$\overline{\mathcal{B}_a} = \overline{\mathrm{O}}(K_a),$$

where K_a *are the* $m\times n$ *matrix polynomials of degree exactly* d *and rank exactly* r *with the generic eigenstructures defined in the first part of the talk.*

...and more next year. **Thank you for your attention!**

The precise result

Theorem (Dmytryshyn, D., and Van Dooren, in progress, 2018)

$$\mathrm{POL}_{d,r}^{m\times n} = \left\{ \begin{array}{c} m \times n \textit{ complex matrix polynomials} \\ \textit{with degree at most } d\textit{, with rank at most } r \end{array} \right\} = \bigcup_{0\le a\le rd} \overline{\mathcal{B}_a}\,,$$

where, for $a = 0, 1, \ldots, rd$*,*

$$\mathcal{B}_a := \left\{ L(\lambda)R(\lambda) : \begin{array}{l} L(\lambda) \in \mathbb{C}[\lambda]^{m\times r},\ R(\lambda) \in \mathbb{C}[\lambda]^{r\times n}, \\ \deg \mathit{row}_i(R) = d_R + 1, \quad \textit{for } i = 1, \ldots, t_R, \\ \deg \mathit{row}_i(R) = d_R, \quad \textit{for } i = t_R + 1, \ldots, r, \\ \deg \mathit{col}_i(L) = d - \deg(R_{i*}), \quad \textit{for } i = 1, \ldots, r \end{array} \right\},$$

with $d_R = \lfloor a/r \rfloor$ *and* $t_R = a \bmod r$*. Moreover,*

$$\overline{\mathcal{B}_a} = \overline{\mathrm{O}}(K_a),$$

where K_a *are the* $m \times n$ *matrix polynomials of degree exactly* d *and rank exactly* r *with the generic eigenstructures defined in the first part of the talk.*

...and more next year. **Thank you for your attention!**

Theorem (Dmytryshyn, D., and Van Dooren, in progress, 2018)

$$\mathrm{POL}_{d,r}^{m\times n} = \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \text{, with rank at most } r \end{array} \right\} = \bigcup_{0 \le a \le rd} \overline{\mathcal{B}_a}\,,$$

where, for $a = 0, 1, \ldots, rd$,

$$\mathcal{B}_a := \left\{ L(\lambda)R(\lambda) \; : \; \begin{array}{l} L(\lambda) \in \mathbb{C}[\lambda]^{m\times r},\; R(\lambda) \in \mathbb{C}[\lambda]^{r\times n}, \\ \deg \mathit{row}_i(R) = d_R + 1, \quad \text{for } i = 1, \ldots, t_R, \\ \deg \mathit{row}_i(R) = d_R, \quad \text{for } i = t_R + 1, \ldots, r, \\ \deg \mathit{col}_i(L) = d - \deg(R_{i*}), \quad \text{for } i = 1, \ldots, r \end{array} \right\},$$

with $d_R = \lfloor a/r \rfloor$ *and* $t_R = a \bmod r$. *Moreover,*

$$\overline{\mathcal{B}_a} = \overline{\mathrm{O}}(K_a),$$

where K_a *are the* $m \times n$ *matrix polynomials of degree exactly* d *and rank exactly* r *with the generic eigenstructures defined in the first part of the talk.*

...and more next year. **Thank you for your attention!**