Solving different rational eigenvalue problems via different types of linearizations

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joint work with **Silvia Marcaida** (U. País Vasco, Spain),
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Minisymposium on
"Nonlinear Eigenvalue Problems and Applications"
SIAM Conference on Applied Linear Algebra (SIAM-ALA18)
4-8 May 2018. Hong Kong Baptist University

- Given a nonsingular rational matrix $G(\lambda) \in \mathbb{C}(\lambda)^{p \times p}$ the REP consists in computing numbers $\lambda_0 \in \mathbb{C}$ and non-zero vectors $x_0 \in \mathbb{C}^p$ such that

$$G(\lambda_0)\, x_0 = 0.$$

- REPs have arisen in applications, directly or as approximations of nonlinear eigenvalue problems (NEP), (surveys Mehrmann-Voss (2004), Betcke et al., NLEVP, (2013), Güttel-Tisseur, (2017)),
- but REPs have been studied since the 60s and 70s in Linear Systems and Control and the more general problem of computing all the structural data of a Rational Matrix was solved using linearizations by Van Dooren in his PhD Thesis (1979) and papers in early 80s for dense problems.
- **A first key difference between REPs and** polynomial eigenvalue problems (**PEPs**) is that, once a scalar polynomial basis is chosen, a PEP is completely determined by the coefficients, while REPs are not determined by the election of a basis and appear in many different forms.
- This is related to the classic theory and computation of realizations of rational matrices in linear systems theory (Rosenbrock (1970), Kailath (1980), Antoulas (2005), etc).

A few words on Rational Eigenvalue Problems (REPs)

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- Loaded elastic string (Betcke et al., NLEVP-collection, (2013)):

$$G(\lambda) = A - \lambda B + \frac{\lambda}{\lambda - \sigma} E = (A + E) - \lambda B + \frac{\sigma}{\lambda - \sigma} E,$$

which almost shows the polynomial and the strictly proper parts of $G(\lambda)$.

- Damped vibration of a structure (Mehrmann & Voss, (2004)):

$$G(\lambda) = \lambda^2 M + K - \sum_{i=1}^{k} \frac{1}{1 + b_i \lambda} \Delta G_i,$$

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A few examples of “modern” REPs with different representations (I)

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A few examples of “modern” REPs with different representations (II)

- NLEIGS-REPs coming from linear rational interpolation of NEPs (Güttel, Van Beeumen, Meerbergen, Michiels (2014)):

$$Q_N(\lambda) = b_0(\lambda)D_0 + b_1(\lambda)D_1 + \cdots + b_N(\lambda)D_N,$$

with $D_j \in \mathbb{C}^{m\times m}$ and $b_j(\lambda) = \dfrac{1}{\beta_0}\displaystyle\prod_{k=1}^{j}\dfrac{\lambda-\sigma_{k-1}}{\beta_k(1-\lambda/\xi_k)}$, $j = 0, 1, \ldots, N$, a sequence of rational scalar functions, with the poles ξ_i all distinct from the nodes σ_j. Some poles ξ_i can be infinite.

- REPs coming from “Automatic Approximation of NEPs” (Lietaert, Pérez, Vandereycken, Meerbergen, 2018 (see Meerbergen’s Talk this Minisymposium)):

$$R(\lambda) = \sum_{i=0}^{k-1}(A_i - \lambda B_i)f_i(\lambda) + \sum_{i=1}^{s}(C_i - \lambda D_i)a_i^T(E_i - \lambda F_i)^{-1}b_i,$$

where $f_i(\lambda)$ are scalar polynomial or rational functions satisfying a linear relation ($f_0(\lambda) = 1$), $a_i, b_i \in \mathbb{C}^{l_i}$ are vectors, A_i, B_i, C_i, D_i matrices, and $l_i \times l_i$ matrices

$$E_i = \begin{bmatrix} w_1 & w_2 & \cdots & w_{l_i-1} & w_{l_i} \\ -z_1 & z_2 & & & \\ & -z_2 & \ddots & & \\ & & \ddots & z_{l_i-1} & \\ & & & -z_{l_i-1} & z_{l_i} \end{bmatrix} \text{ and } F_i = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & & & \\ & 1 & \ddots & & \\ & & \ddots & -1 & \\ & & & 1 & -1 \end{bmatrix}.$$

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- **A second key difference between REPs and PEPs** is that there is no agreement on what is a **linearization** of a rational matrix.
- For regular matrix polynomials, linearizations are just regular pencils with exactly the same finite elementary divisors (same finite eigenvalues with same multiplicities, geometric, algebraic, partial). If a linearization has the same infinite elementary divisors, then it is a strong linearization.
- There are well-known and compact characterizations of linearizations of matrix polys in terms of unimodular transformations.
- In contrast, many authors have developed “linearizations” of rational matrices, but they very rarely prove that such pencils satisfy properties analogous to those of linearizations of matrix polynomials.
- REPs are more difficult than PEPs, so, perhaps, we need to be flexible and to admit **“different types of linearizations in REPs” (sometimes weaker)** that in PEPs, and, in my opinion, each type should have a different name and their properties should be clearly stated.
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Some previous works on "linearizations" of rational matrices

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  1. Works by P. Van Dooren and G. Verghese in late 70s & early 80s, where they construct pencils that have exactly the same structural data as any given rational matrix, including minimal indices. The constructions require some numerical computations.
  2. Y. Su and Z. Bai, SIMAX, 2011, construct a Frobenius-like linearization from a representation of $G(\lambda)$ as polynomial + state-space realization.
- The definitions in this talk are based on and extend those in Amparan, D, Marcaida, and Zaballa, *Strong linearizations of rational matrices*, MIMS Eprint (2016).
- Another approach for defining (non-strong) linearizations of rational matrices can be found in Alam & Behera, SIMAX, 2016.
- NLEIGS linearizations (Güttel, Van Beeumen, Meerbergen, Michiels, SISC (2014)), Automatic Approximation of NEPs (Lietaert, Pérez, Vandereycken, Meerbergen, 2018), Padé Linearization (Bai, this mini),...

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2. **Linearizations of rational matrices: strong, in a set, at infinity**
3. **Block minimal bases linearizations of rational matrices: strong, in a set, at infinity**
4. **The NLEIGS “linearizations” inside this framework**
5. **The “Automatic linearizations” inside this framework**

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Polynomial and strictly proper parts of a rational matrix. Reversal.

- **Any rational matrix $G(\lambda)$ can be uniquely expressed as**

$$G(\lambda) = D(\lambda) + G_{sp}(\lambda),$$

  where

  1. $D(\lambda)$ is a polynomial matrix (polynomial part of $G(\lambda)$), and
  2. the rational matrix $G_{sp}(\lambda)$ is **strictly proper** (strictly proper part of $G(\lambda)$), i.e., $\lim_{\lambda \to \infty} G_{sp}(\lambda) = 0$.

- Let $d = \deg(D)$ if $D(\lambda) \neq 0$ and $d = 0$ otherwise. We define the **reversal of $G(\lambda)$** as

$$\text{rev}\, G(\lambda) = \lambda^d G\left(\frac{1}{\lambda}\right).$$

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Definition (finite zeros, finite poles, finite eigenvalues)

Given the **Smith-McMillan form** of a rational matrix $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$:

$$U(\lambda)G(\lambda)V(\lambda) = \text{diag}\left(\frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)}, \ldots, \frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)}, 0_{(p-r)\times(m-r)}\right),$$

where $U(\lambda), V(\lambda)$ are unimodular matrices and $\varepsilon_1(\lambda)|\cdots|\varepsilon_r(\lambda)$, $\psi_r(\lambda)|\cdots|\psi_1(\lambda)$ are polynomials:

- The **finite zeros** of $G(\lambda)$ are the roots of the numerators $\varepsilon_i(\lambda)$ and the **finite poles** of $G(\lambda)$ are the roots of the denominators $\psi_i(\lambda)$.
- The **finite eigenvalues** of $G(\lambda)$ are the finite zeros that are not poles.

Definition (structural indices or partial multiplicities)

Given any $c \in \mathbb{C}$, one can write for each $i = 1, \ldots, r$,

$$\frac{\varepsilon_i(\lambda)}{\psi_i(\lambda)} = (\lambda - c)^{\sigma_i(c)} \frac{\widetilde{\varepsilon}_i(\lambda)}{\widetilde{\psi}_i(\lambda)}, \qquad \text{with } \widetilde{\varepsilon}_i(c) \neq 0, \widetilde{\psi}_i(c) \neq 0.$$

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$$\frac{\varepsilon_i(\lambda)}{\psi_i(\lambda)} = (\lambda - c)^{\sigma_i(c)} \frac{\widetilde{\varepsilon}_i(\lambda)}{\widetilde{\psi}_i(\lambda)}, \qquad \text{with } \widetilde{\varepsilon}_i(c) \neq 0,\ \widetilde{\psi}_i(c) \neq 0.$$

The structural indices of $G(\lambda)$ at c are $S(G,c) = (\sigma_1(c) \le \sigma_2(c) \le \cdots \le \sigma_r(c))$.

Definition

The structural indices of $G(\lambda)$ at $\lambda = \infty$ are the structural indices of $G(1/\lambda)$ at $\lambda = 0$.

Definition (Rosenbrock, 1970)

Let $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$ be a rational matrix. The polynomial matrix

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{C}[\lambda]^{(n+p)\times(n+m)}$$

is a **polynomial system matrix** of $G(\lambda)$ if

$$G(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda).$$

If, in addition, $\begin{bmatrix} A(\lambda) \\ -C(\lambda) \end{bmatrix}$ and $\begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix}$ do not have finite eigenvalues, then $P(\lambda)$ is a **minimal polynomial system matrix** of $G(\lambda)$.

Theorem (Rosenbrock, 1970)

Each rational matrix has infinitely many minimal polynomial system matrices.

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Minimal polynomial system matrices contain the whole finite structure

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1. *The nontrivial (those different from* 1*) invariant polynomials of* $P(\lambda)$ *are the nontrivial numerators of the Smith-McMillan form of* $G(\lambda)$*.*
2. *The nontrivial invariant polynomials of* $A(\lambda)$ *are the nontrivial denominators of the Smith-McMillan form of* $G(\lambda)$*.*

...in plain words

- The finite eigenvalue structure of $P(\lambda)$ (resp. $A(\lambda)$) (including all types of multiplicities, geometric, algebraic, partial...) coincides exactly with the finite zero (resp. pole) structure of $G(\lambda)$.
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Example: minimality is essential

- $G(\lambda) = \dfrac{\lambda^2 - 1}{\lambda + 2} \in \mathbb{C}(\lambda)^{1\times 1}$ has one finite pole at -2 and two finite zeros at $+1$ and -1.
- Minimal polynomial system matrix of $G(\lambda)$:

$$P(\lambda) = \left[\begin{array}{c|c} \lambda+2 & 1 \\ \hline -3 & \lambda-2 \end{array}\right],$$

since $G(\lambda) = (\lambda - 2) + 3\frac{1}{\lambda+2}$. Note that $\det P(\lambda) = \lambda^2 - 1$.
- Non-minimal polynomial system matrix of $G(\lambda)$ for any $a \in \mathbb{C}$:

$$\widehat{P}(\lambda) = \left[\begin{array}{cc|c} \lambda+a & 0 & 0 \\ 0 & \lambda+2 & 1 \\ \hline 0 & -3 & \lambda-2 \end{array}\right],$$

and since $\det \widehat{P}(\lambda) = (\lambda + a)(\lambda^2 - 1)$, $\widehat{P}(\lambda)$ has an spurious eigenvalue.
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If $P(\lambda)$ is a minimal polynomial system matrix in Σ of $G(\lambda)$, then

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Outline

1. Basics on rational matrices
2. **Linearizations of rational matrices: strong, in a set, at infinity**
3. Block minimal bases linearizations of rational matrices: strong, in a set, at infinity
4. The NLEIGS "linearizations" inside this framework
5. The "Automatic linearizations" inside this framework

Definition (Amparan, D., Marcaida, Zaballa, 2016)

A **linearization of $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$ is a matrix pencil**

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{C}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

such that:

(a) $L(\lambda)$ is a **minimal polynomial system matrix** of

$$\widehat{G}(\lambda) = (D_1\lambda + D_0) + (C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0)$$

(the second term is not present if $n = 0$), and

(b) there exist unimodular matrices $U_1(\lambda)$, $U_2(\lambda)$ such that

$$U_1(\lambda)\ \mathrm{diag}(G(\lambda), I_s)\, U_2(\lambda) = \widehat{G}(\lambda).$$

Remark: In order to guarantee that a pencil is a linearizarion of a rational matrix, it may be several ways to choose the block $A_1\lambda + A_0$. Even more, different selections may have different sizes.

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Definition (D., Marcaida, Quintana, Van Dooren, 2018)

A **linearization at infinity of** $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$ **is a matrix pencil**

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{C}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

such that:

(a) if $n > 0$, then A_1 is invertible, and

(b) if $\widehat{G}(\lambda) = (D_1\lambda + D_0) + (C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0)$, then there exist rational matrices invertible at $\lambda = 0$, $R_1(\lambda)$, $R_2(\lambda)$ (that is, $R_i(\lambda)$ does not have poles at $\lambda = 0$ and $\det R_i(0) \neq 0$) such that

$$R_1(\lambda)\ \mathrm{diag}(\mathrm{rev}\, G(\lambda), I_s)\, R_2(\lambda) = \mathrm{rev}\, \widehat{G}(\lambda).$$

Strong linearization of a rational matrix

Definition (Amparan, D., Marcaida, Zaballa, 2016)

A **strong linearization of $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$ is a matrix pencil** $L(\lambda)$ such that

1. $L(\lambda)$ is a linearization of $G(\lambda)$, and
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Remark

If $G(\lambda)$ is a polynomial matrix, then linearizations and strong linearizations of $G(\lambda)$ according to the definitions above are standard linearizations and strong linearizations of the polynomial matrix $G(\lambda)$.

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A **linearization of $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$ in $\Sigma \subseteq \mathbb{C}$ is a matrix pencil**

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$$W_1(\lambda)\ \mathrm{diag}(G(\lambda), I_s)\, W_2(\lambda) = \widehat{G}(\lambda).$$

Remark: If $\Sigma = \mathbb{C}$, then a linearization in $\mathbb{C}$ is just a linearization as defined above.

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Linearizations (in Σ) contain the whole finite structure (in Σ)

Theorem (Amparan, D., Marcaida, Zaballa, 2016, D., Marcaida, Quintana, Van Dooren, 2018)

If

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is a linearization of $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$ (resp. a linearization in $\Sigma \subseteq \mathbb{C}$), then:

- *The finite eigenvalue structure (resp. in Σ) of $L(\lambda)$ (including all types of multiplicities, geometric, algebraic, partial...) coincides exactly with the finite zero structure (resp. in Σ) of $G(\lambda)$.*
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Linearizations at infinity contain the whole structure at infinity

Theorem (Amparan, D., Marcaida, Zaballa, 2016)

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is a **linearization at infinity of** $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$*, then the structural indices at infinity of* $G(\lambda)$ *can be obtained from the rank*(G) *largest partial multiplicities at infinity of* $L(\lambda)$ *through a constant shift.*

More precisely, if $e_1, \ldots, e_r$ *are these rank*(G) *partial multiplicities of* $L(\lambda)$ *and* $\operatorname{rev} G(\lambda) = \lambda^d G(1/\lambda)$*, then the structural indices at infinity of* $G(\lambda)$ *are*

1. $e_1 - d, \ldots, e_r - d$, *if* $D_1 + C_1 A_1^{-1} B_1 \neq 0$,
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Corollary

Strong linearizations of a rational matrix $G(\lambda)$ contain the whole finite and infinite zero and pole structures of $G(\lambda)$.

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5 The “Automatic linearizations” inside this framework

Definition (D., Lawrence, Pérez, Van Dooren, Numer. Math, to appear)

A matrix pencil

$$\mathcal{L}(\lambda) = \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}$$

is a block minimal bases pencil (BMBP) if $K_1(\lambda)$ and $K_2(\lambda)$ are minimal bases. If, in addition, the row degrees of $K_1(\lambda)$ and $K_2(\lambda)$ are all one, and the row degrees of each of their dual minimal bases $N_1(\lambda)$ and $N_2(\lambda)$ are all equal, then $\mathcal{L}(\lambda)$ is a strong block minimal bases pencil (SBMBP).

Theorem (D., Lawrence, Pérez, Van Dooren, Numer. Math, to appear)

With the notation of the previous definition, if $\mathcal{L}(\lambda)$ is a BMBP (resp. SBMBP), then it is a linearization (resp. strong linearization) of the matrix polynomial

$$Q(\lambda) = N_2(\lambda) M(\lambda) N_1(\lambda)^T.$$

Remark: Frobenius companion linearizations, colleague linearizations in Chebyshev bases and others, Fiedler, Generalized Fiedler lins., etc, are all SBMBP, and their properties can be explained with just one simple theory.

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Remark: Frobenius companion linearizations, colleague linearizations in Chebyshev bases and others, Fiedler, Generalized Fiedler lins., etc, are all SBMBP, and their properties can be explained with just one simple theory.

Theorem (Amparan, D, Marcaida, Zaballa, 2016; Quintana, V. Dooren, 18)

Let

$$\begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}$$

be a SBMBP and $N_1(\lambda), N_2(\lambda)$ be minimal bases dual to $K_1(\lambda), K_2(\lambda)$. Consider for $i = 1, 2$ unimodular matrices

$$U_i(\lambda) = \begin{bmatrix} K_i(\lambda) \\ \widehat{K}_i \end{bmatrix} \quad \textit{and} \quad U_i(\lambda)^{-1} = \begin{bmatrix} \widehat{N}_i(\lambda)^T & N_i(\lambda)^T \end{bmatrix}$$

and a **linear minimal polynomial system matrix**

$$L(\lambda) = \left[\begin{array}{c|cc} (\lambda I_n - A) & B(\lambda)\widehat{K}_1 & 0 \\ \hline -\widehat{K}_2^T C(\lambda) & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array}\right].$$

Then $L(\lambda)$ is a strong linearization of the rational matrix

$$G(\lambda) = N_2(\lambda) M(\lambda) N_1(\lambda)^T + C(\lambda)(\lambda I_n - A)^{-1} B(\lambda) .$$

Example 1 of RSBMBL. Strong linearization based on Frobenius companion linearization for polynomials of Su & Bai (2011)

- Given rational matrix:

$$G(\lambda) = D_d\lambda^d + \cdots + D_1\lambda + D_0 + C(\lambda I_n - A)^{-1}B \in \mathbb{C}(\lambda)^{p\times m}.$$

- Strong linearization (originally introduced by Su & Bai (SIMAX, 2011) without minimal order requirement and without strong nature):

$$L(\lambda) = \left[\begin{array}{c|ccccc} \lambda I_n - A & 0 & 0 & \cdots & 0 & B \\ \hline -C & \lambda D_d + D_{d-1} & D_{d-2} & \cdots & D_1 & D_0 \\ 0 & -I_m & \lambda I_m & & & \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & \ddots & \lambda I_m & \\ 0 & & & & -I_m & \lambda I_m \end{array}\right]$$

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- Given rational matrix:

$$G(\lambda) = D_d\, U_d(\lambda) + \cdots + D_1\, U_1(\lambda) + D_0 + C(\lambda I_n - A)^{-1} B \ \in \mathbb{C}(\lambda)^{p\times m},$$

with polynomial part expressed in Chebyshev basis of the second kind.

- Strong linearization:

$$L(\lambda) = \left[\begin{array}{c|ccccc} \lambda I_n - A & 0 & 0 & 0 & \cdots & B \\ \hline -C & 2\lambda D_d + D_{d-1} & D_{d-2} - D_d & D_{d-3} & \cdots & D_0 \\ 0 & -I_m & 2\lambda I_m & -I_m & & \\ \vdots & & \ddots & \ddots & \ddots & \\ \vdots & & & -I_m & 2\lambda I_m & -I_m \\ 0 & & & & -I_m & 2\lambda I_m \end{array}\right]$$

Example 2 of RSBMBL. Strong linearization based on Chebyshev colleague linearization for polynomials (Amparan, D., et al, 2016)

- Given rational matrix:

$$G(\lambda) = D_d\,U_d(\lambda) + \cdots + D_1\,U_1(\lambda) + D_0 + C(\lambda I_n - A)^{-1}B \;\in \mathbb{C}(\lambda)^{p\times m},$$

with polynomial part expressed in Chebyshev basis of the second kind.

- Strong linearization:

$$L(\lambda) = \left[\begin{array}{c|ccccc} \lambda I_n - A & 0 & 0 & 0 & \cdots & B \\ \hline -C & 2\lambda D_d + D_{d-1} & D_{d-2} - D_d & D_{d-3} & \cdots & D_0 \\ 0 & -I_m & 2\lambda I_m & -I_m & & \\ \vdots & & \ddots & \ddots & \ddots & \\ \vdots & & & -I_m & 2\lambda I_m & -I_m \\ 0 & & & & -I_m & 2\lambda I_m \end{array}\right]$$

- Given rational matrix:

$$G(\lambda) = \lambda^5 D_5 + \lambda^4 D_4 + \lambda^3 D_3 + \lambda^2 D_2 + \lambda D_1 + D_0 + C(\lambda I_n - A)^{-1} B \in \mathbb{F}(\lambda)^{p\times m}$$

- Strong linearization:

$$L(\lambda) = \left[\begin{array}{c|ccc|cc} \lambda I_n - A & 0 & 0 & B & 0 & 0 \\ \hline 0 & \lambda P_5 + P_4 & 0 & 0 & -I_p & 0 \\ 0 & 0 & \lambda P_3 + P_2 & 0 & \lambda I_p & -I_p \\ -C & 0 & 0 & \lambda P_1 + P_0 & 0 & \lambda I_p \\ \hline 0 & -I_m & \lambda I_m & 0 & 0 & 0 \\ 0 & 0 & -I_m & \lambda I_m & 0 & 0 \end{array}\right]$$

Example 3 of RSBMBL. Strong linearization based on another block Kronecker pencil (Amparan, D., Marcaida, Zaballa, 2016)

- Given rational matrix:

$$G(\lambda) = \lambda^5 D_5 + \lambda^4 D_4 + \lambda^3 D_3 + \lambda^2 D_2 + \lambda D_1 + D_0 + C(\lambda I_n - A)^{-1} B \in \mathbb{F}(\lambda)^{p\times m}$$

- Strong linearization:

$$L(\lambda) = \left[\begin{array}{c|ccc|cc} \lambda I_n - A & 0 & 0 & B & 0 & 0 \\ \hline 0 & \lambda P_5 + P_4 & 0 & 0 & -I_p & 0 \\ 0 & 0 & \lambda P_3 + P_2 & 0 & \lambda I_p & -I_p \\ -C & 0 & 0 & \lambda P_1 + P_0 & 0 & \lambda I_p \\ \hline 0 & -I_m & \lambda I_m & 0 & 0 & 0 \\ 0 & 0 & -I_m & \lambda I_m & 0 & 0 \end{array}\right]$$

Theorem (D., Marcaida, Quintana, Van Dooren, 2018)

Let

$$\begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}$$

be a BMBP and $N_1(\lambda)$, $N_2(\lambda)$ be minimal bases dual to $K_1(\lambda)$, $K_2(\lambda)$. Consider a linear polynomial system matrix

$$L(\lambda) = \left[\begin{array}{c|cc} A(\lambda) & B(\lambda) & 0 \\ \hline -C(\lambda) & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array}\right]. \qquad (1)$$

Let $\Omega := \{\lambda_0 \in \mathbb{C} : A(\lambda_0)$ is invertible$\}$. **Then $L(\lambda)$ is a linearization of the rational matrix**

$$H(\lambda) = N_2(\lambda)[M(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)]N_1(\lambda)^T$$

in Ω.

Theorem (D., Marcaida, Quintana, Van Dooren, 2018)

Moreover, if

- $\begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}$ *is a SBMBP, and*
- $A(\lambda) = A_1\lambda + A_0$ *with* A_1 *nonsingular,*

then $L(\lambda)$ is also a linearization of

$$H(\lambda) = N_2(\lambda)[M(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)]N_1(\lambda)^T$$

at infinity.

Outline

1. Basics on rational matrices
2. Linearizations of rational matrices: strong, in a set, at infinity
3. Block minimal bases linearizations of rational matrices: strong, in a set, at infinity
4. **The NLEIGS "linearizations" inside this framework**
5. The "Automatic linearizations" inside this framework

For the rational matrix

$$Q_N(\lambda) = b_0(\lambda)D_0 + b_1(\lambda)D_1 + \cdots + b_N(\lambda)D_N,$$

with $D_j \in \mathbb{C}^{m\times m}$ and

$$b_j(\lambda) = \frac{1}{\beta_0}\prod_{k=1}^{j} \frac{\lambda - \sigma_{k-1}}{\beta_k(1-\lambda/\xi_k)} =: \frac{n_j(\lambda)}{d_j(\lambda)}, \quad j = 0, 1, \ldots, N,$$

a sequence of rational scalar functions, with the poles ξ_i all distinct from the nodes σ_j, and some poles ξ_i can be infinite,

Güttel, Van Beeumen, Meerbergen, Michiels (2014) construct the following pencil associated to $Q_N(\lambda)$

$$L_N(\lambda) = \begin{bmatrix} \left(1-\frac{\lambda}{\xi_N}\right)D_0 & \left(1-\frac{\lambda}{\xi_N}\right)D_1 & \cdots & \left(1-\frac{\lambda}{\xi_N}\right)D_{N-2} & \left(1-\frac{\lambda}{\xi_N}\right)D_{N-1} + \frac{\lambda-\sigma_{N-1}}{\beta_N}D_N \\ (\sigma_0-\lambda)I_m & \beta_1(1-\frac{\lambda}{\xi_1})I_m & & & \\ & & \ddots & \ddots & \\ & & & (\sigma_{N-2}-\lambda)I_m & \beta_{N-1}(1-\frac{\lambda}{\xi_{N-1}})I_m \end{bmatrix}$$

For the rational matrix

$$Q_N(\lambda) = b_0(\lambda)D_0 + b_1(\lambda)D_1 + \cdots + b_N(\lambda)D_N,$$

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Theorem (D., Marcaida, Quintana, Van Dooren, 2018)

$L_N(\lambda)$ is a SBMBP and, so, a strong linearization, of the polynomial matrix

$$d_N(\lambda)\, Q_N(\lambda)\,.$$

Theorem (D., Marcaida, Quintana, Van Dooren, 2018)

- *$L_N(\lambda)$ is a linearization of the rational matrix*

  $$\left(1 - \frac{\lambda}{\xi_N}\right) Q_N(\lambda)$$

  in the set $\Omega := \mathbb{C} \setminus \Theta$, where Θ is the set of finite poles in $\{\xi_1, \ldots, \xi_{N-1}\}$. In fact, $L_N(\lambda)$ is a RBMBP in Ω.

- *Moreover, if all the poles in $\{\xi_1, \ldots, \xi_{N-1}\}$ are finite, then $L_N(\lambda)$ is a linearization of $\left(1 - \frac{\lambda}{\xi_N}\right) Q_N(\lambda)$ at infinity. In fact, $L_N(\lambda)$ is a RSBMBP in Ω.*

NLEIGS "linearizations" without low-rank structure (II)

Theorem (D., Marcaida, Quintana, Van Dooren, 2018)

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- *Moreover, if all the poles in $\{\xi_1, \ldots, \xi_{N-1}\}$ are finite, then $L_N(\lambda)$ is a linearization of $\left(1 - \frac{\lambda}{\xi_N}\right) Q_N(\lambda)$ at infinity. In fact, $L_N(\lambda)$ is a RSBMBP in Ω.*

- To get the second theorem in the previous slide, the invertible $A(\lambda)$ has to be placed in the $(2,2)$ position and partition $L_N(\lambda)$ as

$$L_N(\lambda) = \left[\begin{array}{c|ccccc} \left(1-\frac{\lambda}{\xi_N}\right) D_0 & \left(1-\frac{\lambda}{\xi_N}\right) D_1 & \dots & \left(1-\frac{\lambda}{\xi_N}\right) D_{N-2} & \left(1-\frac{\lambda}{\xi_N}\right) D_{N-1} + \frac{\lambda-\sigma_{N-1}}{\beta_N} D_N \\ \hline (\sigma_0-\lambda)I_m & \beta_1(1-\frac{\lambda}{\xi_1})I_m & & & \\ & & \ddots & \ddots & \\ & & & (\sigma_{N-2}-\lambda)I_m & \beta_{N-1}(1-\frac{\lambda}{\xi_{N-1}})I_m \end{array}\right]$$

$$= \left[\begin{array}{c|c} M(\lambda) & -C(\lambda) \\ \hline B(\lambda) & A(\lambda) \end{array}\right].$$

Is this what we want?

Theorem (D., Marcaida, Quintana, Van Dooren, 2018)

$L_N(\lambda)$ is a strong linearization of the polynomial matrix $d_N(\lambda)\, Q_N(\lambda)$.

- This result (and others that we will see) guarantees that $L_N(\lambda)$ has the information that the authors of NLEIGS need, but
- $L_N(\lambda)$ may contain also non-desired information, because $d_N(\lambda)\, Q_N(\lambda)$ has eigenvalues in (some) of the poles $\xi_1, \xi_2, \ldots, \xi_N$ except in the case that all the denominators in the Smith-McMillan form of $Q_N(\lambda)$ are equal to $d_N(\lambda)$.
- Fortunately, for $Q_N(\lambda)$, this happens "almost always".

Theorem (D., Marcaida, Quintana, Van Dooren, 2018)

$Q_N(\lambda)$ is regular and the denominators of its Smith-McMillan form are all equal to $d_N(\lambda)$ if and only if the matrices $C_k(\xi_k)$ are nonsingular for every finite pole ξ_k in $\{\xi_1, \ldots, \xi_N\}$, where

$$C_N(\lambda) = D_N \text{ and } \; C_k(\lambda) = \prod_{j=k+1}^{N} \left(1 - \frac{\lambda}{\xi_i}\right) D_k + \frac{\lambda - \sigma_k}{\beta_{k+1}} C_{k+1}(\lambda) \text{ for } k = N-1, \ldots, 1.$$

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For the rational matrix

$$\widetilde{Q}_N(\lambda) = \sum_{i=0}^{p} (\widetilde{B}_i + \widetilde{C}_i) b_i(\lambda) + \sum_{i=p+1}^{N} \widetilde{C}_i b_i(\lambda),$$

with the same $b_i(\lambda)$ as before, and

$$\widetilde{B}_i = \sum_{j=0}^{p} \beta_{ij} B_j \in \mathbb{C}^{m\times m}, \quad \widetilde{C}_i = \sum_{j=i}^{n} \gamma_{ij} L_j U_j^* \in \mathbb{C}^{m\times m},$$

Güttel, Van Beeumen, Meerbergen, Michiels (2014) define low-rank matrices

$$\widetilde{L}_i = [\gamma_{i1} L_1 \quad \gamma_{i2} L_2 \quad \cdots \quad \gamma_{in} L_n] \quad \text{and} \quad \widetilde{U} = [U_1 \quad U_2 \quad \cdots \quad U_n]$$

and the pencil

$$\widetilde{\mathcal{L}}_N(\lambda) = \left[\begin{array}{c|c} W(\lambda) & V(\lambda) \\ \hline U(\lambda) & T(\lambda) \end{array}\right],$$

where

For the rational matrix

$$\widetilde{Q}_N(\lambda) = \sum_{i=0}^{p}(\widetilde{B}_i + \widetilde{C}_i)b_i(\lambda) + \sum_{i=p+1}^{N} \widetilde{C}_i b_i(\lambda),$$

with the same $b_i(\lambda)$ as before, and

$$\widetilde{B}_i = \sum_{j=0}^{p} \beta_{ij} B_j \in \mathbb{C}^{m\times m}, \quad \widetilde{C}_i = \sum_{j=i}^{n} \gamma_{ij} L_j U_j^* \in \mathbb{C}^{m\times m},$$

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and the pencil

$$\widetilde{\mathcal{L}}_N(\lambda) = \left[\begin{array}{c|c} W(\lambda) & V(\lambda) \\ \hline U(\lambda) & T(\lambda) \end{array}\right],$$

where

- $W(\lambda) =$
$$\begin{bmatrix} \left(1-\frac{\lambda}{\xi_N}\right)(\widetilde{B}_0+\widetilde{C}_0) & \left(1-\frac{\lambda}{\xi_N}\right)(\widetilde{B}_1+\widetilde{C}_1) & \cdots & \left(1-\frac{\lambda}{\xi_N}\right)(\widetilde{B}_p+\widetilde{C}_p) \\ (\sigma_0-\lambda)I_m & \beta_1\left(1-\frac{\lambda}{\xi_1}\right)I_m & & \\ & \ddots & \ddots & \\ & & (\sigma_{p-1}-\lambda)I_m & \beta_p\left(1-\frac{\lambda}{\xi_p}\right)I_m \end{bmatrix},$$
- $V(\lambda) = e_1 \otimes \left[\ \left(1-\frac{\lambda}{\xi_N}\right)\widetilde{L}_{p+1} \quad \left(1-\frac{\lambda}{\xi_N}\right)\widetilde{L}_{p+2} \quad \cdots \quad \left(1-\frac{\lambda}{\xi_N}\right)\widetilde{L}_{N-1} + \frac{\lambda-\sigma_{N-1}}{\beta_N}\widetilde{L}_N\ \right],$
  with $e_1 \in \mathbb{C}^{p+1}$,
- $U(\lambda) = \begin{bmatrix} 0 & (\sigma_p-\lambda)\widetilde{U}^* \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{(N-p-1)r\times(p+1)n},$
- $T(\lambda) =$
$$\begin{bmatrix} \beta_{p+1}\left(1-\frac{\lambda}{\xi_{p+1}}\right)I_r & & & \\ (\sigma_{p+1}-\lambda)I_r & \beta_{p+2}\left(1-\frac{\lambda}{\xi_{p+2}}\right)I_r & & \\ & \ddots & \ddots & \\ & & (\sigma_{N-2}-\lambda)I_r & \beta_{N-1}\left(1-\frac{\lambda}{\xi_{N-1}}\right)I_r \end{bmatrix}.$$

Theorem (D., Marcaida, Quintana, Van Dooren, 2018)

- *$\widetilde{\mathcal{L}}_N(\lambda)$ is a linearization of the rational matrix*

$$d_p(\lambda)\left(1-\frac{\lambda}{\xi_N}\right)\widetilde{Q}_N(\lambda)$$

in the set $\Omega_p := \mathbb{C} \setminus \Theta_p$, where Θ_p is the set of finite poles in $\{\xi_{p+1},\ldots,\xi_{N-1}\}$. In fact, $\widetilde{\mathcal{L}}_N(\lambda)$ is a RBMBP in Ω_p.

- *Moreover, if all the poles in $\{\xi_{p+1},\ldots,\xi_{N-1}\}$ are finite, then $\widetilde{\mathcal{L}}_N(\lambda)$ is a linearization of $d_p(\lambda)\left(1-\frac{\lambda}{\xi_N}\right)\widetilde{Q}_N(\lambda)$ at infinity. In fact, $\widetilde{\mathcal{L}}_N(\lambda)$ is a RSBMBP in Ω_p.*

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For the rational matrix

$$R(\lambda) = \sum_{i=0}^{k-1} (A_i - \lambda B_i) f_i(\lambda) + \sum_{i=1}^{s} (C_i - \lambda D_i) a_i^T (E_i - \lambda F_i)^{-1} b_i,$$

where $f_i(\lambda)$ are scalar polynomial or rational functions satisfying a linear relation ($f_0(\lambda) = 1$)

$$(M - \lambda N) f(\lambda) = 0, \quad f(\lambda) := [f_0(\lambda), f_1(\lambda), \ldots, f_{k-1}(\lambda)]^T \neq 0,$$

with $M - \lambda N$ of size $(k-1) \times k$ and rank$(M - \lambda N) = k - 1$ for all $\lambda \in \mathbb{C}$, $a_i, b_i \in \mathbb{C}^{l_i}$ are vectors, $A_i, B_i, C_i, D_i \in \mathbb{C}^{m \times m}$ matrices, and $l_i \times l_i$ matrices

$$E_i = \begin{bmatrix} w_1 & w_2 & \cdots & w_{l_i-1} & w_{l_i} \\ -z_1 & z_2 & & & \\ & -z_2 & \ddots & & \\ & & \ddots & z_{l_i-1} & \\ & & & -z_{l_i-1} & z_{l_i} \end{bmatrix} \text{ and } F_i = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & & & \\ & 1 & \ddots & & \\ & & \ddots & -1 & \\ & & & 1 & -1 \end{bmatrix},$$

Lietaert, Pérez, Vandereycken, Meerbergen (2018) define the pencil

$$\mathcal{L}_R(\lambda) = \left[\begin{array}{ccc|ccc} A_0 - \lambda B_0 & \cdots & A_{k-1} - \lambda B_{k-1} & a_1^T \otimes (C_1 - \lambda D_1) & \cdots & a_s^T \otimes (C_s - \lambda D_s) \\ \hline \multicolumn{3}{c|}{(M - \lambda N) \otimes I_n} & \multicolumn{3}{c}{0} \\ \hline \multicolumn{2}{c}{-b \otimes I_n} & 0 & \multicolumn{3}{c}{(E - \lambda F) \otimes I_n} \end{array}\right],$$

where $b := [b_1^T \quad \cdots \quad b_s^T]^T$ and $E - \lambda F := \operatorname{diag}(E_1 - \lambda F_1, \ldots, E_s - \lambda F_s)$.

Theorem (D., Marcaida, Quintana, Van Dooren, 2018)

Let $\Omega := \{\lambda_0 \in \mathbb{C} : E - \lambda_0 F$ is invertible$\}$.

- *$\mathcal{L}_R(\lambda)$ is a linearization of $d_f(\lambda) R(\lambda)$ in Ω, where $d_f(\lambda)$ is the least common denominator of $f_0(\lambda), f_1(\lambda), \ldots, f_{k-1}(\lambda)$.*
- *If $M - \lambda N$ is a minimal basis, then $\mathcal{L}_R(\lambda)$ is a RBMBP in Ω.*
- *$\mathcal{L}_R(\lambda)$ is NOT a linearization at infinity of $d_f(\lambda) R(\lambda)$.*

Remark

Analogous results hold for the “automatic low-rank exploiting linearizations”.

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