Conditioning and backward errors of eigenvalues of homogeneous matrix polynomials under Möbius transformations

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22nd Conference of the International Linear Algebra Society
Minisymposium "Matrices over elementary divisor domains"
July 8-12, 2019. Rio de Janeiro, Brazil

Outline

1. Homogeneous matrix polynomials and their eigenvalues
2. Homogeneous eigenvalue condition numbers
3. Möbius transformations of homogeneous matrix polynomials
4. The effect of Möbius transformations on eigenvalue condition numbers
5. The effect of Möbius transformations on backward errors of approximate eigenpairs
6. Conclusions

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- In this talk, we consider **regular homogeneous matrix polynomials** of degree k

$$P(\alpha,\beta)=\sum_{i=0}^{k}\alpha^i\beta^{k-i}B_i,\quad B_i\in\mathbb{C}^{n\times n},$$

where α,β are complex scalar variables.

- In contrast to the more standard non-homogeneous formulation

$$P(\lambda)=\sum_{i=0}^{k}\lambda^i B_i,\quad B_i\in\mathbb{C}^{n\times n}.$$

- From several points of view, in particular for the purpose of this talk, **the homogeneous formulation has nicer mathematical properties**, but
- **the non-homogeneous formulation is more meaningful in applications**.

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- The **polynomial eigenvalue problem (PEP) associated to** $P(\alpha, \beta)$ consists of finding scalars α_0 and β_0, at least one nonzero, and nonzero vectors $x, y \in \mathbb{C}^n$ such that

$$y^* P(\alpha_0, \beta_0) = 0 \quad \text{and} \quad P(\alpha_0, \beta_0) x = 0.$$

- The previous equalities hold if and only if the equalities

$$y^* P(a\alpha_0, a\beta_0) = 0 \quad \text{and} \quad P(a\alpha_0, a\beta_0) x = 0$$

hold for any complex number $a \neq 0$.

- This motivates defining the corresponding **eigenvalue** of $P(\alpha, \beta)$ as **the set** (line in $\mathbb{C}^2$ passing through the origin)

$$(\alpha_0, \beta_0) := \{[a\alpha_0, a\beta_0]^T : a \in \mathbb{C}\} \subset \mathbb{C}^2.$$

- x and y are called **right and left eigenvectors** associated with (α_0, β_0),
- and $(x, (\alpha_0, \beta_0))$, $(y^*, (\alpha_0, \beta_0))$ are called **right and left eigenpairs**.
- A specific (nonzero) representative of (α_0, β_0) is denoted by $[\alpha_0, \beta_0]^T$.
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From homogeneous to non-homogeneous eigenvalues

- If $[\alpha_0, \beta_0]^T$ is any nonzero representative of an eigenvalue of $P(\alpha, \beta) = \sum_{i=0}^{k} \alpha^i \beta^{k-i} B_i$, then
- $\lambda_0 = \alpha_0/\beta_0 \in \mathbb{C}$ is the corresponding eigenvalue of $P(\lambda) = \sum_{i=0}^{k} \lambda^i B_i$, (which may be $\lambda_0 = \infty$).
- This indicates that non-homogeneous eigenvalues are more sensitive to perturbations of the matrix coefficients than homogeneous eigenvalues,
- since small (norm) variations in $[\alpha_0, \beta_0]^T$ may produce large variations in the quotient α_0/β_0.
- This has been rigourously proved in Anguas, Bueno, D., *A comparison of eigenvalue condition numbers for matrix polynomials*, LAA (2019),
- and hints why the homogeneous formulation simplifies the analysis of eigenvalue condition number problems in PEPs.
- However, note also that homogeneous eigenvalues are the natural outcome of solving PEPs via linearization + QZ algorithm:

$$\underbrace{A - \lambda B \longrightarrow T - \lambda U}_{QZ} \Rightarrow \lambda_i = t_{ii}/u_{ii}$$

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Two options:

- **Dedieu-Tisseur condition number** $\kappa_h((\alpha_0,\beta_0),P)$ (Dedieu, Tisseur, LAA (2003)):
  1. Complicated definition.
  2. Not so easily related to the non-homogeneous Wilkinson-like condition numbers.
- **Stewart-Sun condition number** $\kappa_\theta((\alpha_0,\beta_0),P)$ (Berhanu, PhD Thesis Manchester (2005)), (Anguas, Bueno, D, LAA (2019)):
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- **Both are equivalent**

Corollary (Anguas, Bueno, D, LAA (2019))

Let (α_0,β_0) *be a simple eigenvalue of* $P(\alpha,\beta)=\sum_{i=0}^{k}\alpha^i\beta^{k-i}B_i$. *Then,*

$$\frac{1}{\sqrt{k+1}}\leq\frac{\kappa_h((\alpha_0,\beta_0),P)}{\kappa_\theta((\alpha_0,\beta_0),P)}\leq 1.$$

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$$\kappa_\theta((\alpha_0, \beta_0), P) := \lim_{\epsilon \to 0} \sup \left\{ \frac{\sin \theta((\alpha_0, \beta_0), (\alpha_0 + \Delta\alpha_0, \beta_0 + \Delta\beta_0))}{\epsilon} : \right.$$
$$[P(\alpha_0 + \Delta\alpha_0, \beta_0 + \Delta\beta_0) + \Delta P(\alpha_0 + \Delta\alpha_0, \beta_0 + \Delta\beta_0)](x + \Delta x) = 0,$$
$$\left. \|\Delta B_i\|_2 \leq \epsilon\, \omega_i, i = 0 : k \right\},$$

where $\Delta P(\alpha, \beta) = \sum_{i=0}^{k} \alpha^i \beta^{k-i} \Delta B_i$ and ω_i, $i = 0 : k$, are weights.

- $\sin\ \theta(\langle x \rangle, \langle y \rangle) = \|x - \text{proj}_y x\|_2 / \|x\|_2$, for $\langle x \rangle, \langle y \rangle$ lines in $\mathbb{C}^2$.
- We will use two types of weights:
  1. $\omega_i = \|P\|_\infty = \max_{j=0:k}\{\|B_j\|_2\} \longrightarrow$ relative eigenvalue cond. number with respect to the norm of P: $\kappa_\theta^p((\alpha_0, \beta_0), P)$.
  2. $\omega_i = \|B_i\|_2 \longrightarrow$ relative eigenvalue cond. number: $\kappa_\theta^r((\alpha_0, \beta_0), P)$.

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  1. $\omega_i = \|P\|_\infty = \max_{j=0:k}\{\|B_j\|_2\} \longrightarrow$ relative eigenvalue cond. number with respect to the norm of P: $\kappa_\theta^p((\alpha_0,\beta_0),P)$.
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Let (α_0, β_0) be a simple eigenvalue of $P(\alpha, \beta) = \sum_{i=0}^{k} \alpha^i \beta^{k-i} B_i$ and let x be a right eigenvector of $P(\alpha, \beta)$ associated with (α_0, β_0). We define

$$\kappa_\theta((\alpha_0, \beta_0), P) := \lim_{\epsilon \to 0} \sup \left\{ \frac{\sin\theta((\alpha_0, \beta_0), (\alpha_0 + \Delta\alpha_0, \beta_0 + \Delta\beta_0))}{\epsilon} : \right.$$
$$[P(\alpha_0 + \Delta\alpha_0, \beta_0 + \Delta\beta_0) + \Delta P(\alpha_0 + \Delta\alpha_0, \beta_0 + \Delta\beta_0)](x + \Delta x) = 0,$$
$$\left. \|\Delta B_i\|_2 \leq \epsilon\, \omega_i, i = 0 : k \right\},$$

where $\Delta P(\alpha, \beta) = \sum_{i=0}^{k} \alpha^i \beta^{k-i} \Delta B_i$ and ω_i, $i = 0 : k$, are weights.

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Explicit expression of Stewart-Sun condition number

Theorem (Anguas, Bueno, D, LAA (2019))

Let (α_0, β_0) be a simple eigenvalue of $P(\alpha, \beta) = \sum_{i=0}^{k} \alpha^i \beta^{k-i} B_i$, and let y and x be associated left and right eigenvectors of $P(\alpha, \beta)$. Then, the Stewart-Sun eigenvalue condition number of (α_0, β_0) is

$$\kappa_\theta((\alpha_0, \beta_0), P) = \left(\sum_{i=0}^{k} |\alpha_0|^i |\beta_0|^{k-i} \omega_i \right) \frac{\|y\|_2 \|x\|_2}{|y^*(\overline{\beta_0} D_\alpha P(\alpha_0, \beta_0) - \overline{\alpha_0} D_\beta P(\alpha_0, \beta_0))x|},$$

where $D_z \equiv \frac{\partial}{\partial z}$ denotes the partial derivative with respect to $z \in \{\alpha, \beta\}$

Remark

The explicit expression for $\kappa_\theta((\alpha_0, \beta_0), P)$ does not depend on the choice of representative of the eigenvalue (α_0, β_0).

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Outline

1. Homogeneous matrix polynomials and their eigenvalues
2. Homogeneous eigenvalue condition numbers
3. **Möbius transformations of homogeneous matrix polynomials**
4. The effect of Möbius transformations on eigenvalue condition numbers
5. The effect of Möbius transformations on backward errors of approximate eigenpairs
6. Conclusions

Definition

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{C})$ and $\mathbb{C}[\alpha, \beta]_k^{m \times n}$ be the vector space of $m \times n$ homogeneous matrix polynomials of degree k. Then the Möbius transformation on $\mathbb{C}[\alpha, \beta]_k^{m \times n}$ induced by A is the map

$$M_A : \mathbb{C}[\alpha, \beta]_k^{m \times n} \to \mathbb{C}[\alpha, \beta]_k^{m \times n}$$

given by

$$M_A \left(\sum_{i=0}^{k} \alpha^i \beta^{k-i} B_i \right) (\gamma, \delta) = \sum_{i=0}^{k} (a\gamma + b\delta)^i (c\gamma + d\delta)^{k-i} B_i.$$

The matrix polynomial $M_A(P)(\gamma, \delta)$, that is, the image of $P(\alpha, \beta)$ under M_A, is said to be the Möbius transform of $P(\alpha, \beta)$ under M_A.

- Standard tool in Rational Matrices since the 1950's (McMillan).
- They are often used in the theory of matrix polynomials for transforming a polynomial with infinite eigenvalues into one without infinite eigenvalues, which simplifies some problems.
- Cayley transformations are the most important ones

$$A_{+1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad A_{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

- since they transform matrix polynomials with relevant structures into matrix polynomials with other structures. Important references on this
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The effect of Möbius transformations on eigenvalues

Theorem

Let $P(\alpha,\beta) \in \mathbb{C}[\alpha,\beta]_k^{n\times n}$ *and let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,\mathbb{C})$.

- *If* $(x,(\alpha_0,\beta_0))$ *is a right eigenpair of* $P(\alpha,\beta)$, *then* $(x,\langle A^{-1}[\alpha_0,\beta_0]^T\rangle)$ *is a right eigenpair of* $M_A(P)(\gamma,\delta)$.
- *Same for left eigenpairs.*
- *Moreover,* (α_0,β_0) *is a simple eigenvalue of* $P(\alpha,\beta)$ *if and only if* $\langle A^{-1}[\alpha_0,\beta_0]^T\rangle$ *is a simple eigenvalue of* $M_A(P)(\gamma,\delta)$.

Remark

A much stronger result holds, since in the case of eigenvalues that are not simple the partial multiplicities are preserved.

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But, this is not enough for numerical stability, because

- when the numerical solution of a problem is obtained by transforming the problem into another one,
- a fundamental question is whether or not such transformation deteriorates the conditioning of the problem and/or
- the backward errors of approximate solutions,
- because a significant deterioration of such quantities may lead to unreliable solutions.
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6 Conclusions

- We focus on two Stewart-Sun condition numbers
- with respect to two types of perturbations of the matrix coefficients $\|\Delta B_i\|_2 \leq \epsilon\, \omega_i$
  1. $\omega_i = \|P\|_\infty = \max_{j=0:k}\{\|B_j\|_2\} \longrightarrow$ relative eigenvalue cond. number with respect to the norm of P: $\kappa_\theta^p((\alpha_0, \beta_0), P)$.
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- For measuring the effect of Möbius transformations on these eigenvalue condition numbers, the following two quotients are bounded
  1. $$Q_\theta^p := \frac{\kappa_\theta^p(\langle A^{-1}[\alpha_0, \beta_0]^T\rangle, M_A(P))}{\kappa_\theta^p((\alpha_0, \beta_0), P)},$$
     "the relative quotient with respect to the norms of $M_A(P)$ and P".
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Bounds for “the relative with respect to the norms” quotient

Theorem (Anguas, Bueno, D, Math. Comp., to appear)

Let $P(\alpha, \beta) \in \mathbb{C}[\alpha, \beta]_k^{n\times n}$ *and let* $A \in GL(2, \mathbb{C})$*. Let* (α_0, β_0) *be a simple eigenvalue of* $P(\alpha, \beta)$ *and let* $\langle A^{-1}[\alpha_0, \beta_0]^T \rangle$ *be the associated eigenvalue of* $M_A(P)(\gamma, \delta)$*. Let* $Z_k := 4(k+1)^2\binom{k}{\lfloor k/2 \rfloor}$ *and* $\mathrm{cond}_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$*.*

1. *If* $k = 1$*, then*

$$\frac{1}{4\,\mathrm{cond}_\infty(A)} \leq Q_\theta^p \leq 4\,\mathrm{cond}_\infty(A).$$

2. *If* $k \geq 2$*, then*

$$\frac{1}{Z_k\,\mathrm{cond}_\infty(A)^{k-1}} \leq Q_\theta^p \leq Z_k\,\mathrm{cond}_\infty(A)^{k-1}.$$

Remark

- **Neat and universal “extremely *a priori*” bounds depending only on the degree and the condition number of** A.
- **Great result if** $\mathrm{cond}_\infty(A) \approx 1$**!!!!!** (at least for moderate k).
- $\mathrm{cond}_\infty(A) = 2$ for Cayley transformations.

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$$\frac{1}{4\,\mathrm{cond}_\infty(A)} \leq Q_\theta^p \leq 4\,\mathrm{cond}_\infty(A), \qquad \text{for } k = 1,$$
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- The factor $Z_k := 4(k+1)^2\binom{k}{\lfloor k/2 \rfloor}$ increases very fast with k:

k	Z_k
2	72
3	192
4	600
5	1440
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- This makes the lower and upper bounds very different from each other for moderate values of k, even if $\mathrm{cond}_\infty(A) \approx 1$.
- However, many numerical random experiments confirm that the factor Z_k is very pessimistic, since although Q_θ^p typically increases slowly with k, it is much smaller than the corresponding upper bound.

Comments on the bounds for the relative w.r.t. the norms quotient (I)

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Let $P(\alpha,\beta)=\sum_{i=0}^{k}\alpha^i\beta^{k-i}B_i$ *and* $A\in GL(2,\mathbb{C})$. *Let* (α_0,β_0) *be a simple eigenvalue of* $P(\alpha,\beta)$ *and let* $\langle A^{-1}[\alpha_0,\beta_0]^T\rangle$ *be the corresponding eigenvalue of* $M_A(P)(\gamma,\delta)=\sum_{i=0}^{k}\gamma^i\delta^{k-i}\widetilde{B}_i$. *Let* $Z_k := 4(k+1)^2\binom{k}{\lfloor k/2\rfloor}$. *Assume that* $B_0\neq 0, B_k\neq 0, \widetilde{B}_0\neq 0$, *and* $\widetilde{B}_k\neq 0$ *and define*

$$\rho := \frac{\max_{i=0:k}\{\|B_i\|_2\}}{\min\{\|B_0\|_2,\|B_k\|_2\}},\quad \widetilde{\rho} := \frac{\max_{i=0:k}\{\|\widetilde{B}_i\|_2\}}{\min\{\|\widetilde{B}_0\|_2,\|\widetilde{B}_k\|_2\}}.$$

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5. **The effect of Möbius transformations on backward errors of approximate eigenpairs**
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- Note that if $(\widehat{x},(\widehat{\gamma_0},\widehat{\delta_0}))$ is a computed *approximate* right eigenpair of $M_A(P)$, and $(\widehat{\alpha_0},\widehat{\beta_0}) := (a\widehat{\gamma_0}+b\widehat{\delta_0}, c\widehat{\gamma_0}+d\widehat{\delta_0})$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,
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- **a natural question in this setting is whether** $(\widehat{x},(\widehat{\alpha_0},\widehat{\beta_0}))$ **is also an approximate eigenpair of** P **with small backward errors**.

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Main results on backward errors

- This would happen if the quotient

$$Q_{\eta,\text{right}} := \frac{\eta_P(\widehat{x}, (\widehat{\alpha_0}, \widehat{\beta_0}))}{\eta_{M_A(P)}(\widehat{x}, (\widehat{\gamma_0}, \widehat{\delta_0}))}$$

  is a moderate number not much larger than one,

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  with $\Delta P(\alpha, \beta) = \sum_{i=0}^{k} \alpha^i \beta^{k-i} \Delta B_i$.

- **Bounds analogous to those of the quotients of condition numbers hold for the two classes of weights ω_i that we have considered.**
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  1. Möbius transformations induced by well-conditioned matrices A do not change significatively the eigenvalue condition numbers and backward errors of any eigenvalue of any matrix polynomial.
  2. This is completely rigorous for matrix polynomials with low degree,
  3. but also happens in practice for larger degrees.
- For perturbations of each coefficient small w.r.t. the norm of each coefficient:
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