Polynomial, rational and nonlinear eigenvalue problems and their solution via linearizations

Froilán M. Dopico

Departamento de Matemáticas
Universidad Carlos III de Madrid, Spain

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Collaborators

- University of California at Santa Barbara (USA): María I. Bueno.
- Universidad Carlos III de Madrid (Spain): Fernando De Terán, María C. Quintana.
- Université Catholique de Louvain (Belgium): Piers Lawrence, Paul Van Dooren.
- Universidad Católica del Norte (Chile): Javier González-Pizarro.
- University of Montana (USA): Javier Pérez.
- Universidad del País Vasco/Euskal Herriko Unibertsitatea: Agurtzane Amparan, Silvia Marcaida, Ion Zaballa.
- Universidade do Porto (Portugal): Susana Furtado.
- Western Michigan University (USA): Steve Mackey.

The basic eigenvalue problem (BEP). Given $A \in \mathbb{C}^{n\times n}$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that

$$Av = \lambda v \iff \boxed{(\lambda I_n - A)\, v = 0}$$

- It arises in many applications. For instance, if one looks for solutions of the form $y(t) = e^{\lambda t} v$ in the system of first order ODEs

$$\frac{dy(t)}{dt} = Ay(t) \Longrightarrow \lambda v = Av$$

- There are stable algorithms for its numerical solution.
- QR algorithm (Francis-Kublanovskaya 1961) for small to medium size dense matrices.
- Arnoldi method (1951) and (many) other variants of Krylov methods for large-scale problems and sparse matrices.
- Easy to use software. For instance MATLAB's commands `eig(A)` or `eigs(A)`.

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under the regularity assumption $\det(zB - A)$ is not zero for all $z \in \mathbb{C}$.

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$$(P_d\lambda^d + \cdots + P_1\lambda + P_0)v = 0,$$

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- Easy to use software for small to medium size dense matrices: MATLAB's commands `polyeig(P0,P1,...,Pd)` (Van Dooren, 1979).
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The RATIONAL eigenvalue problem (REP). Given a rational matrix $G(z) \in \mathbb{C}(z)^{n\times n}$, i.e.,

$$\begin{array}{rccc} G: & \mathbb{C} & \to & \mathbb{C}^{n\times n} \\ & z & \mapsto & G(z), \end{array}$$

such that $G(z)_{ij}$ is a scalar rational function of $z \in \mathbb{C}$, for $1 \leq i, j \leq n$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that λ is not a pole of any $G(z)_{ij}$ and

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- It arises in applications either directly (multivariable system theory and control theory) or as an approximation.
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The NONLINEAR eigenvalue problem (NEP). Given a non-empty open set $\Omega \subseteq \mathbb{C}$ and a holomorphic matrix-valued function

$$\begin{array}{rccc} F: & \Omega & \rightarrow & \mathbb{C}^{n\times n} \\ & z & \mapsto & F(z), \end{array}$$

compute scalars $\lambda \in \Omega$ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that

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The NONLINEAR eigenvalue problem (NEP). Given a non-empty open set $\Omega \subseteq \mathbb{C}$ and a holomorphic matrix-valued function

$$\begin{array}{rccl} F: & \Omega & \rightarrow & \mathbb{C}^{n\times n} \\ & z & \mapsto & F(z), \end{array}$$

compute scalars $\lambda \in \Omega$ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that

$$\boxed{F(\lambda)v = 0},$$

under the regularity assumption $\det(F(z)) \not\equiv 0$.

- It arises in applications. For instance, if one looks for solutions $y(t) = e^{\lambda t} v$ in the system of first order DELAYED differential equations

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- There are different algorithms for the numerical solution of NEP.
- One of the most important family of algorithms is based on the following two step strategy
  1. Approximate $F(z)$ by a rational matrix $G(z)$ with poles outside Ω.
  2. Solve the REP associated to $G(z)$.
- There is software available for NEPs developed by the authors of some key papers that follow the previous strategy:
  1. NLEIGS (Güttel, Van Beeumen, Meerbergen, Michiels, 2014) (not easy to use),
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- **1st KEY IDEA: ALL THESE PROBLEMS CAN BE SOLVED BY TRANSFORMING THE PROBLEM INTO A GEP $\rightarrow$ LINEARIZATION.**
- For **PEPs** and **REPs**, this transformation is **exact!!!!!**.
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Outline

1. The “flavor” of applied PEPs, REPs, NEPs: examples
2. Additional “difficulties” of GEPs, PEPs, and REPs over BEPs
3. Linearizations and numerical solution of PEPs
4. Linearizations of REPs
5. Local and Strong Linearizations of REPs and their numerical solution
6. Global backward stability of PEPs solved with linearizations
7. Global backward stability of REPs solved with linearizations
8. Conclusions

Outline

- Every matrix $F(z)$ defining an $n \times n$ PEP, REP or NEP can be written in **"split form"** with at most n^2 terms, i.e.,

$$F(z) = f_1(z)\, C_1 + f_2(z)\, C_2 + \cdots + f_\ell(z)\, C_\ell,$$

where $f_i : \mathbb{C} \to \mathbb{C}$, $C_i \in \mathbb{C}^{n \times n}$, and $\ell \leq n^2$.

- This result is, of course, a triviality,

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How large is the degree of $P(z) = P_d z^d + \cdots + P_1 z + P_0$ in practical PEPs?

- In most direct applications coming from vibrational problems in mechanics $\mathbf{d = 2}$: **the quadratic eigenvalue problem (QEP)**

$$(z^2 M + zC + K)v = 0,$$

while $M, C, K \in \mathbb{C}^{n \times n}$ with $\mathbf{n = 10^2, 10^3, 10^4, 10^5, 10^6, \ldots}$.

- Betcke, Higham, Mehrmann, Schröder, Tisseur, *"NLEVP: A Collection of Nonlinear Eigenvalue Problems"*, (2013) reports on applications with
  - $\mathbf{d = 4}$: Hamiltonian control problems, homography-based method for calibrating a central cadioptric vision system, spatial stability analysis of the Orr-Sommerfeld equation, and finite element solution of the equation for the modes of a planar waveguide using piecewise linear basis functions.
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- Loaded elastic string (Betcke et al., NLEVP-collection, (2013)):

$$G(\lambda) = A - \lambda B + \frac{\lambda}{\lambda - \sigma} E = (A + E) - \lambda B + \frac{\sigma}{\lambda - \sigma} E,$$

which almost shows the polynomial and the strictly proper parts of $G(\lambda)$. Only 3 functions (terms) in split form, $A, B \in \mathbb{R}^{n \times n}$ symmetric tridiagonal matrices, E only one nonzero entry in (n, n) position. $n \geq 10^2$ large.

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$$G(\lambda) = \lambda^2 M + K - \sum_{i=1}^{k} \frac{1}{1 + b_i \lambda} \Delta G_i,$$

which shows the polynomial and the strictly proper parts of $G(\lambda)$. Only $k + 2$ functions in split form, M, K positive definite, $n = 10704$ large.

- NLEIGS-REPs coming from linear rational interpolation of NEPs (Güttel, Van Beeumen, Meerbergen, Michiels (2014)):

$$Q_N(\lambda) = b_0(\lambda)D_0 + b_1(\lambda)D_1 + \cdots + b_N(\lambda)D_N,$$

with $D_j \in \mathbb{C}^{n\times n}$,

$$b_0(\lambda) = \frac{1}{\beta_0}, \qquad b_j(\lambda) = \frac{1}{\beta_0}\prod_{k=1}^{j} \frac{\lambda - \sigma_{k-1}}{\beta_k(1 - \lambda/\xi_k)},$$

$j = 1, \ldots, N$, rational scalar functions, with the “poles” ξ_i different from zero and all distinct from the nodes σ_j. $N \leq 140$, $n = 16281$.

"Approximating" REPs have been used to approximate...

among many others, **the following NEPs**:

- **The radio-frequency gun cavity problem**:

$$\left[(K - \lambda M) + i\sqrt{\lambda - \sigma_1^2}\, W_1 + i\sqrt{\lambda - \sigma_2^2}\, W_2\right] v = 0,$$

  where M, K, W_1, W_2 are real sparse symmetric 9956×9956 matrices (only 4 scalar functions involved in split form).

- **Bound states in semiconductor devices problems**:

$$\left[(H - \lambda I) + \sum_{j=0}^{80} e^{i\sqrt{\lambda - \alpha_j}}\, S_j\right] v = 0,$$

  where $H, S_j \in \mathbb{R}^{16281 \times 16281}$, H symmetric and the matrices S_j have low rank (only 83 scalar functions involved in split form).

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For the rest of the talk...

- since NEPs can be approximated by REPs, **I will focus on PEPs and REPs**.
- Though PEPs are mathematically a particular case of REPs,
- PEPs and REPs have been always considered separately from the point of view of numerical algorithms,
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Outline

1. **BEP**: $(\lambda I_n - A)\, v = 0$
2. **GEP**: $(\lambda B - A)\, v = 0$
3. **PEP**: $(P_d \lambda^d + \cdots + P_1 \lambda + P_0) v = 0$
4. **REP**: $G(\lambda) v = 0$

- So far, we have only considered **finite eigenvalues**, but
- regular **GEPs, PEPs, REPs** may have also **infinite eigenvalues**.
- **GEPs, PEPs, REPs** may be **singular** (BEPs are always regular) and to have, in addition to eigenvalues, **minimal indices**.
- **REPs** have **poles**. In modern applications, the poles are usually known (even chosen in approximating NEPs by REPs), but in other applications (Control), poles are not known and must be computed.
- We illustrate informally some of these concepts on matrix polynomials...

GEPs-PEPs-REPs have more spectral "structural" data than BEPs

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Infinite eigenvalues in PEPs

- Given the **regular ($\det P(\lambda) \not\equiv 0$) polynomial matrix**

$$P(\lambda) = P_d \lambda^d + \cdots + P_1 \lambda + P_0 , \qquad P_i \in \mathbb{C}^{n\times n},$$

then the finite eigenvalues of the PEP $P(\lambda_0)\, v = 0, \quad 0 \neq v \in \mathbb{C}^n$

are the roots of the scalar polynomial $\det P(\lambda)$.

- Thus, $P(\lambda)$ has **at most dn finite eigenvalues** since

$$\det P(\lambda) = (\det P_d)\, \lambda^{dn} + \text{lower degree terms in } \lambda \,.$$

- If $\det P_d = 0$, then the number of finite eigenvalues of the PEP is degree $(\det P(\lambda))$ and it is said that
- the PEP has dn – degree $(\det P(\lambda))$ **infinite eigenvalues**.

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- Another way to define infinite eigenvalues of a PEP that can be generalized to non-regular or **singular polynomial matrices** is through **the reversal polynomial**.
- Given $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0$, its reversal is

$$\text{rev}P(\lambda) := \lambda^d P(\tfrac{1}{\lambda}) = P_0\lambda^d + \cdots + P_{d-1}\lambda + P_d \, .$$

- Then the **infinite eigenvalues** of $P(\lambda)$ correspond to the **zero eigenvalues** of rev$P(\lambda)$.
- Why the name **infinite eigenvalues**? A possible reason is that if a polynomial with infinite eigenvalues, i.e., with P_d singular, is perturbed a bit, then eigenvalues with very large absolute values often appears.

The reversal polynomial and more on infinite eigenvalues

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Example 1

Let ϵ be a small parameter and consider the quadratic matrix polynomial

$$P(\lambda) = \begin{bmatrix} (\lambda-1)(\lambda-2) & 0 \\ 0 & \lambda(\epsilon\lambda - 1) \end{bmatrix}$$
$$= \lambda^2 \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix} + \lambda \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

- If $\epsilon \neq 0$, then the eigenvalues are $\{1, 2, 0, 1/\epsilon\}$, (very large if $|\epsilon| \ll 1$).
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- An additional high step of difficulty is that PEPs can be **singular**, which happens when

$$P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0$$

is either **rectangular or square with** $\det P(\lambda) \equiv 0$, i.e., zero for all λ.

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- In addition to eigenvalues, **singular matrix polynomials have** other "interesting numbers" attached to them called **minimal indices**.
- Recall that eigenvalues are related to the existence of nontrivial **null spaces**. For instance, $\mathcal{N}_r(\lambda_0 I_n - A) \neq \{0\}$ in BEPs.
- **Minimal indices** are related to the fact that a singular $m \times n$ matrix polynomial $P(\lambda)$ has non-trivial left and/or right null-spaces over the field $\mathbb{C}(\lambda)$ of rational functions:

$$\begin{aligned} \mathcal{N}_\ell(P) &:= \left\{ y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times m} \,:\, y(\lambda)^T P(\lambda) \equiv 0^T \right\}, \\ \mathcal{N}_r(P) &:= \left\{ x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} \,:\, P(\lambda) x(\lambda) \equiv 0 \right\}, \end{aligned}$$

- which have bases consisting entirely of vector polynomials.
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Example III: right minimal bases and right minimal indices

$$P(\lambda) = \begin{bmatrix} \lambda & -\lambda^4 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1 & -\lambda \end{bmatrix} \in \mathbb{C}[\lambda]^{3\times 5}$$

$$\mathcal{N}_r(P) = \text{Span}\{ \underbrace{\begin{bmatrix} \lambda^3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ \lambda^2 \\ \lambda \\ 1 \end{bmatrix}}_{u_2} \} = \text{Span}\{ \underbrace{\begin{bmatrix} \lambda^3 \\ 1 \\ \lambda^3 \\ \lambda^2 \\ \lambda \end{bmatrix}}_{w_1}, \underbrace{\begin{bmatrix} \lambda^5 \\ \lambda^2 \\ \lambda^2 \\ \lambda \\ 1 \end{bmatrix}}_{w_2} \}$$

Sum of degrees of $\{u_1, u_2\} = 3 + 2 = 5$ (right minimal bases of $P(\lambda)$)

Sum of degrees of $\{w_1, w_2\} = 3 + 5 = 8$

Right minimal indices of $P(\lambda)$ = $\{2, 3\}$

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Sum of degrees of $\{u_1, u_2\} = 3 + 2 = 5$ (right minimal bases of $P(\lambda)$)

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As a consequence of the previous discussion, we define:

Definition

The **complete "eigenstructure"** of a polynomial matrix $P(\lambda)$ is comprised of:

- its finite eigenvalues, together with their partial multiplicities,
- its infinite eigenvalue, together with its partial multiplicities,
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Remarks

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The **complete “eigenstructure”** of a rational matrix $G(\lambda)$ is comprised of:

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Outline

As said before, the most reliable methods for solving numerically PEPs are based on the concept of linearization.

Definition

- A **linear polynomial matrix (or matrix pencil)** $L(\lambda)$ is a **linearization** of $P(\lambda) = P_d \lambda^d + \cdots + P_1 \lambda + P_0$ if there exist **unimodular** polynomial matrices $U(\lambda), V(\lambda)$ such that

$$U(\lambda)\, L(\lambda)\, V(\lambda) = \begin{bmatrix} I_s & 0 \\ 0 & P(\lambda) \end{bmatrix}.$$

- $L(\lambda)$ is a **strong linearization** of $P(\lambda)$ if, in addition, $\mathrm{rev}\, L(\lambda)$ is a linearization for $\mathrm{rev}\, P(\lambda)$.

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Theorem

A matrix pencil $L(\lambda)$ is a linearization *of a polynomial matrix* $P(\lambda)$ **if and only if**

(1) $L(\lambda)$ *and* $P(\lambda)$ **have the same number of right minimal indices***.*

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$L(\lambda)$ **is a strong linearization** *of* $P(\lambda)$ **if and only if** *(1), (2), (3) and*

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The classical **Frobenius companion form** of the $m \times n$ matrix polynomial

$$P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0$$

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$$C_1(\lambda) := \begin{bmatrix} \lambda P_d + P_{d-1} & P_{d-2} & \cdots & P_1 & P_0 \\ -I_n & \lambda I_n & & & \\ & \ddots & \ddots & & \\ & & \ddots & \lambda I_n & \\ & & & -I_n & \lambda I_n \end{bmatrix} \in \mathbb{C}[\lambda]^{(m+n(d-1))\times nd}$$

Additional property of $C_1(\lambda)$: Example of strong linearization whose right (resp. left) minimal indices allow us to recover the ones of the polynomial via addition of a constant.

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Theorem (recovery of eigenvectors from $C_1(\lambda)$)

Let $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0$ be a **regular** *matrix polynomial,* $\lambda_0 \in \mathbb{C}$ **be a finite eigenvalue of** $P(\lambda)$*, and $C_1(\lambda)$ be the Frobenius companion form of $P(\lambda)$. Then,* **any eigenvector** v **of** $C_1(\lambda)$ **associated to** λ_0 **has the form**

$$v = \begin{bmatrix} \lambda_0^{d-1}\, x \\ \vdots \\ \lambda_0\, x \\ x \end{bmatrix} = \begin{bmatrix} \lambda_0^{d-1} \\ \vdots \\ \lambda_0 \\ 1 \end{bmatrix} \otimes x$$

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- Since 2006 (Mackey, Mackey, Mehl, Mehrmann), many "new" strong linearizations of matrix polynomials have been developed by many authors all around the world
- which also allow us to recover minimal indices and eigenvectors of PEPs without any computational cost.
- One relevant motivation for developing new classes of linearizations is to preserve structures appearing in applications, which is important for saving operations in algorithms and for preserving properties of the eigenvalues in floating point arithmetic.
- For instance, if $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0$ is Hermitian, i.e., it has Hermitian coefficients, **the Frobenius companion form is not!!**

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Parenthesis: There are many other strong linearizations of PEPs (I)

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$$\widetilde{L}(\lambda) = \begin{bmatrix} \lambda P_1 + P_0 & \lambda I_n & & & & & 0 \\ \lambda I_n & 0 & I_n & & & & \\ & I_n & \lambda P_3 + P_2 & \lambda I_n & & & \\ & & \lambda I_n & 0 & I_n & & \\ & & & I_n & \lambda P_5 + P_4 & \lambda I_n & \\ & & & & \lambda I_n & 0 & I_n \\ 0 & & & & & I_n & \lambda P_7 + P_6 \end{bmatrix},$$

is a **Hermitian strong linearization** of the $n \times n$ Hermitian matrix polynomial $P(\lambda) = P_7\lambda^7 + \cdots + P_1\lambda + P_0$ (Antoniou-Vologiannidis 2004; De Terán-D-Mackey 2010; Mackey-Mackey-Mehl-Mehrmann 2010).

Linearizations transform PEPs into GEPs ($P(\lambda) \longrightarrow \lambda B - A$)

- "Good" strong linearizations of a matrix polynomial $P(\lambda)$ are linear matrix polynomials (matrix pencils) that have the same eigenvalues as $P(\lambda)$ and that allow us to recover the eigenvectors when $P(\lambda)$ is regular, and the minimal indices when $P(\lambda)$ is singular.
- **They allow to solve numerically PEPs because there exist excellent algorithms for solving linear PEPs, i.e., GEPs.**
- The fundamental proposed approach

  "linearization + linear eigenvalue algorithm on the linearization"

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  - QZ is implemented in MATLAB command `eig(A,B)` and cost $O(n^3)$.
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Algorithms for solving “not too large” GEPs $\lambda B - A$

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- and on **Krylov subspace methods on** $\lambda B - A$ (Arnoldi on $B^{-1}A$, Rational-Krylov with shifts on $(A - \theta_j B)^{-1}B$) for computing a few desired eigenvalues,
- **but the application of these Krylov methods is NOT direct**,
- **since this would be very expensive in terms of memory and orthogonalization costs**, because
- if $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0 \in \mathbb{C}[\lambda]^{n\times n}$ then its Frobenius companion form (and any other strong linearization) has size $nd \times nd$

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- Therefore, Krylov subspace methods for PEPs take advantage, **in a sophisticated way**, of the structure of the linearization and of the bases of their Krylov subspaces
- to obtain **memory** and orthogonalization costs of the same order of those of an $n \times n$ standard matrix problem (almost no influence of d).
- The most stable and efficient methods in this family are
  1. TOAR (Two level Orthogonal ARnoldi) for QEPs (Su-Bai-Lu, 2008 and 2016) based on $C_1(\lambda)$,
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A few words on algorithms for solving large-scale regular PEPs (II)

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  3. CORK (COmpact Rational Krylov) for arbitrary PEPs (Van Beeumen-Meerbergen-Michiels, 2015) very general, it can use many linearizations and bases for expressing the PEP.
- **Available HPC software:** parallel implementations of TOAR for any degree (including symm. versions) in SLEPc (Roman, UPV, 2016).

Most of the linearizations of polynomial matrices available in the literature are inside (or very closely connected to) the following class of pencils.

Definition (D, Lawrence, Pérez, Van Dooren, Numer. Math., 2018)

A matrix pencil

$$\mathcal{L}(\lambda) = \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}$$

is a block minimal bases pencil (BMBP) if $K_1(\lambda)$ and $K_2(\lambda)$ are minimal bases. If, in addition, the row degrees of $K_1(\lambda)$ and $K_2(\lambda)$ are all one, and the row degrees of each of their dual minimal bases $N_1(\lambda)$ and $N_2(\lambda)$ are all equal, then $\mathcal{L}(\lambda)$ is a strong block minimal bases pencil (SBMBP).

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If $\mathcal{L}(\lambda)$ is a BMBP (resp. SBMBP), then it is a linearization (resp. strong linearization) of the matrix polynomial

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- In the complex linear space of matrix pencils of size $m \times n$ with $m < n$ endowed with the Euclidean metric, the set of pencils that are minimal bases is open and dense,
- even more is the complement of a proper algebraic set.
- If $m = (n - m)\eta$ with η integer, then the set of pencils that are minimal bases with all their row degrees equal to one and with their dual minimal bases having all the row degrees equal to η is open and dense, even more is the complement of a proper algebraic set.
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These definitions are not restrictive (Van Dooren, D, Lin. Alg., 2018)

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Two fundamental auxiliary matrix polynomials in the rest of the talk are the pair of dual minimal bases

$$L_k(\lambda) := \begin{bmatrix} -1 & \lambda & & & \\ & -1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & -1 & \lambda \end{bmatrix} \in \mathbb{C}[\lambda]^{k\times(k+1)},$$

$$\Lambda_k(\lambda)^T := \begin{bmatrix} \lambda^k & \lambda^{k-1} & \cdots & \lambda & 1 \end{bmatrix} \in \mathbb{C}[\lambda]^{1\times(k+1)},$$

and their Kronecker products by identities

$$L_k(\lambda) \otimes I_n := \begin{bmatrix} -I_n & \lambda I_n & & & \\ & -I_n & \lambda I_n & & \\ & & \ddots & \ddots & \\ & & & -I_n & \lambda I_n \end{bmatrix} \in \mathbb{C}[\lambda]^{nk\times n(k+1)},$$

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The Frobenius companion form of the $m \times n$ matrix polynomial $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0$ is

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Let $M(\lambda)$ be an arbitrary pencil. Then any pencil of the form

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is called a block Kronecker pencil (one-block row and column cases included).

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Any block Kronecker pencil $\mathcal{L}(\lambda)$ is a SBMBP and, so, a strong linearization of the matrix polynomial

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Examples of SBMBP: block-Kronecker pencils (III)

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Let $M(\lambda)$ be an arbitrary pencil. Then any pencil of the form

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Examples of block Kronecker pencils (I)

$$P(\lambda) = \lambda^5 P_5 + \lambda^4 P_4 + \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0 \in \mathbb{C}[\lambda]^{m\times n}$$

$$\left[\begin{array}{ccc|cc} \lambda P_5 + P_4 & 0 & 0 & -I_m & 0 \\ 0 & \lambda P_3 + P_2 & 0 & \lambda I_m & -I_m \\ 0 & 0 & \lambda P_1 + P_0 & 0 & \lambda I_m \\ \hline -I_n & \lambda I_n & 0 & 0 & 0 \\ 0 & -I_n & \lambda I_n & 0 & 0 \end{array}\right]$$

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Outline

- **A difference between REPs and PEPs** is that there is no agreement yet on what is a **linearization** of a rational matrix.
- Many authors have developed "linearizations" of rational matrices, but they very rarely prove that properties analogous to those of linearizations of polynomial matrices are satisfied → MORE DIFFICULT PROBLEM.
- Pioneering works on linearizations of rational matrices:
  1. Van Dooren and Verghese in late 70s & early 80s construct pencils that have exactly the same eigenstructure as any given rational matrix. Their constructions require numerical computations.
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- The definitions in this section are based on Amparan, D, Marcaida, and Zaballa, *Strong linearizations of rational matrices*, SIMAX (2018).
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- **Any rational matrix $G(\lambda)$ can be uniquely expressed as**

$$G(\lambda) = D(\lambda) + G_{sp}(\lambda),$$

where

1. $D(\lambda)$ is a polynomial matrix (polynomial part of $G(\lambda)$), and
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- Let $d = \deg(D)$ if $D(\lambda) \neq 0$ and $d = 0$ otherwise. We define the **reversal of $G(\lambda)$** as

$$\operatorname{rev} G(\lambda) = \lambda^d \, G\left(\frac{1}{\lambda}\right)$$

- and the **g-reversal of $G(\lambda)$** as

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Minimal polynomial system matrices of rational matrices

Definition (Rosenbrock, 1970)

Let $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$ be a rational matrix. The polynomial matrix

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{C}[\lambda]^{(n+p)\times(n+m)}$$

is a **polynomial system matrix** of $G(\lambda)$ if

$$G(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda).$$

If, in addition, $\begin{bmatrix} A(\lambda_0) \\ -C(\lambda_0) \end{bmatrix}$ and $\begin{bmatrix} A(\lambda_0) & B(\lambda_0) \end{bmatrix}$ have full column and row ranks, respectively, for any $\lambda_0 \in \mathbb{C}$, then $P(\lambda)$ is a **minimal polynomial system matrix** of $G(\lambda)$.

Theorem (Rosenbrock, 1970)

Each rational matrix has infinitely many minimal polynomial system matrices and, in particular, has minimal polynomial system matrices in **space-state form**, *i.e.*, $A(\lambda) = \lambda I_n - A, \quad B(\lambda) = B, \quad C(\lambda) = C$.

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1. *The finite eigenvalue structure of $P(\lambda)$ (including all types of multiplicities) coincides exactly with the finite zero structure of $G(\lambda)$.*
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Remark:

- Nothing can be guaranteed on the structure at infinity.

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Example of (minimal) polynomial system matrix

Consider the rational matrix

$$G(\lambda) = -B_0 + \lambda A_0 + \frac{B_1}{\lambda - \sigma_1} + \cdots + \frac{B_s}{\lambda - \sigma_s} \in \mathbb{C}(\lambda)^{p \times p},$$

$A_0, B_i \in \mathbb{C}^{p \times p}$ and $\sigma_i \neq \sigma_j$ if $i \neq j$, from Saad, El-Guide, Miedlar, 2019.
Then, these authors introduce the pencil,

$$P(\lambda) = \left[\begin{array}{cccc|c} (\lambda - \sigma_1)I & & & & I \\ & (\lambda - \sigma_2)I & & & I \\ & & \ddots & & \vdots \\ & & & (\lambda - \sigma_s)I & I \\ \hline -B_1 & -B_2 & \cdots & -B_s & \lambda A_0 - B_0 \end{array}\right]$$

which **is a polynomial system matrix of** $G(\lambda)$.

Moreover, $P(\lambda)$ **is minimal if and only if all the matrices** $B_1, \ldots, B_s$ **are nonsingular**.

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A **linearization of $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$ is a matrix pencil**

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{C}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

such that:

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and

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Very simple example of linearization

Consider again the rational matrix

$$G(\lambda) = -B_0 + \lambda A_0 + \frac{B_1}{\lambda - \sigma_1} + \cdots + \frac{B_s}{\lambda - \sigma_s} \in \mathbb{C}(\lambda)^{p\times p},$$

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Outline

We need more...

- The previous definition of linearization of rational matrices follows the spirit of the well-known and widely-accepted definition of linearization of polynomial matrices.
- In fact such definition coincides with the one for polynomial matrices when it is applied to a polynomial matrix.
- **The key goal is to construct a pencil that contains all the information about the (finite) poles and zeros of rational matrices.**
- But in contrast to the polynomial case, in the rational case, **this requires to impose conditions** on the matrices used to represent the rational matrix and to construct the linearization (as we have illustrated in an example).
- Such conditions cannot be always guaranteed (checked) in modern applications of REPs related to approximating NEPs.
- Even more, some of the “linearizations” that have been used in modern packages (NLEIGS) for solving large-scale NEPs do NOT contain all the information of the rational matrix.

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Very informally, after reading a number of “classic” and “modern” references on rational matrices, I share some personal feelings:

- **In the “classic” days** (dominated by applications in Linear Systems and Control):
  1. Rational matrices were often transfer functions of time invariant linear systems.
  2. All the zeros and poles of the rational matrices were of interest.
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Example of difficulty of checking conditions in a “modern” REP

REPs coming from approximating scalar holomorphic functions through numerical quadrature of their Cauchy integral representations (Saad, El-Guide, Miedlar, 2019). For solving a NEP in a certain region Ω

$$T(\lambda_0)v = 0, \quad \lambda_0 \in \mathbb{C}, v \in \mathbb{C}^p,$$

where

$$T(\lambda) = -B_0 + \lambda A_0 + f_1(\lambda)A_1 + \cdots + f_q(\lambda)A_q,$$

with $B_0, A_0, \ldots, A_q \in \mathbb{C}^{p\times p}$ and $f_j : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$, each scalar function is approximated as

$$f_j(z) = -\frac{1}{2\pi i}\int_\Gamma \frac{f_j(t)}{z-t}dt \approx \sum_{i=1}^{m} \frac{\alpha_{ij}}{z-\sigma_i}, \quad z \in \Omega$$

where σ_i are quadrature nodes on the contour Γ, and the nonlinear matrix $T(\lambda)$ is approximated by a rational matrix of the type

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Let $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$ be a rational matrix and

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be a polynomial system matrix of $G(\lambda)$. If $\begin{bmatrix} A(\lambda_0) \\ -C(\lambda_0) \end{bmatrix}$ and $\begin{bmatrix} A(\lambda_0) & B(\lambda_0) \end{bmatrix}$ have full rank n for all $\lambda_0 \in \Sigma \subseteq \mathbb{C}$, then $P(\lambda)$ is a **minimal polynomial system matrix in Σ** of $G(\lambda)$.

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Remark: If $\Sigma = \mathbb{C}$, then a linearization in $\mathbb{C}$ is just a **linearization** in the sense of Amparan, D, Marcaida and Zaballa, SIMAX, 2018.

Linearization of a rational matrix in a set

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Then, without any assumption, $P(\lambda)$ is a linearization of $G(\lambda)$ in Σ, with $\widehat{G}(\lambda) = G(\lambda)$.

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Examples of linearizations in a set (II)

- The linearization in the previous slide can be seen as a particular case of the next one.
- Given the rational matrix:

$$G(\lambda) = D_d\lambda^d + \cdots + D_1\lambda + D_0 + C(\lambda I_n - A)^{-1}B \in \mathbb{C}(\lambda)^{p\times m},$$

- Su & Bai (2011) introduced the Frobenius-like companion pencil

$$L(\lambda) = \left[\begin{array}{c|cccccc} \lambda I_n - A & 0 & 0 & \cdots & 0 & B \\ \hline -C & \lambda D_d + D_{d-1} & D_{d-2} & \cdots & D_1 & D_0 \\ 0 & -I_m & \lambda I_m & & & \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & \ddots & \lambda I_m & \\ 0 & & & & -I_m & \lambda I_m \end{array}\right],$$

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A matrix pencil with degree 1

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is a **linearization at infinity of grade g of** $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$ if $\text{rev}_1 L(\lambda)$ is a linearization of $\text{rev}_g G(\lambda)$ in $\{0\}$.

Example of linearization at infinity

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Then, without any assumption, $P(\lambda)$ is a linearization of $G(\lambda)$ at ∞ of grade 1, with $\widehat{G}(\lambda) = G(\lambda)$.

Example of linearization at infinity

Consider again the rational matrix

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Theorem (D., Marcaida, Quintana, Van Dooren, 2019)

If

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{C}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

is a **linearization at infinity of grade g of $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$**, *the normal rank of $G(\lambda)$ is r, and*

- $e_1 \leq \cdots \leq e_t$ *are the (nonzero) partial multiplicities of* rev$_1(A_1\lambda + A_0)$ *at* 0, *and*
- $\widetilde{e}_1 \leq \cdots \leq \widetilde{e}_u$ *are the (nonzero) partial multiplicities of* rev$_1 L(\lambda)$ *at* 0,

then

$$(q_1, q_2, \ldots, q_r) = (-e_t, -e_{t-1}, \ldots, -e_1, \underbrace{0, \ldots, 0}_{r-t-u}, \widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_u) - (g, g, \ldots, g)$$

are the structural indices at infinity of $G(\lambda)$.

Definition (D., Marcaida, Quintana, Van Dooren, in preparation, 2019)

A **g-strong linearization of $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$ is a matrix pencil** $L(\lambda)$ such that

1. $L(\lambda)$ is a linearization of $G(\lambda)$ in $\mathbb{C}$, and
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g-strong linearizations of a rational matrix $G(\lambda)$ contain the whole finite and infinite zero and pole structures of $G(\lambda)$.

- This is a consequence of the theorem in the next slide,
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There exist infinitely many strong linearizations of rational matrices

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Strong block minimal bases linearizations of rational matrices (RSBMBL)

Theorem (Amparan, D, Marcaida, Zaballa, SIMAX, 2018)

Let

$$\begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}$$

be a SBMBP and $N_1(\lambda), N_2(\lambda)$ *be minimal bases dual to* $K_1(\lambda), K_2(\lambda)$. *Consider for* $i = 1, 2$ *unimodular matrices*

$$U_i(\lambda) = \begin{bmatrix} K_i(\lambda) \\ \widehat{K}_i \end{bmatrix} \quad \text{and} \quad U_i(\lambda)^{-1} = \begin{bmatrix} \widehat{N}_i(\lambda)^T & N_i(\lambda)^T \end{bmatrix},$$

a **linear minimal polynomial system matrix**

$$L(\lambda) = \left[\begin{array}{c|cc} (\lambda I_n - A) & B\widehat{K}_1 & 0 \\ \hline -\widehat{K}_2^T C & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array}\right],$$

and the rational matrix $G(\lambda) = \underbrace{N_2(\lambda)M(\lambda)N_1(\lambda)^T}_{Q(\lambda) := \text{poly. part}} + \underbrace{C(\lambda I_n - A)^{-1}B}_{\text{strict. proper. part}}$.

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Examples of g-strong linearizations (II)

Let us consider the rational matrix

$$G(\lambda) = D_d\lambda^d + \cdots + D_1\lambda + D_0 + C(\lambda I_n - A)^{-1}B \in \mathbb{C}(\lambda)^{p\times m}$$

and the Frobenius-like companion pencil introduced by Su & Bai (2011)

$$L(\lambda) = \left[\begin{array}{c|cccccc} \lambda I_n - A & 0 & 0 & \cdots & 0 & B \\ \hline -C & \lambda D_d + D_{d-1} & D_{d-2} & \cdots & D_1 & D_0 \\ 0 & -I_m & \lambda I_m & & & \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & \ddots & \lambda I_m & \\ 0 & & & & -I_m & \lambda I_m \end{array}\right].$$

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Step 1. Construct one of the previous (strong) linearizations $L(\lambda)$ of $G(\lambda)$.

Step 2. For computing the zeros (and minimal indices, if singular):

- **Step 2.1** Apply to $L(\lambda)$ the QZ algorithm for not too large regular problems.
- **Step 2.2** Apply to $L(\lambda)$ the Staircase algorithm for not too large singular problems.
- **Step 2.3** Apply to $L(\lambda)$ the structured rational Krylov algorithm R-CORK (D, González-Pizarro, 2018) for large-scale regular problems.

Step 3. If the poles are unknown and desired:

- **Step 3.1** Apply to the (1,1)-block of $L(\lambda)$ the QZ algorithm for not too large regular problems.
- **Step 3.2** Apply to the (1,1)-block of $L(\lambda)$ a rational Krylov algorithm for large-scale pencils.

But...

- There are linearizations that **cannot be strong** and, more important,
- that **do not have structures** as the ones described in previous slides,
- since they are constructed from **rational matrices that are NOT represented as the sum of their polynomial and strictly proper parts**.
- Such linearizations can be easily analyzed with the **new theory of local linearizations**.

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- Güttel, Van Beeumen, Meerbergen, Michiels, NLEIGS: a class of fully rational Krylov methods for nonlinear eigenvalue problems, SISC (2014),

a NEP

$$T(\lambda_0)v = 0, \quad \lambda_0 \in \mathbb{C},\ v \in \mathbb{C}^m$$

is approximated in a certain region via Hermite's rational interpolation by a rational matrix of the type

$$Q_N(\lambda) = b_0(\lambda)D_0 + b_1(\lambda)D_1 + \cdots + b_N(\lambda)D_N,$$

with $D_j \in \mathbb{C}^{m\times m}$ and

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a sequence of rational scalar functions. **The poles ξ_i are outside the region of interest, are known,** and are all distinct from the nodes σ_j, some poles ξ_i can be infinite, and β_i are nonzero scaling parameters.

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It can be proved that the NLEIGS "linearizations" satisfy

Theorem (D., Marcaida, Quintana, Van Dooren, in preparation, 2019)

- *$L_N(\lambda)$ is a linearization with empty state matrix of*

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This and many other results on the NLEIGS linearizations have been proved with the **new theory of local linearizations of rational matrices**.

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Outline

1. The "flavor" of applied PEPs, REPs, NEPs: examples
2. Additional "difficulties" of GEPs, PEPs, and REPs over BEPs
3. Linearizations and numerical solution of PEPs
4. Linearizations of REPs
5. Local and Strong Linearizations of REPs and their numerical solution
6. **Global backward stability of PEPs solved with linearizations**
7. Global backward stability of REPs solved with linearizations
8. Conclusions

- We consider a **general** $m \times n$ **polynomial matrix** of degree d

$$P(\lambda) = P_d \lambda^d + \cdots + P_1 \lambda + P_0 \quad , \qquad P_i \in \mathbb{C}^{m \times n},$$

- and we assume that its **complete eigenstructure**
- has been computed by applying a **backward stable algorithm** (QZ for regular, Staircase for singular)
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$$\frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F} = O(\mathbf{u}),$$

where $\mathbf{u} \approx 10^{-16}$ is the unit roundoff and

- $\|\cdot\|_F$ is the Frobenius norm, i.e., for any matrix polynomial

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because block Kronecker linearizations are highly structured pencils and perturbations destroy the structure!!

Example: The Frobenius Companion Form

$$C_1(\lambda) := \begin{bmatrix} \lambda P_d + P_{d-1} & P_{d-2} & \cdots & P_1 & P_0 \\ -I_n & \lambda I_n & & & \\ & \ddots & \ddots & & \\ & & \ddots & \lambda I_n & \\ & & & -I_n & \lambda I_n \end{bmatrix}$$

$$C_1(\lambda) + \Delta\mathcal{L}(\lambda) =$$

$$\begin{bmatrix} \lambda(P_d + E_{11}) + (P_{d-1} + F_{11}) & \lambda E_{12} + P_{d-2} + F_{12} & \cdots & \lambda E_{1,d-1} + P_1 + F_{1,d-1} & \cdots \\ \lambda E_{21} - I_n + F_{21} & \lambda(I_n + E_{22}) + F_{22} & \lambda E_{23} + F_{23} & & \\ \lambda E_{31} + F_{31} & \lambda E_{32} + F_{32} & \ddots & & \\ \vdots & \vdots & \ddots & \lambda(I_n + E_{d-1,d-1}) + F_{d-1,d-1} & \\ \lambda E_{d1} + F_{d1} & \lambda E_{d2} + F_{d2} & & \lambda E_{d,d-1} + F_{d,d-1} - I_n & \cdots \end{bmatrix}$$

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  with $C_{P,\mathcal{L}}$ a number depending on $P(\lambda)$ and $\mathcal{L}(\lambda)$.
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Previous works on this type of backward error analyses

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Theorem (D, Lawrence, Pérez, Van Dooren, Numer. Math., 2018)

Let $\mathcal{L}(\lambda)$ be a block Kronecker pencil for $P(\lambda) = \sum_{i=0}^{d} P_i \lambda^i \in \mathbb{C}[\lambda]^{m\times n}$, i.e.,

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M(\lambda) & L_\eta(\lambda)^T \otimes I_m \\ \hline L_\varepsilon(\lambda) \otimes I_n & 0 \end{array}\right].$$

If $\Delta\mathcal{L}(\lambda)$ is any pencil with the same size as $\mathcal{L}(\lambda)$ and such that

$$\|\Delta\mathcal{L}(\lambda)\|_F < \frac{(\sqrt{2}-1)^2}{d^{5/2}} \frac{1}{1+\|M(\lambda)\|_F},$$

then $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ is a strong linearization of a polynomial matrix $P(\lambda) + \Delta P(\lambda)$ with grade d and such that

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Discussion of the perturbation bounds for block Kronecker pencils

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M(\lambda) & L_\eta(\lambda)^T \otimes I_m \\ \hline L_\varepsilon(\lambda) \otimes I_n & 0 \end{array}\right].$$

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- Therefore, for getting "backward stability" from Block Kronecker linearizations, one needs to normalize the matrix poly $\|P(\lambda)\|_F = 1$ and to use pencils such that $\|M(\lambda)\|_F \approx \|P(\lambda)\|_F$, then

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Outline

The setting

- We consider a **general** $p \times m$ **rational matrix** expressed as

$$G(\lambda) = D_d \lambda^d + \cdots + D_1 \lambda + D_0 + C(\lambda I_n - A)^{-1} B \quad ,$$

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$$\mathcal{L}(\lambda) = \left[\begin{array}{c|cc} \lambda I_n - A & B(e_{\varepsilon+1}^T \otimes I_m) & 0 \\ \hline -(e_{\eta+1} \otimes I_p)C & M(\lambda) & L_\eta(\lambda)^T \otimes I_p \\ 0 & L_\varepsilon(\lambda) \otimes I_m & 0 \end{array}\right].$$

An **example** we have already seen is for

$$G(\lambda) = \lambda^5 D_5 + \lambda^4 D_4 + \lambda^3 D_3 + \lambda^2 D_2 + \lambda D_1 + D_0 + C(\lambda I_n - A)^{-1} B$$

the strong linerization

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Some auxiliary definitions

- Given $G(\lambda) = \sum_{i=0}^{d} \lambda^i D_i + C(\lambda I_n - A)^{-1}B$, we define

$$\|G(\lambda)\|_F = \sqrt{\sum_{i=0}^{d} \|D_i\|_F^2 + \|C\|_F^2 + \|I_n\|_F^2 + \|A\|_F^2 + \|B\|_F^2},$$

- which is the norm of the polynomial system matrix of $G(\lambda)$

$$P(\lambda) = \begin{bmatrix} \lambda I_n - A & B \\ -C & \sum_{i=0}^{d} \lambda^i D_i \end{bmatrix}.$$

- Given a perturbation of $G(\lambda)$, $\widehat{G}(\lambda) = \sum_{i=0}^{d} \lambda^i \widehat{D}_i + \widehat{C}(\lambda I_n - \widehat{A})^{-1}\widehat{B}$, we define (it is a definition!!)

$$\|G(\lambda) - \widehat{G}(\lambda)\|_F := \|\Delta G(\lambda)\|_F$$

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Theorem (D, Quintana, Van Dooren, in progress, 2018)

Let $\mathcal{L}(\lambda)$ be a rational block Kronecker strong linearization of

$$G(\lambda) = \sum_{i=0}^{d} \lambda^i D_i + C(\lambda I_n - A)^{-1} B \in \mathbb{C}(\lambda)^{p\times m}.$$

If $\Delta\mathcal{L}(\lambda)$ is any sufficiently small pencil with the same size as $\mathcal{L}(\lambda)$, then the EIGENVALUE AND MINIMAL INDEX STRUCTURE OF $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ corresponds exactly to the ZERO AND MINIMAL INDEX STRUCTURE of a rational matrix

$$\widehat{G}(\lambda) = \sum_{i=0}^{d} \lambda^i \widehat{D}_i + \widehat{C}(\lambda I_n - \widehat{A})^{-1} \widehat{B} \in \mathbb{C}(\lambda)^{p\times m},$$

such that, to first order in $\|\Delta\mathcal{L}(\lambda)\|_F$,

$$\frac{\|\Delta G(\lambda)\|_F}{\|G(\lambda)\|_F} \leq q(d)\, \frac{\|\mathcal{L}(\lambda)\|_F}{\|G(\lambda)\|_F}\, \mathbf{C_G}\, (1 + \|M(\lambda)\|_F + \|M(\lambda)\|_F^2)\, \frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F},$$

where

$$\mathbf{C_G} = \|C\|_2 + \|A\|_2^{\max\{\varepsilon,\eta\}} + \|B\|_2 \,.$$

There is a penalty with respect to the polynomial case!!!

- $\mathbf{C_G} = \|C\|_2 + \|A\|_2^{\max\{\varepsilon,\eta\}} + \|B\|_2$ depends on the particular state-space realization of the strictly proper part that is used, which **is natural** since there are infinitely many of such realizations:
- $G(\lambda) = \sum_{i=0}^{d} \lambda^i D_i + CT^{-1}(\lambda I_n - TAT^{-1})^{-1}TB.$
- This effect **has been observed** in numerical tests!! (next slide)
- However, for block Kronecker strong linearizations such that $\|M(\lambda)\|_F \approx \|D(\lambda)\|_F$, we have proved that:
  1. There exists a scaling, $G_s(\lambda_s) = d_r\, G(d_\lambda \lambda)$, and
  2. and a balancing diagonal T,
- that transform the original REP into another REP such that

$$\frac{\|\mathcal{L}(\lambda)\|_F}{\|G(\lambda)\|_F}\, \mathbf{C_G}\, (1 + \|M(\lambda)\|_F + \|M(\lambda)\|_F^2) \approx f(d,p,m),$$

with $f(d,p,m)$ a slowing increasing function of $d, p,$ and m.

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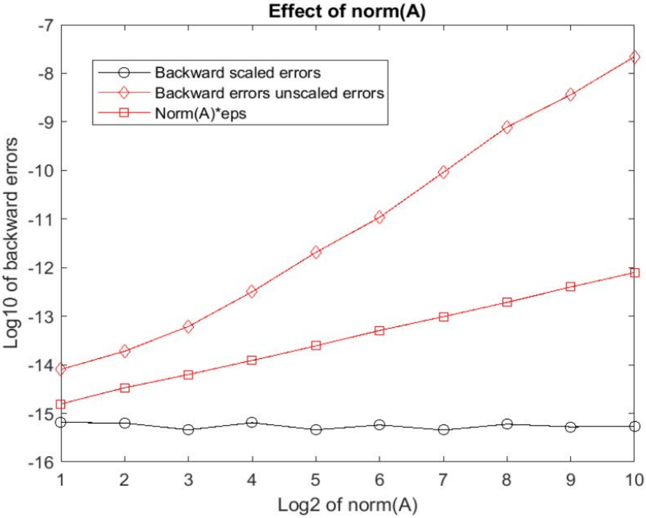
Numerical test on backward errors for zeros of REPs



Outline

Conclusions

- There are many matrix eigenvalue problems in addition to the basic one that are attracting a lot of attention in the last 15 years.
- There are still many open problems in this area: development of algorithms, approximation of NEPs by REPs, theoretical understanding of REPs, different ways of representing rational matrices, and stability analyses (in particular for REPs and for PEPs in non-monomial bases).
- We have developed new classes of linearizations of PEPs that unify and extend the previous ones and, for the first time in the literature, a theory of **local and strong** linearizations of REPs.
- We have have performed a backward stability analysis of PEPs solved with linearizations that improve previous analyses in generality and quality, but more general analyses, including PEPs represented in other bases, are necessary.
- We have performed for the first time in the literature a backward stability analysis of REPs solved with linearizations, which confirms (from another perspective) that REPs are more difficult than PEPs, but this is just the beginning of these analyses...