Conditioning and backward errors of eigenvalues of homogeneous matrix polynomials under Möbius transformations

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1. Homogeneous matrix polynomials and their eigenvalues
2. Homogeneous eigenvalue condition numbers
3. Möbius transformations of homogeneous matrix polynomials
4. The effect of Möbius transformations on eigenvalue condition numbers
5. The effect of Möbius transformations on backward errors of approximate eigenpairs
6. Conclusions
Outline

1. Homogeneous matrix polynomials and their eigenvalues
2. Homogeneous eigenvalue condition numbers
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In this talk, we consider regular homogeneous matrix polynomials of degree \( k \):

\[
P(\alpha, \beta) = \sum_{i=0}^{k} \alpha^{i} \beta^{k-i} B_{i}, \quad B_{i} \in \mathbb{C}^{n \times n},
\]

where \( \alpha, \beta \) are complex scalar variables.

In contrast to the more standard non-homogeneous formulation

\[
P(\lambda) = \sum_{i=0}^{k} \lambda^{i} B_{i}, \quad B_{i} \in \mathbb{C}^{n \times n}.
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From several points of view, in particular for the purpose of this talk, the homogeneous formulation has nicer mathematical properties, but the non-homogeneous formulation is more meaningful in applications.
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Eigenvalues and eigenvectors of homogeneous matrix polynomials

- The **polynomial eigenvalue problem (PEP)** associated to \( P(\alpha, \beta) \) consists of finding scalars \( \alpha_0 \) and \( \beta_0 \), at least one nonzero, and nonzero vectors \( x, y \in \mathbb{C}^n \) such that

\[
y^* P(\alpha_0, \beta_0) = 0 \quad \text{and} \quad P(\alpha_0, \beta_0)x = 0.
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- The previous equalities hold if and only if the equalities

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- This motivates defining the corresponding **eigenvalue** of \( P(\alpha, \beta) \) as the set (line in \( \mathbb{C}^2 \) passing through the origin)

\[
(\alpha_0, \beta_0) := \{ [a\alpha_0, a\beta_0]^T : a \in \mathbb{C} \} \subset \mathbb{C}^2.
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- \( x \) and \( y \) are called **right and left eigenvectors** associated with \( (\alpha_0, \beta_0) \), and \( (x, (\alpha_0, \beta_0)), (y^*, (\alpha_0, \beta_0)) \) are called **right and left eigenpairs**.

- A specific (nonzero) representative of \( (\alpha_0, \beta_0) \) is denoted by \([\alpha_0, \beta_0]^T\).

- We will also use \( \langle x \rangle \), where \( x \in \mathbb{C}^2 \), to denote the line generated by \( x \).
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From homogeneous to non-homogeneous eigenvalues

- If $[\alpha_0, \beta_0]^T$ is any nonzero representative of an eigenvalue of
  $P(\alpha, \beta) = \sum_{i=0}^{k} \alpha^i \beta^{k-i} B_i$, then

- $\lambda_0 = \alpha_0 / \beta_0 \in \mathbb{C}$ is the corresponding eigenvalue of
  $P(\lambda) = \sum_{i=0}^{k} \lambda^i B_i$, (which may be $\lambda_0 = \infty$).

- This indicates that non-homogeneous eigenvalues are more sensitive to
  perturbations of the matrix coefficients than homogeneous eigenvalues,
  since small (norm) variations in $[\alpha_0, \beta_0]^T$ may produce large variations in
  the quotient $\alpha_0 / \beta_0$.

- This has been rigourously proved in Anguas, Bueno, D., *A comparison of eigenvalue condition numbers for matrix polynomials*, LAA (2019),

- and hints why the homogeneous formulation simplifies the analysis of
  eigenvalue condition number problems in PEPs.

- However, note also that homogeneous eigenvalues are the natural
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Two options:

- **Dedieu-Tisseur condition number** $\kappa_h((\alpha_0, \beta_0), P)$ (Dedieu, Tisseur, LAA (2003)):
  1. Complicated definition.

- **Stewart-Sun condition number** $\kappa_\theta((\alpha_0, \beta_0), P)$ (Berhanu, PhD Thesis Manchester (2005)), (Anguas, Bueno, D, LAA (2019)):
  1. Easy and natural definition.
  2. Easily related to the non-homogeneous Wilkinson-like condition numbers.

- **Both are equivalent**

**Corollary (Anguas, Bueno, D, LAA (2019))**

Let $(\alpha_0, \beta_0)$ be a *simple eigenvalue* of $P(\alpha, \beta) = \sum_{i=0}^{k} \alpha^i \beta^{k-i} B_i$. Then,

$$\frac{1}{\sqrt{k + 1}} \leq \frac{\kappa_h((\alpha_0, \beta_0), P)}{\kappa_\theta((\alpha_0, \beta_0), P)} \leq 1.$$
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Definition of Stewart-Sun condition number

Definition

Let \((\alpha_0, \beta_0)\) be a simple eigenvalue of \(P(\alpha, \beta) = \sum_{i=0}^{k} \alpha^i \beta^{k-i} B_i\) and let \(x\) be a right eigenvector of \(P(\alpha, \beta)\) associated with \((\alpha_0, \beta_0)\). We define

\[
\kappa_\theta((\alpha_0, \beta_0), P) := \lim_{\epsilon \to 0} \sup \left\{ \frac{\sin \theta((\alpha_0, \beta_0), (\alpha_0 + \Delta \alpha_0, \beta_0 + \Delta \beta_0))}{\epsilon} : \right. \\
\left. [P(\alpha_0 + \Delta \alpha_0, \beta_0 + \Delta \beta_0) + \Delta P(\alpha_0 + \Delta \alpha_0, \beta_0 + \Delta \beta_0)](x + \Delta x) = 0, \right. \\
\|\Delta B_i\|_2 \leq \epsilon \omega_i, i = 0 : k \right\},
\]

where \(\Delta P(\alpha, \beta) = \sum_{i=0}^{k} \alpha^i \beta^{k-i} \Delta B_i\) and \(\omega_i, i = 0 : k\), are weights.

- \(\sin \theta(\langle x \rangle, \langle y \rangle) = \|x - \text{proj}_y x\|_2 / \|x\|_2\), for \(\langle x \rangle, \langle y \rangle\) lines in \(\mathbb{C}^2\).
- We will use two types of weights:
  1. \(\omega_i = \|P\|_\infty = \max_{j=0:k} \{\|B_j\|_2\}\) \(\longrightarrow\) relative eigenvalue cond. number with respect to the norm of \(P\): \(\kappa^P_\theta((\alpha_0, \beta_0), P)\).
  2. \(\omega_i = \|B_i\|_2 \longrightarrow\) relative eigenvalue cond. number: \(\kappa^r_\theta((\alpha_0, \beta_0), P)\).
Definition of Stewart-Sun condition number

Let \((\alpha_0, \beta_0)\) be a simple eigenvalue of \(P(\alpha, \beta) = \sum_{i=0}^{k} \alpha^i \beta^{k-i} B_i\) and let \(x\) be a right eigenvector of \(P(\alpha, \beta)\) associated with \((\alpha_0, \beta_0)\). We define

\[
\kappa_\theta((\alpha_0, \beta_0), P) := \lim_{\epsilon \to 0} \sup \left\{ \frac{\sin \theta((\alpha_0, \beta_0), (\alpha_0 + \Delta \alpha_0, \beta_0 + \Delta \beta_0))}{\epsilon} : [P(\alpha_0 + \Delta \alpha_0, \beta_0 + \Delta \beta_0) + \Delta P(\alpha_0 + \Delta \alpha_0, \beta_0 + \Delta \beta_0)](x + \Delta x) = 0, \right. \\
\left. \|\Delta B_i\|_2 \leq \epsilon \omega_i, i = 0 : k \right\},
\]

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\kappa_\theta((\alpha_0, \beta_0), P) := \lim_{\epsilon \to 0} \sup \left\{ \frac{\sin \theta((\alpha_0, \beta_0), (\alpha_0 + \Delta \alpha_0, \beta_0 + \Delta \beta_0))}{\epsilon} : \right.
\]

\[
[P(\alpha_0 + \Delta \alpha_0, \beta_0 + \Delta \beta_0) + \Delta P(\alpha_0 + \Delta \alpha_0, \beta_0 + \Delta \beta_0)](x + \Delta x) = 0,
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Explicit expression of Stewart-Sun condition number

Theorem (Anguas, Bueno, D, LAA (2019))

Let \((\alpha_0, \beta_0)\) be a simple eigenvalue of \(P(\alpha, \beta) = \sum_{i=0}^{k} \alpha^i \beta^{k-i} B_i\), and let \(y\) and \(x\) be associated left and right eigenvectors of \(P(\alpha, \beta)\). Then, the Stewart-Sun eigenvalue condition number of \((\alpha_0, \beta_0)\) is

\[
\kappa_\theta((\alpha_0, \beta_0), P) = \left( \sum_{i=0}^{k} |\alpha_0|^i |\beta_0|^{k-i} \omega_i \right) \frac{\|y\|_2 \|x\|_2}{|y^*(\beta_0 D_\alpha P(\alpha_0, \beta_0) - \alpha_0 D_\beta P(\alpha_0, \beta_0))x|},
\]

where \(D_z \equiv \frac{\partial}{\partial z}\) denotes the partial derivative with respect to \(z \in \{\alpha, \beta\}\).

Remark

The explicit expression for \(\kappa_\theta((\alpha_0, \beta_0), P)\) does not depend on the choice of representative of the eigenvalue \((\alpha_0, \beta_0)\).
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Definition of Möbius transformations

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{C})$ and $\mathbb{C}[\alpha, \beta]_{k}^{m\times n}$ be the vector space of $m \times n$ homogeneous matrix polynomials of degree $k$. Then the Möbius transformation on $\mathbb{C}[\alpha, \beta]_{k}^{m\times n}$ induced by $A$ is the map

$$M_A : \mathbb{C}[\alpha, \beta]_{k}^{m\times n} \rightarrow \mathbb{C}[\alpha, \beta]_{k}^{m\times n}$$

given by

$$M_A \left( \sum_{i=0}^{k} \alpha^i \beta^{k-i} B_i \right) (\gamma, \delta) = \sum_{i=0}^{k} (a \gamma + b \delta)^i (c \gamma + d \delta)^{k-i} B_i.$$

The matrix polynomial $M_A(P)(\gamma, \delta)$, that is, the image of $P(\alpha, \beta)$ under $M_A$, is said to be the Möbius transform of $P(\alpha, \beta)$ under $M_A$. 
Comments, applications, properties for Möbius transformations (I)

- **Standard tool in Rational Matrices since the 1950’s (McMillan).**
- They are often used in the theory of matrix polynomials for transforming a polynomial with infinite eigenvalues into one without infinite eigenvalues, which simplifies some problems.
- **Cayley transformations** are the most important ones

\[
A_{+1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad A_{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},
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since they transform matrix polynomials with relevant structures into matrix polynomials with other structures. Important references on this

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Cayley transformations, and some variants, have been used explicitly in important numerical algorithms for eigenvalue problems as:

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The effect of Möbius transformations on eigenvalues

**Theorem**

Let $P(\alpha, \beta) \in \mathbb{C}[\alpha, \beta]_{k}^{n \times n}$ and let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{C})$.

- If $(x, (\alpha_0, \beta_0))$ is a right eigenpair of $P(\alpha, \beta)$, then $(x, \langle A^{-1}[\alpha_0, \beta_0]^T \rangle)$ is a right eigenpair of $M_A(P)(\gamma, \delta)$.
- Same for left eigenpairs.
- Moreover, $(\alpha_0, \beta_0)$ is a simple eigenvalue of $P(\alpha, \beta)$ if and only if $\langle A^{-1}[\alpha_0, \beta_0]^T \rangle$ is a simple eigenvalue of $M_A(P)(\gamma, \delta)$.

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A much stronger result holds, since in the case of eigenvalues that are not simple the partial multiplicities are preserved.
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But, this is not enough for numerical stability, because

- when the numerical solution of a problem is obtained by transforming the problem into another one,
- a fundamental question is whether or not such transformation deteriorates the conditioning of the problem and/or
- the backward errors of approximate solutions,
- because a significant deterioration of such quantities may lead to unreliable solutions.

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The fundamental quotients

We focus on two Stewart-Sun condition numbers with respect to two types of perturbations of the matrix coefficients
\[ \| \Delta B_i \|_2 \leq \epsilon \omega_i \]

1. \[ \omega_i = \left\| P \right\|_\infty = \max_{j=0:k} \{ \| B_j \|_2 \} \quad \text{relative eigenvalue cond. number} \]

with respect to the norm of \( P \): \( \kappa^P_\theta((\alpha_0, \beta_0), P) \).

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For measuring the effect of Möbius transformations on these eigenvalue condition numbers, the following two quotients are bounded

1. \[ Q^p_\theta := \frac{\kappa^P_\theta(\langle A^{-1}[\alpha_0, \beta_0]^T \rangle, M_A(P))}{\kappa^P_\theta((\alpha_0, \beta_0), P)} \]

“the relative quotient with respect to the norms of \( M_A(P) \) and \( P \)”.

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Conditioning Möbius transformations  
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- We focus on two Stewart-Sun condition numbers

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Bounds for “the relative with respect to the norms” quotient


Let $P(\alpha, \beta) \in \mathbb{C}[\alpha, \beta]_{k}^{n \times n}$ and let $A \in \text{GL}(2, \mathbb{C})$. Let $(\alpha_{0}, \beta_{0})$ be a simple eigenvalue of $P(\alpha, \beta)$ and let $\langle A^{-1}[\alpha_{0}, \beta_{0}]^{T} \rangle$ be the associated eigenvalue of $M_{A}(P)(\gamma, \delta)$. Let $Z_{k} := 4(k + 1)^{2}\left(\frac{k}{\lfloor k/2 \rfloor}\right)$ and $\text{cond}_{\infty}(A) = \|A\|_{\infty}\|A^{-1}\|_{\infty}$.

1. If $k = 1$, then
   $\frac{1}{4 \text{ cond}_{\infty}(A)} \leq Q_{\theta}^{p} \leq 4 \text{ cond}_{\infty}(A)$.

2. If $k \geq 2$, then
   $\frac{1}{Z_{k} \text{ cond}_{\infty}(A)^{k-1}} \leq Q_{\theta}^{p} \leq Z_{k} \text{ cond}_{\infty}(A)^{k-1}$.

**Remark**

- Neat and universal “extremely a priori” bounds depending only on the degree and the condition number of $A$.
- **Great result if** $\text{cond}_{\infty}(A) \approx 1$!!!!!! (at least for moderate $k$).
- $\text{cond}_{\infty}(A) = 2$ for Cayley transformations.
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\frac{1}{4 \ \text{cond}_\infty(A)} \leq Q_{\theta}^p \leq 4 \ \text{cond}_\infty(A), \quad \text{for } k = 1,
\]
\[
\frac{1}{Z_k \ \text{cond}_\infty(A)^{k-1}} \leq Q_{\theta}^p \leq Z_k \ \text{cond}_\infty(A)^{k-1}, \quad \text{for } k \geq 2.
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- The factor \( Z_k := 4(k + 1)^2\left(\frac{k}{k/2}\right) \) increases very fast with \( k \):

<table>
<thead>
<tr>
<th>( k )</th>
<th>( Z_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>72</td>
</tr>
<tr>
<td>3</td>
<td>192</td>
</tr>
<tr>
<td>4</td>
<td>600</td>
</tr>
<tr>
<td>5</td>
<td>1440</td>
</tr>
<tr>
<td>10</td>
<td>121968</td>
</tr>
</tbody>
</table>

- This makes the lower and upper bounds very different from each other for moderate values of \( k \), even if \( \text{cond}_\infty(A) \approx 1 \).

- However, many numerical random experiments confirm that the factor \( Z_k \) is very pessimistic, since although \( Q_{\theta}^p \) typically increases slowly with \( k \), it is much smaller than the corresponding upper bound.
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</tr>
<tr>
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<td>10</td>
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</tr>
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<table>
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<tr>
<th>$k$</th>
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Bounds for the relative quotient


Let $P(\alpha, \beta) = \sum_{i=0}^{k} \alpha^i \beta^{k-i} B_i$ and $A \in GL(2, \mathbb{C})$. Let $(\alpha_0, \beta_0)$ be a simple eigenvalue of $P(\alpha, \beta)$ and let $\langle A^{-1}[\alpha_0, \beta_0]^T \rangle$ be the corresponding eigenvalue of $M_A(P)(\gamma, \delta) = \sum_{i=0}^{k} \gamma^i \delta^{k-i} \tilde{B}_i$. Let $Z_k := 4(k + 1)^2 \left( \frac{k}{[k/2]} \right)$. Assume that $B_0 \neq 0, B_k \neq 0, \tilde{B}_0 \neq 0, \text{ and } \tilde{B}_k \neq 0$ and define

$$\rho := \frac{\max_{i=0:k} \{ \| B_i \|_2 \}}{\min \{ \| B_0 \|_2, \| B_k \|_2 \}}, \quad \tilde{\rho} := \frac{\max_{i=0:k} \{ \| \tilde{B}_i \|_2 \}}{\min \{ \| \tilde{B}_0 \|_2, \| \tilde{B}_k \|_2 \}}.$$

1. If $k = 1$, then
   $$\frac{1}{4 \ \text{cond}_\infty(A) \tilde{\rho}} \leq Q^r_\theta \leq 4 \ \text{cond}_\infty(A) \rho.$$

2. If $k \geq 2$, then
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**Remark**

- Penalty w.r.t. $Q^P_\theta$ due to $\rho \geq 1$ and $\tilde{\rho} \geq 1$, which is observed in practice.
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The scenario is: we want to compute eigenpairs of $P(\alpha, \beta)$, but, for some reason, it is advantageous to compute eigenpairs of $M_A(P)(\gamma, \delta)$.

A motivation for this might be that $P(\alpha, \beta)$ has a structure that we would like to preserve in the computation for efficiency, accuracy, symmetries of eigenvalues, ... but there are no specific algorithms available for such structure, although there are for the structure of $M_A(P)(\gamma, \delta)$.

Note that if $(\hat{x}, (\hat{\gamma}_0, \hat{\delta}_0))$ is a computed approximate right eigenpair of $M_A(P)$, and $(\hat{\alpha}_0, \hat{\beta}_0) := (a\hat{\gamma}_0 + b\hat{\delta}_0, c\hat{\gamma}_0 + d\hat{\delta}_0)$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

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Assuming that $(\hat{x}, (\hat{\gamma}_0, \hat{\delta}_0))$ has been computed with small backward errors,

a natural question in this setting is whether $(\hat{x}, (\hat{\alpha}_0, \hat{\beta}_0))$ is also an approximate eigenpair of $P$ with small backward errors.
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Main results on backward errors

- This would happen if the quotient

\[ Q_{\eta, \text{right}} := \frac{\eta_P(\hat{x}, (\hat{\alpha}_0, \hat{\beta}_0))}{\eta_{MA}(\hat{x}, (\hat{\gamma}_0, \hat{\delta}_0))} \]

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- For perturbations of each coefficient small w.r.t. the norm of the whole homogeneous matrix polynomial:
  1. Möbius transformations induced by well-conditioned matrices $A$ do not change significantly the eigenvalue condition numbers and backward errors of any eigenvalue of any matrix polynomial.
  2. This is completely rigorous for matrix polynomials with low degree, but also happens in practice for larger degrees.

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  - The results on condition numbers do NOT hold for the standard non-homogeneous formulation.
Conclusions

- For perturbations of each coefficient small w.r.t. the norm of the whole homogeneous matrix polynomial:
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