Structural backward stability in rational eigenvalue problems solved via linearization

Froilán M. Dopico

joint work with **María C. Quintana** (U. Carlos III de Madrid, Spain) and **Paul Van Dooren** (U. catholique Louvain, Belgium)

Departamento de Matemáticas
Universidad Carlos III de Madrid, Spain

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Outline

1. Rational matrices and representations
2. Linearizations for the considered representation
3. Statement of the backward stability problem to be solved
4. The main two structured backward error (perturbation) results
5. Interpretation of the results: scalings
6. Conclusions

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Rational matrices and some of their applications

- In this talk, we want to compute zeros of rational matrices (and, perhaps, other data as poles and minimal indices) in a "stable" way.
- A rational matrix $R(\lambda) \in \mathbb{C}(\lambda)^{m \times n}$ is a matrix whose entries are rational functions of a scalar complex variable.
- To compute the zeros (and the rest of structural data) of a rational matrix (transfer function) is a classical topic in linear systems and control theories and some of the most reliable algorithms in these areas proceed by linearization (Van Dooren, 1979, 1981, 1983).
- Recently to compute the zeros that are not poles (eigenvalues) of a rational matrix has received considerable attention in the community working on NLEPs (nonlinear eigenvalue problems),
- since one of the most important strategies for computing the eigenvalues of a NLEP, $F(\lambda)x = 0$, is based on the following two steps:
  1. Approximate $F(\lambda)$ by a rational matrix $R(\lambda)$ with poles outside Ω, where Ω is the region of interest.
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- Rational matrices can be represented in different forms and the chosen representation is related to the numerical methods used for computing their zeros.
- In this talk, we consider that the rational matrix is represented as

$$R(\lambda) = R_p(\lambda) + D(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i \lambda^i \in \mathbb{C}(\lambda)^{m\times n},$$

where the triple $\{A, B, C\}$ is a minimal state-space realization of the strictly proper part $R_p(\lambda)$, and d is the degree of the polynomial part.
- This minimality means that $\begin{bmatrix} \lambda_0 I_\ell - A \\ C \end{bmatrix}$ and $\begin{bmatrix} \lambda_0 I_\ell - A & B \end{bmatrix}$ have full column and row ranks, respectively, for any $\lambda_0 \in \mathbb{C}$.
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- Loaded elastic string (Betcke et al., NLEVP, (2013); Solov'ëv (2006)):

$$R(\lambda) = A - \lambda B + \frac{\lambda}{\lambda - \sigma} E = (A + E) - \lambda B + \frac{\sigma}{\lambda - \sigma} E \in \mathbb{C}(\lambda)^{n \times n}.$$

- Damped vibration of a viscoelastic structure (Mehrmann & Voss, (2004)):

$$R(\lambda) = \lambda^2 M + K - \sum_{i=1}^{k} \frac{1}{1 + b_i \lambda} \Delta G_i \in \mathbb{C}(\lambda)^{n \times n},$$

M, K positive definite.

- Saad, El-Guide, Miedlar, (2019) consider for approximating some NLEPs

$$R(\lambda) = -B_0 + \lambda A_0 + \frac{B_1}{\lambda - \sigma_1} + \cdots + \frac{B_s}{\lambda - \sigma_s} \in \mathbb{C}(\lambda)^{p \times p},$$

$$= -B_0 + \lambda A_0 + \begin{bmatrix} B_1 & \cdots & B_s \end{bmatrix} \begin{bmatrix} (\lambda - \sigma_1) I_p & & \\ & \ddots & \\ & & (\lambda - \sigma_s) I_p \end{bmatrix}^{-1} \begin{bmatrix} I_p \\ \vdots \\ I_p \end{bmatrix}$$

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... but does not capture all interesting cases

- NLEIGS-Rational Eigenvalue Problems (REPs) coming from linear rational interpolation of NLEPs (Güttel, Van Beeumen, Meerbergen, Michiels (2014)):

$$Q_N(\lambda) = b_0(\lambda)D_0 + b_1(\lambda)D_1 + \cdots + b_N(\lambda)D_N,$$

with $D_j \in \mathbb{C}^{n\times n}$,

$$b_0(\lambda) = \frac{1}{\beta_0}, \qquad b_j(\lambda) = \frac{1}{\beta_0}\prod_{k=1}^{j}\frac{\lambda - \sigma_{k-1}}{\beta_k(1-\lambda/\xi_k)},$$

$j = 1, \ldots, N$, rational scalar functions, with the "poles" ξ_i different from zero and all distinct from the nodes σ_j.

- $Q_N(\lambda)$ cannot be "easily" represented as $C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i\lambda^i$,
- but yes "easily" as $C(\lambda)A(\lambda)^{-1}B(\lambda) + D(\lambda)$ for certain pencils $A(\lambda), B(\lambda), C(\lambda), D(\lambda)$ "easily" constructed from $Q_N(\lambda)$. (M. C. Quintana's talk).

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A few words on linearizations of rational matrices

- In simple words, a linearization $L(\lambda)$ of a rational matrix $R(\lambda)$ is a matrix pencil whose eigenvalues are the finite zeros of $R(\lambda)$ and such that the eigenvalues of a certain submatrix of $L(\lambda)$ are the poles of $R(\lambda)$,
- with equal multiplicities (geometric, algebraic, partial) in both cases, i.e., for zeros and poles.
- If $L(\lambda)$ contains also the pole-zero information of $R(\lambda)$ at infinity, then it is said to be a strong linearization of $R(\lambda)$.
- "Local" linearizations of $R(\lambda)$ have also been defined recently, which are guaranteed to have the complete information of poles and zeros of $R(\lambda)$ in a certain subset $\Sigma \subseteq \mathbb{C} \cup \{\infty\}$. They are not considered in this talk. (M. C. Quintana's talk).
- There are many families of strong linearizations of rational matrices.
- The family considered in this talk includes (modulo some obvious permutations), for example, the following ones among many others:

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- Given the rational matrix:

$$R(\lambda) = C(\lambda I_\ell - A)^{-1} B + \sum_{i=0}^{d} D_i \lambda^i \in \mathbb{C}(\lambda)^{m\times n},$$

where the triple $\{A, B, C\}$ is a minimal state-space realization,

- the following pencil

$$L(\lambda) = \left[\begin{array}{c|ccccc} A-\lambda I_\ell & & & & & B \\ \hline & I_n & -\lambda I_n & & & \\ & & I_n & \ddots & & \\ & & & \ddots & -\lambda I_n & \\ & & & & I_n & -\lambda I_n \\ C & \lambda D_d & \dots & \dots & \lambda D_2 & \lambda D_1 + D_0 \end{array}\right]$$

- is a strong linearization of $R(\lambda)$.
- It is a highly structured pencil with a very particular identity/zero pattern.

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$$R(\lambda) = -B_0 + \lambda A_0 + \frac{B_1}{\lambda - \sigma_1} + \cdots + \frac{B_s}{\lambda - \sigma_s} \in \mathbb{C}(\lambda)^{p\times p},$$
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- The family of linearizations to be considered includes as limit case the simple pencil

$$L(\lambda) = \left[\begin{array}{cccc|c} (\lambda-\sigma_1)I & & & & I \\ & (\lambda-\sigma_2)I & & & I \\ & & \ddots & & \vdots \\ & & & (\lambda-\sigma_s)I & I \\ \hline -B_1 & -B_2 & \cdots & -B_s & \lambda A_0 - B_0 \end{array}\right] = \left[\begin{array}{c|c} \lambda I_\ell - A & B \\ \hline -C & D(\lambda) \end{array}\right],$$

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Auxiliary pencils for defining the considered family of linearizations

$$L_k(\lambda) := \begin{bmatrix} 1 & -\lambda & & & \\ & 1 & -\lambda & & \\ & & \ddots & \ddots & \\ & & & 1 & -\lambda \end{bmatrix} \in \mathbb{C}[\lambda]^{k\times(k+1)},$$

$$\Lambda_k(\lambda)^T := \begin{bmatrix} \lambda^k & \lambda^{k-1} & \cdots & \lambda & 1 \end{bmatrix} \in \mathbb{C}[\lambda]^{1\times(k+1)}.$$

More important, their Kronecker products by identities

$$K_1(\lambda) := L_\epsilon(\lambda) \otimes I_n,$$
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Theorem (Amparan, D, Marcaida, Zaballa, SIMAX, 2018)

Let

- $A \in \mathbb{C}^{\ell \times \ell}$, $B \in \mathbb{C}^{\ell \times n}$, $C \in \mathbb{C}^{m \times \ell}$ *be arbitrary constant matrices and* $M(\lambda)$ *be an arbitrary pencil of adequate size, and*
- $K_1(\lambda)$, $K_2(\lambda)$, $\widehat{K}_1$, $\widehat{K}_2$ *be the pencils and matrices in the previous slide.*

Let us consider the pencil

$$S(\lambda) = \left[\begin{array}{c|cc} A - \lambda I_\ell & B\widehat{K}_1 & 0 \\ \hline \widehat{K}_2^T C & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array}\right],$$

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If $\{A, B, C\}$ *is a minimal state-space realization,* **then** $S(\lambda)$ **is a strong linearization of** $R(\lambda)$**.**

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Comments on the family of (rational) block Kronecker linearizations

- It is a very wide family with infinitely many elements, which is very adequate for the representation of rational matrices considered in this talk.
- It is a subfamily of the much larger class of "strong block minimal bases linearizations of rational matrices" (Amparan,D, Marcaida, Zaballa, SIMAX, 2018), which
  1. is defined by replacing $K_1(\lambda)$ and $K_2(\lambda)$ by any minimal bases with all their row degrees equal to one and by modifying $\widehat{K}_1$ and $\widehat{K}_2$ consistently,
  2. allows to deal with representations of rational matrices where the polynomial part is not expressed in the monomial bases (among many other things).
- However, we do not have yet backward stability analyses related to other strong block minimal bases linearizations of rational matrices (or to any other class of linearizations of rational matrices).
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Outline

We need some norms to state and solve the backward error problem

- **Norm of a matrix polynomial (including matrix pencils).** For $D(\lambda) = \sum_{i=0}^{d} D_i \lambda^i$, we define

$$\|D(\lambda)\|_F := \sqrt{\sum_{i=0}^{d} \|D_i\|_F^2}.$$

- **"Norm" of a rational matrix represented as**

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Statement of the problem (I)

- Given a rational matrix represented as

$$R(\lambda) = C(\lambda I_\ell - A)^{-1} B + \sum_{i=0}^{d} D_i \lambda^i \, ,$$

- its zeros are computed by applying a backward stable algorithm (QZ,...) for computing the eigenvalues of its block Kronecker linearization

$$S(\lambda) = \begin{bmatrix} M(\lambda) & \widehat{K}_2^T C & K_2(\lambda)^T \\ B\widehat{K}_1 & A - \lambda I_\ell & 0 \\ K_1(\lambda) & 0 & 0 \end{bmatrix},$$

where $\sum_{i=0}^{d} D_i \lambda^i = (\Lambda_\eta(\lambda)^T \otimes I_m) M(\lambda) (\Lambda_\epsilon(\lambda) \otimes I_n)$.

- This means that **we have computed the exact eigenvalues of a pencil**

$$\widehat{S}(\lambda) := S(\lambda) + \Delta_S(\lambda), \quad \text{with} \quad \frac{\|\Delta_S(\lambda)\|_F}{\|S(\lambda)\|_F} = O(\mathbf{u}),$$

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- But, **does this imply that we have computed the exact zeros of a nearby rational matrix?**
- **Nearby in the following structural sense:**

$$\widetilde{R}(\lambda) = (C + \Delta C)(\lambda I_\ell - (A + \Delta A))^{-1}(B + \Delta B) + \sum_{i=0}^{d}(D_i + \Delta D_i)\lambda^i \,,$$

with

$$\frac{\|\Delta_R\|_F}{\|R(\lambda)\|_F} := \frac{\sqrt{\|\Delta A\|_F^2 + \|\Delta B\|_F^2 + \|\Delta C\|_F^2 + \sum_{i=0}^{d}\|\Delta D_i\|_F^2}}{\|R(\lambda)\|_F} = O(\mathbf{u}) \;??$$

- This problem is not easy because **the perturbation pencil** $\Delta_S(\lambda)$ **destroys completely the delicate structure of** $S(\lambda)$.

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- This problem is not easy because **the perturbation pencil** $\Delta_S(\lambda)$ **destroys completely the delicate structure of** $S(\lambda)$.

- But, **does this imply that we have computed the exact zeros of a nearby rational matrix?**
- **Nearby in the following structural sense:**

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Outline

Theorem

Given $R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i \lambda^i$, *let* $S(\lambda)$ *be any of its block Kronecker linearizations, and let us perturb it with a pencil to get*

$$\widehat{S}(\lambda) = \underbrace{\begin{bmatrix} M(\lambda) & \widehat{K}_2^T C & K_2(\lambda)^T \\ B\widehat{K}_1 & A - \lambda I_\ell & 0 \\ K_1(\lambda) & 0 & 0 \end{bmatrix}}_{S(\lambda)} + \Delta_S(\lambda).$$

- *If* $\|\Delta_S(\lambda)\|_F$ *is sufficiently small, then* $\widehat{S}(\lambda)$ *is strictly equivalent (and has the same eigenvalues) to the (rational) block Kronecker pencil*

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- *moreover, to first order in* $\|\Delta_S(\lambda)\|_F$,

$$\frac{\|\widetilde{S}(\lambda) - S(\lambda)\|_F}{\|\Delta_S(\lambda)\|_F} \leq 486\, d^4 \max\{1, \|A\|_2^{2(d-1)}\}\, (\|B\|_2 + \|C\|_2 + 1)\|S(\lambda)\|_F^3$$

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From the perturbed pencil to another (rational) block Kronecker pencil

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From the new (rational) block Kronecker pencil to a perturbed rational matrix

- From the theory of block Kronecker pencils, we get that, if $\|\Delta_S(\lambda)\|_F$ is sufficiently small,

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- Thus, the eigenvalues of the perturbed Kronecker pencil, $\widehat{S}(\lambda) = S(\lambda) + \Delta_s(\lambda)$, are equal to those of the exact Kronecker pencil $\widetilde{S}(\lambda)$, which are the zeros of the rational matrix $\widetilde{R}(\lambda)$.
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Theorem

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such that, to first order in $\|\Delta_S(\lambda)\|_F$,

$$\frac{\sqrt{\|\Delta A\|_F^2 + \|\Delta B\|_F^2 + \|\Delta C\|_F^2 + \sum_{i=0}^{d} \|\Delta D_i\|_F^2}}{\|R(\lambda)\|_F} \leq C_{S,R} \, \frac{\|\Delta_S(\lambda)\|_F}{\|S(\lambda)\|_F},$$

where

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Outline

Extra work is needed for guaranteeing structural backward stability (1)!!

- We should have

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- This can be achieved but **requires to scale both the variable λ and the whole rational matrix**:

$$R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i \lambda^i.$$

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Outline

- We have performed for the first time in the literature a global backward stability analysis of rational eigenvalue problems (REPs) solved with linearizations,
- which holds for a very general representation of rational matrices and for a wide class of linearizations.
- The analysis is highly technical and
- indicates that scalings of the variable and the whole rational matrix are necessary in order to get global backward stability in the sense of this talk.
- This is in contrast with polynomial eigenvalue problems, where the scaling of the variable is not needed.
- We expect that this analysis is just the first of a series of structural backward error analyses of REPs, where other representations of rational matrices and other linearizations are used.

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