Local linearizations of rational matrices with application to nonlinear eigenvalue problems

Froilán M. Dopico

joint work with **Silvia Marcaida** (U. País Vasco, Spain),
M^a del Carmen Quintana (U. Carlos III, Spain), and
Paul Van Dooren (U. C. Louvain, Belgium)

Departamento de Matemáticas
Universidad Carlos III de Madrid, Spain

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A few words on Rational Eigenvalue Problems (REPs)

- Given a nonsingular rational matrix $G(\lambda) \in \mathbb{C}(\lambda)^{p\times p}$ the REP consists in computing numbers $\lambda_0 \in \mathbb{C}$ and non-zero vectors $x_0 \in \mathbb{C}^p$ such that

$$G(\lambda_0)\, x_0 = 0 \qquad \text{(so } \lambda_0 \text{ is not a pole).}$$

- REPs are appearing recently in applications and in approximations of nonlinear eigenvalue problems (NEP), (surveys Mehrmann-Voss (2004), Betcke et al., NLEVP, (2013), Güttel-Tisseur, (2017)),
- but REPs have been studied since the 60s and 70s in Linear Systems and Control and the more general problem of computing all the structural data of a Rational Matrix was solved using linearizations by Van Dooren in his PhD Thesis (1979) and papers in early 80s for dense problems.
- **A key difference between REPs and** polynomial eigenvalue problems (**PEPs**) is that, once a scalar polynomial basis is chosen, a PEP is completely determined by the coefficients, while REPs are not determined by the election of a basis and appear in many different forms.
- This is related to the classic theory and computation of realizations of rational matrices in linear systems theory (Rosenbrock (1970), Kailath (1980), Antoulas (2005), etc).

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- Loaded elastic string (Betcke et al., NLEVP-collection, (2013)):

$$G(\lambda) = A - \lambda B + \frac{\lambda}{\lambda - \sigma} E = (A + E) - \lambda B + \frac{\sigma}{\lambda - \sigma} E,$$

which almost shows the polynomial and the strictly proper parts of $G(\lambda)$.

- Damped vibration of a structure (Mehrmann & Voss, (2004)):

$$G(\lambda) = \lambda^2 M + K - \sum_{i=1}^{k} \frac{1}{1 + b_i \lambda} \Delta G_i,$$

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A few examples of "modern" REPs with different representations (I)

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A few examples of "modern" REPs with different representations (II)

- NLEIGS-REPs coming from linear rational interpolation of NEPs (Güttel, Van Beeumen, Meerbergen, Michiels, 2014):

$$Q_N(\lambda) = b_0(\lambda) D_0 + b_1(\lambda) D_1 + \cdots + b_N(\lambda) D_N,$$

  with $D_j \in \mathbb{C}^{m \times m}$ and $b_j(\lambda) = \dfrac{1}{\beta_0} \displaystyle\prod_{k=1}^{j} \dfrac{\lambda - \sigma_{k-1}}{\beta_k (1 - \lambda/\xi_k)}$, $j = 0, 1, \ldots, N$, a sequence of rational scalar functions, with the poles ξ_i all distinct from the nodes σ_j. Some poles ξ_i can be infinite.

- REPs coming from "Automatic Approximation of NEPs" (Lietaert, Pérez, Vandereycken, Meerbergen, 2018):

$$R(\lambda) = \sum_{i=0}^{k-1} (A_i - \lambda B_i) f_i(\lambda) + \sum_{i=1}^{s} (C_i - \lambda D_i) a_i^T (E_i - \lambda F_i)^{-1} b_i,$$

  where $f_i(\lambda)$ are scalar polynomial or rational functions satisfying a linear relation ($f_0(\lambda) = 1$), $a_i, b_i \in \mathbb{C}^{l_i}$ are vectors, A_i, B_i, C_i, D_i matrices, and $l_i \times l_i$ matrices

$$E_i = \begin{bmatrix} w_1 & w_2 & \cdots & w_{l_i-1} & w_{l_i} \\ -z_1 & z_2 & & & \\ & -z_2 & \ddots & & \\ & & \ddots & z_{l_i-1} & \\ & & & -z_{l_i-1} & z_{l_i} \end{bmatrix} \text{ and } F_i = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & & & \\ & 1 & \ddots & & \\ & & \ddots & -1 & \\ & & & 1 & -1 \end{bmatrix}.$$

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- REPs coming from approximating scalar holomorphic functions through numerical quadrature of their Cauchy integral representations (Saad, El-Guide, Miedlar, 2019).

For solving a NEP in a certain region

$$T(\lambda_0)v = 0, \quad \lambda_0 \in \mathbb{C}, v \in \mathbb{C}^p,$$

where

$$T(\lambda) = -B_0 + \lambda A_0 + f_1(\lambda)A_1 + \cdots + f_q(\lambda)A_q,$$

with $B_0, A_0, \ldots, A_q \in \mathbb{C}^{p\times p}$ and $f_i : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$, the nonlinear matrix $T(\lambda)$ is approximated by a rational matrix of the type

$$G(\lambda) = -B_0 + \lambda A_0 + \frac{B_1}{\lambda - \sigma_1} + \cdots + \frac{B_s}{\lambda - \sigma_s},$$

where $A_0, B_0, \ldots, B_s \in \mathbb{C}^{p\times p}$ and $\sigma_i \neq \sigma_j$ if $i \neq j$ are quadrature points on the contour of the region of interest.

Remark: $G(\lambda)$ is a rational matrix with polynomial part of degree one.

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- One of the most common and reliable methods for solving REPs and PEPs is by computing the eigenvalues of **linearizations**, i.e., matrix pencils (polynomials of degree 1) that have the "same" eigenvalues.
- **Another key difference between REPs and PEPs** is that there is no clear agreement on what is a **linearization** of a rational matrix.
- For regular matrix polynomials, linearizations are regular pencils with exactly the same finite eigenvalues with the same multiplicities (geometric, algebraic, partial). If a linearization has the same infinite eigenvalues and multiplicities, then it is a strong linearization.
- There are well-known compact characterizations of linearizations of matrix polys in terms of unimodular transformations.
- In contrast, the "linearizations" of REPs in the literature are rarely proved to have the same properties as the linearizations of PEPs.
- REPs are more difficult than PEPs: we need **"different types of linearizations in REPs" (sometimes weaker)** that in PEPs. Each type should have a different name and its properties should be clearly stated.
- **This talk is a step in this direction with a strong local emphasis.**

REPs different representations and (apparently) different “linearizations”

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Some previous works on "linearizations" of rational matrices

- Pioneering works on linearizations of rational matrices:
  1. P. Van Dooren and G. Verghese in late 70s & early 80s constructed pencils that have exactly the same structural data as any given rational matrix. The constructions require numerical computations.
  2. Y. Su and Z. Bai, SIMAX, 2011, construct a Frobenius-like linearization from a representation of $G(\lambda)$ as polynomial + state-space realization.
- This talk **extends in a local sense** results in Amparan, D, Marcaida, and Zaballa, *Strong linearizations of rational matrices*, SIMAX (2018).
- Another approach for defining (non-strong) linearizations of rational matrices can be found in Alam & Behera, SIMAX, 2016.
- NLEIGS linearizations (Güttel, Van Beeumen, Meerbergen, Michiels, SISC (2014)), Automatic Approximation of NEPs (Lietaert, Pérez, Vandereycken, Meerbergen, 2018), Padé Linearization (Bai), other approximations of NEPs (Saad, El-Guide, Miedlar, 2019)...

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“Old” versus “new” days for rational matrices

Very informally, after reading a number of “old” and “new” references on rational matrices, I share some personal feelings:

- **In the “old” days** (dominated by applications in Linear Systems and Control):
  1. Rational matrices were often transfer functions of time invariant linear systems.
  2. All the zeros and poles of the rational matrices were of interest.
  3. The structure at infinity of a rational matrix was important because of its physical meaning.
- **In the “new” days** (dominated by approximating NEPs):
  1. Rational matrices often arise in approximating a NEP in a certain region.
  2. Only the zeros that are not poles (eigenvalues?) in that region are of interest and the poles are often known.
  3. The structure at infinity of the rational matrix does not receive attention.

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- **In the “new” days** (dominated by approximating NEPs):
  1. Rational matrices often arise in approximating a NEP in a certain region.
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  3. The structure at infinity of the rational matrix does not receive attention.

“Old” versus “new” days for rational matrices

Very informally, after reading a number of “old” and “new” references on rational matrices, I share some personal feelings:

- **In the “old” days** (dominated by applications in Linear Systems and Control):
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1. Basics on rational matrices
2. Polynomial system matrices minimal in subsets of $\mathbb{C}$
3. Linearizations of rational matrices: in a set, at infinity, strong
4. Block full rank pencils: linearizations with empty state matrices
5. The NLEIGS "linearizations" as block full rank pencils
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Polynomial and strictly proper parts of a rational matrix. Reversal. Rank

- **Any rational matrix $G(\lambda)$ can be uniquely expressed as**

$$G(\lambda)=D(\lambda)+G_{sp}(\lambda)\in\mathbb{C}(\lambda)^{p\times m},$$

where

1. $D(\lambda)$ is a polynomial matrix (polynomial part of $G(\lambda)$), and
2. the rational matrix $G_{sp}(\lambda)$ is **strictly proper** (strictly proper part of $G(\lambda)$), i.e., $\lim_{\lambda\to\infty} G_{sp}(\lambda)=0$.

- We define the **g-reversal of $G(\lambda)$** as

$$\text{rev}_g\, G(\lambda)=\lambda^g\, G\left(\frac{1}{\lambda}\right).$$

Often $g=\deg(D)$ if $D(\lambda)\neq 0$ and $g=0$ otherwise, but not always.

- The **normal rank of $G(\lambda)$** is the size of the largest non-identically zero minor of $G(\lambda)$.

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Definition (finite zeros, finite poles, finite eigenvalues)

Given the **Smith-McMillan form** of a rational matrix $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$:

$$U(\lambda)G(\lambda)V(\lambda) = \text{diag}\left(\frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)}, \ldots, \frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)}, 0_{(p-r)\times(m-r)}\right),$$

where $U(\lambda), V(\lambda)$ are unimodular matrices and $\varepsilon_1(\lambda)|\cdots|\varepsilon_r(\lambda)$, $\psi_r(\lambda)|\cdots|\psi_1(\lambda)$ are monic scalar polynomials:

- The **finite zeros** of $G(\lambda)$ are the roots of the numerators $\varepsilon_i(\lambda)$ and the **finite poles** of $G(\lambda)$ are the roots of the denominators $\psi_i(\lambda)$.
- The **finite eigenvalues** of $G(\lambda)$ are the finite zeros that are not poles.

Definition (structural indices or partial multiplicities)

Given any $c \in \mathbb{C}$, one can write for each $i = 1, \ldots, r$,

$$\frac{\varepsilon_i(\lambda)}{\psi_i(\lambda)} = (\lambda - c)^{\sigma_i(c)} \frac{\widetilde{\varepsilon}_i(\lambda)}{\widetilde{\psi}_i(\lambda)}, \qquad \text{with } \widetilde{\varepsilon}_i(c) \neq 0,\ \widetilde{\psi}_i(c) \neq 0.$$

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Definition

The structural indices of $G(\lambda)$ at $\lambda = \infty$ are the structural indices of $G(1/\lambda)$ at $\lambda = 0$.

Definition (Rosenbrock, 1970)

Let $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$ be a rational matrix. The polynomial matrix

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{C}[\lambda]^{(n+p)\times(n+m)}$$

is a **polynomial system matrix** of $G(\lambda)$ if

$$G(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda).$$

If, in addition, $\begin{bmatrix} A(\lambda) \\ -C(\lambda) \end{bmatrix}$ and $\begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix}$ do not have finite eigenvalues (i.e., they have respectively full column and row ranks when evaluated in any $\lambda_0 \in \mathbb{C}$), then $P(\lambda)$ is a **minimal polynomial system matrix** of $G(\lambda)$.

Theorem (Rosenbrock, 1970)

Each rational matrix has infinitely many minimal polynomial system matrices.

The position of the state matrix $A(\lambda)$ is not important: it may be anywhere, the point is to take the Schur complement with respect to it.

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The position of the state matrix $A(\lambda)$ is not important: it may be anywhere, the point is to take the Schur complement with respect to it.

Consider the rational matrix

$$G(\lambda) = -B_0 + \lambda A_0 + \frac{B_1}{\lambda - \sigma_1} + \cdots + \frac{B_s}{\lambda - \sigma_s} \in \mathbb{C}(\lambda)^{p \times p},$$

$A_0, B_i \in \mathbb{C}^{p \times p}$ and $\sigma_i \neq \sigma_j$ if $i \neq j$, from Saad, El-Guide, Miedlar, 2019.
Then, these authors introduce the pencil,

$$P(\lambda) = \left[\begin{array}{cccc|c} (\lambda-\sigma_1)I & & & & I \\ & (\lambda-\sigma_2)I & & & I \\ & & \ddots & & \vdots \\ & & & (\lambda-\sigma_s)I & I \\ \hline -B_1 & -B_2 & \cdots & -B_s & \lambda A_0 - B_0 \end{array}\right]$$

which **is a polynomial system matrix of** $G(\lambda)$ of degree 1.

Moreover, $P(\lambda)$ **is minimal if and only if all the matrices** $B_1, \ldots, B_s$ **are nonsingular**.

Example of (minimal) polynomial system matrix

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Minimal polynomial system matrices contain the whole finite structure

Theorem (Rosenbrock, 1970)

If

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{C}[\lambda]^{(n+p)\times(n+m)}$$

is a **minimal** **polynomial system matrix** *of* $G(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$, *then:*

1. *The nontrivial (those different from* 1*) invariant polynomials of* $P(\lambda)$ *are the nontrivial numerators of the Smith-McMillan form of* $G(\lambda)$*.*
2. *The nontrivial invariant polynomials of* $A(\lambda)$ *are the nontrivial denominators of the Smith-McMillan form of* $G(\lambda)$*.*

...in plain words

- The finite eigenvalue structure of $P(\lambda)$ (resp. $A(\lambda)$) (including all types of multiplicities, geometric, algebraic, partial) coincides exactly with the finite zero (resp. pole) structure of $G(\lambda)$.

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Example of minimal polynomial system matrix (continuation)

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$A_0, B_i \in \mathbb{C}^{p\times p}$, $\sigma_i \neq \sigma_j$ if $i \neq j$ (Saad, El-Guide, Miedlar, 2019) and the pencil

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Corollary (from Rosenbrock, 1970)

If all the matrices $B_1, \ldots, B_s$ are nonsingular, then

- *The eigenvalues of the pencil $P(\lambda)$ are the finite zeros of $G(\lambda)$ with exactly the same partial multiplicities.*
- *The poles of $G(\lambda)$ are $\underbrace{\sigma_1, \ldots, \sigma_1}_{p}, \ldots, \underbrace{\sigma_s, \ldots, \sigma_s}_{p}$ each of them with partial multiplicity one.*

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Example: minimality is essential for having all the information

- $G(\lambda) = \dfrac{\lambda^2 - 1}{\lambda + 2} \in \mathbb{C}(\lambda)^{1\times 1}$ has one finite pole at -2 and two finite zeros at $+1$ and -1.
- Minimal polynomial system matrix of $G(\lambda)$:

$$P(\lambda) = \left[\begin{array}{c|c} \lambda + 2 & 1 \\ \hline -3 & \lambda - 2 \end{array}\right],$$

  since $G(\lambda) = (\lambda - 2) + 3\frac{1}{\lambda+2}$. Note that $\det P(\lambda) = \lambda^2 - 1$.
- Non-minimal polynomial system matrix of $G(\lambda)$ for any $a \in \mathbb{C}$:

$$\widehat{P}(\lambda) = \left[\begin{array}{cc|c} \lambda + a & 0 & 0 \\ 0 & \lambda + 2 & 1 \\ \hline 0 & -3 & \lambda - 2 \end{array}\right],$$

  and since $\det \widehat{P}(\lambda) = (\lambda + a)(\lambda^2 - 1)$, $\widehat{P}(\lambda)$ has an spurious eigenvalue and $\widehat{A}(\lambda)$ an spurious pole.
- **Can we relax minimality and guarantee that we have all the information that is needed in REPs or in NEPs?**

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be a polynomial system matrix of $G(\lambda)$. If $\begin{bmatrix} A(\lambda_0) \\ -C(\lambda_0) \end{bmatrix}$ and $\begin{bmatrix} A(\lambda_0) & B(\lambda_0) \end{bmatrix}$ have full rank n for all $\lambda_0 \in \Sigma \subseteq \mathbb{C}$, then $P(\lambda)$ is a **minimal polynomial system matrix in Σ** of $G(\lambda)$.

Theorem (D., Marcaida, Quintana, Van Dooren, 2019)

If $P(\lambda)$ is a minimal polynomial system matrix in Σ of $G(\lambda)$, then

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Minimal polynomial system matrices in a set

Definition (D., Marcaida, Quintana, Van Dooren, 2019)

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$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{C}[\lambda]^{(n+p)\times(n+m)}$$

be a polynomial system matrix of $G(\lambda)$. If $\begin{bmatrix} A(\lambda_0) \\ -C(\lambda_0) \end{bmatrix}$ and $\begin{bmatrix} A(\lambda_0) & B(\lambda_0) \end{bmatrix}$ have full rank n for all $\lambda_0 \in \Sigma \subseteq \mathbb{C}$, then $P(\lambda)$ is a **minimal polynomial system matrix in Σ** of $G(\lambda)$.

Theorem (D., Marcaida, Quintana, Van Dooren, 2019)

If $P(\lambda)$ is a minimal polynomial system matrix in Σ of $G(\lambda)$, then

- *The finite eigenvalue structure in Σ of $P(\lambda)$ (including all types of multiplicities, geometric, algebraic, partial) coincides exactly with the finite zero structure in Σ of $G(\lambda)$.*
- *The finite eigenvalue structure in Σ of $A(\lambda)$ (including all types of multiplicities, geometric, algebraic, partial) coincides exactly with the finite pole structure in Σ of $G(\lambda)$.*

Consider again the rational matrix

$$G(\lambda) = -B_0 + \lambda A_0 + \frac{B_1}{\lambda - \sigma_1} + \cdots + \frac{B_s}{\lambda - \sigma_s} \in \mathbb{C}(\lambda)^{p\times p},$$

$A_0, B_i \in \mathbb{C}^{p\times p}$, $\sigma_i \neq \sigma_j$ if $i \neq j$ (Saad, El-Guide, Miedlar, 2019), the pencil

$$P(\lambda) = \left[\begin{array}{cccc|c} (\lambda-\sigma_1)I & & & & I \\ & (\lambda-\sigma_2)I & & & I \\ & & \ddots & & \vdots \\ & & & (\lambda-\sigma_s)I & I \\ \hline -B_1 & -B_2 & \cdots & -B_s & \lambda A_0 - B_0 \end{array}\right],$$

and **the set** $\Sigma = \mathbb{C} \setminus \{\sigma_1, \ldots, \sigma_s\}$.

Corollary

Then, without any assumption,

- *$P(\lambda)$ is a minimal polynomial system matrix in Σ of $G(\lambda)$.*
- *The eigenvalues of the pencil $P(\lambda)$ in Σ coincide with the finite zeros of $G(\lambda)$ in Σ with exactly the same partial multiplicities.*

Example of minimal polynomial system matrix in a set

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Minimal polynomial system matrices at infinity

Definition (D., Marcaida, Quintana, Van Dooren, 2019)

Let $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$ be a rational matrix and

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{C}[\lambda]^{(n+p)\times(n+m)}$$

be a polynomial system matrix of $G(\lambda)$ with degree d. If $\text{rev}_d P(\lambda)$ is minimal at 0, we say that $P(\lambda)$ **is a minimal polynomial system matrix at ∞ of $G(\lambda)$**.

Theorem (D., Marcaida, Quintana, Van Dooren, 2019)

If $P(\lambda)$ is a minimal poly. system matrix at ∞ of $G(\lambda)$ with normal rank r,

- *$e_1 \leq \cdots \leq e_s$ are the (nonzero) partial multiplicities of $\text{rev}_d A(\lambda)$ at 0, and*
- *$\widetilde{e}_1 \leq \cdots \leq \widetilde{e}_u$ are the (nonzero) partial multiplicities of $\text{rev}_d P(\lambda)$ at 0,*

then

$$(q_1, q_2, \ldots, q_r) = (-e_s, -e_{s-1}, \ldots, -e_1, \underbrace{0, \ldots, 0}_{r-s-u}, \widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_u) - (d, d, \ldots, d)$$

are the structural indices at infinity of $G(\lambda)$.

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Consider again the same rational matrix and pencil

$$G(\lambda) = -B_0 + \lambda A_0 + \frac{B_1}{\lambda - \sigma_1} + \cdots + \frac{B_s}{\lambda - \sigma_s} \in \mathbb{C}(\lambda)^{p\times p},$$

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Then,

$$\mathrm{rev}_1 P(\lambda) = \left[\begin{array}{cccc|c} (1-\lambda\sigma_1)I & & & & \lambda I \\ & (1-\lambda\sigma_2)I & & & \lambda I \\ & & \ddots & & \vdots \\ & & & (1-\lambda\sigma_s)I & \lambda I \\ \hline -\lambda B_1 & -\lambda B_2 & \cdots & -\lambda B_s & A_0 - \lambda B_0 \end{array}\right]$$

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Example of minimal polynomial system matrix at infinity

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and $P(\lambda)$ is a minimal polynomial system matrix at ∞ of $G(\lambda)$.

Outline

1. Basics on rational matrices
2. Polynomial system matrices minimal in subsets of $\mathbb{C}$
3. **Linearizations of rational matrices: in a set, at infinity, strong**
4. Block full rank pencils: linearizations with empty state matrices
5. The NLEIGS "linearizations" as block full rank pencils
6. Conclusions

Definition (D., Marcaida, Quintana, Van Dooren, 2019)

A **linearization of** $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$ **in** $\Sigma \subseteq \mathbb{C}$ **is a matrix pencil**

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{C}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

such that:

(a) $L(\lambda)$ is a **minimal polynomial system matrix** in Σ of

$$\widehat{G}(\lambda) = (D_1\lambda + D_0) + (C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0),$$

(b) and, there exist rational matrices invertible in Σ, $W_1(\lambda)$, $W_2(\lambda)$ such that

$$W_1(\lambda)\ \mathrm{diag}(G(\lambda), I_s)\, W_2(\lambda) = \widehat{G}(\lambda).$$

Remark: If $\Sigma = \mathbb{C}$, then a linearization in $\mathbb{C}$ is called just a **linearization** and it was defined by Amparan, D, Marcaida and Zaballa, SIMAX, 2018.

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of a rational matrix in a set Σ are allowed, are important in applications, and make the previous definition very general:

1. $\widehat{G}(\lambda) = G(\lambda)$, then condition **(b)** can be removed, because it is automatically satisfied with $W_1(\lambda) = I_p$, $W_2(\lambda) = I_m$ and $s = 0$.
2. $n = 0$, i.e., empty state matrix, then condition **(a)** can be removed and $\widehat{G}(\lambda) = D_1\lambda + D_0 = L(\lambda)$.

Remarks:

- In the second case, we use the expression "$L(\lambda)$ is a linearization of $G(\lambda)$ in Σ with empty state matrix".
- In general, when one says that a pencil is a linearization, one has to specify which is the considered state matrix (as for any other polynomial system matrix).
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Consider again the rational matrix

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Examples of linearizations in a set (II)

- The linearization in the previous slide can be seen as a particular case of the next one.
- Given the rational matrix:

$$G(\lambda) = D_d\lambda^d + \cdots + D_1\lambda + D_0 + C(\lambda I_n - A)^{-1}B \in \mathbb{C}(\lambda)^{p\times m},$$

- Su & Bai (2011) introduced the Frobenius-like companion pencil

$$L(\lambda) = \left[\begin{array}{c|ccccc} \lambda I_n - A & 0 & 0 & \cdots & 0 & B \\ \hline -C & \lambda D_d + D_{d-1} & D_{d-2} & \cdots & D_1 & D_0 \\ 0 & -I_m & \lambda I_m & & & \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & \ddots & \lambda I_m & \\ 0 & & & & -I_m & \lambda I_m \end{array}\right],$$

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Linearizations in Σ contain the whole finite structure in Σ

Theorem (D., Marcaida, Quintana, Van Dooren, 2019)

If

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{C}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

is a linearization of $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$ in $\Sigma \subseteq \mathbb{C}$, then:

- *The finite eigenvalue structure in Σ of $L(\lambda)$ (including all types of multiplicities, geometric, algebraic, partial) coincides exactly with the finite zero structure in Σ of $G(\lambda)$.*
- *The finite eigenvalue structure in Σ of $A_1\lambda + A_0$ (including all types of multiplicities, geometric, algebraic, partial) coincides exactly with the finite pole structure in Σ of $G(\lambda)$.*

Definition (D., Marcaida, Quintana, Van Dooren, 2019)

A matrix pencil with degree 1

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{C}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

is a **linearization at infinity of grade g of** $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$ if $\text{rev}_1 L(\lambda)$ is a linearization of $\text{rev}_g G(\lambda)$ at 0.

Consider again the rational matrix

$$G(\lambda) = -B_0 + \lambda A_0 + \frac{B_1}{\lambda - \sigma_1} + \cdots + \frac{B_s}{\lambda - \sigma_s} \in \mathbb{C}(\lambda)^{p\times p},$$

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Then, without any assumption, $P(\lambda)$ is a linearization of $G(\lambda)$ at ∞ of grade 1, with $\widehat{G}(\lambda) = G(\lambda)$.

Consider again the rational matrix

$$G(\lambda) = -B_0 + \lambda A_0 + \frac{B_1}{\lambda - \sigma_1} + \cdots + \frac{B_s}{\lambda - \sigma_s} \in \mathbb{C}(\lambda)^{p\times p},$$

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Linearizations at infinity contain the whole structure at infinity

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is a **linearization at infinity of grade g of $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$**, *the normal rank of $G(\lambda)$ is r, and*

- $e_1 \leq \cdots \leq e_t$ *are the (nonzero) partial multiplicities of* $\text{rev}_1(A_1\lambda + A_0)$ *at* 0, *and*
- $\widetilde{e}_1 \leq \cdots \leq \widetilde{e}_u$ *are the (nonzero) partial multiplicities of* $\text{rev}_1 L(\lambda)$ *at* 0,

then

$$(q_1, q_2, \ldots, q_r) = (-e_t, -e_{t-1}, \ldots, -e_1, \underbrace{0, \ldots, 0}_{r-t-u}, \widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_u) - (g, g, \ldots, g)$$

are the structural indices at infinity of $G(\lambda)$.

Definition (D., Marcaida, Quintana, Van Dooren, 2019)

A **g-strong linearization of $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$ is a matrix pencil** $L(\lambda)$ such that

1. $L(\lambda)$ is a linearization of $G(\lambda)$ in $\mathbb{C}$, and
2. $L(\lambda)$ is a linearization at infinity of grade g of $G(\lambda)$.

Corollary

g-strong linearizations of a rational matrix $G(\lambda)$ contain the whole finite and infinite zero and pole structures of $G(\lambda)$.

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Corollary

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Examples of g-strong linearizations (I)

Consider once again the rational matrix

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$A_0, B_i \in \mathbb{C}^{p\times p}$ and $\sigma_i \neq \sigma_j$ if $i \neq j$, from Saad, El-Guide, Miedlar, 2019, and the pencil

$$P(\lambda) = \left[\begin{array}{cccc|c} (\lambda-\sigma_1)I & & & & I \\ & (\lambda-\sigma_2)I & & & I \\ & & \ddots & & \vdots \\ & & & (\lambda-\sigma_s)I & I \\ \hline -B_1 & -B_2 & \cdots & -B_s & \lambda A_0 - B_0 \end{array}\right].$$

Then, $P(\lambda)$ is a 1-strong linearization of $G(\lambda)$ if and only if all the matrices $B_1, \ldots, B_s$ are nonsingular.

Examples of g-strong linearizations (I)

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Let us consider the rational matrix

$$G(\lambda) = D_d\lambda^d + \cdots + D_1\lambda + D_0 + C(\lambda I_n - A)^{-1}B \in \mathbb{C}(\lambda)^{p\times m}$$

and the Frobenius-like companion pencil introduced by Su & Bai (2011)

$$L(\lambda) = \left[\begin{array}{c|cccccc} \lambda I_n - A & 0 & 0 & \cdots & 0 & B \\ \hline -C & \lambda D_d + D_{d-1} & D_{d-2} & \cdots & D_1 & D_0 \\ 0 & -I_m & \lambda I_m & & & \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & \ddots & \lambda I_m & \\ 0 & & & & -I_m & \lambda I_m \end{array}\right].$$

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Examples of g-strong linearizations (III)

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Outline

1. Basics on rational matrices
2. Polynomial system matrices minimal in subsets of $\mathbb{C}$
3. Linearizations of rational matrices: in a set, at infinity, strong
4. **Block full rank pencils: linearizations with empty state matrices**
5. The NLEIGS "linearizations" as block full rank pencils
6. Conclusions

Why to consider linearizations with empty state matrices?

- In modern applications of REPs as approximations of NEPs, the poles of the REPs are often chosen to construct a “good approximation” and thus they are known.
- There is no need to compute them (in contrast with “classic” applications of REPs in linear system theory and control).
- In addition, the eigenvalues of REPs (and NEPs) are not poles by definition.
- Therefore, it makes sense to look for simple criteria that guarantee that a pencil is a linearization of a rational matrix $G(\lambda)$ in a set $\Sigma \subseteq \mathbb{C}$ which does not contain poles.
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Definition (D., Marcaida, Quintana, Van Dooren, 2019)

A **block full rank pencil** is a linear polynomial matrix with the following structure

$$L(\lambda) = \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}$$

where $K_1(\lambda)$ **and** $K_2(\lambda)$ **are pencils with full row normal rank**.

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The position of $M(\lambda)$ is not relevant and this definition includes the cases

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Block full rank pencils include as particular cases block minimal bases pencils introduced by D., Lawrence, Pérez, Van Dooren, Numer. Math., 2018, which are linerizations of matrix polynomials.

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now partitioned in a different way.

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A rational matrix $R(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$ **has full row rank in $\Sigma \subseteq \mathbb{C}$ if, for all $\lambda_0 \in \Sigma$**,

1. $R(\lambda_0) \in \mathbb{C}^{p\times m}$, i.e., $R(\lambda)$ is defined or bounded at λ_0, and
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Observe that this implies that $R(\lambda)$ has no poles in Σ.

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Two rational matrices $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$ and $H(\lambda) \in \mathbb{C}(\lambda)^{q\times m}$ are **dual rational bases** if

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Block full rank pencils are linearizations of certain matrices in some sets

Theorem (D., Marcaida, Quintana, Van Dooren, 2019)

Let $L(\lambda) = \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}$ *be a block full rank pencil,*

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If $L(\lambda) = \begin{bmatrix} M(\lambda) \\ K_1(\lambda) \end{bmatrix}$, then take $N_2(\lambda) = I$.

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Theorem (D., Marcaida, Quintana, Van Dooren, 2019)

Let $L(\lambda)$ be a block full rank pencil

$$L(\lambda) = \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix},$$

with degree 1 and let $N_1(\lambda)$ and $N_2(\lambda)$ be rational bases dual to $K_1(\lambda)$ and $K_2(\lambda)$, respectively. If, for $i = 1, 2$,

- $\mathrm{rev}_1 K_i(\lambda)$ *has full row rank at zero, and*
- *there exists an integer number t_i such that $\mathrm{rev}_{t_i} N_i(\lambda)$ has full row rank at zero,*

then $L(\lambda)$ is a **linearization with empty state matrix** *of the rational matrix*

$$G(\lambda) = N_2(\lambda) M(\lambda) N_1(\lambda)^T \qquad \text{at } \infty \text{ of grade } 1 + t_1 + t_2.$$

Remark: this can be applied, of course, to prove that our favorite pencil is a linearization with empty state matrix of our favorite rational matrix at ∞ of grade 1.

Block full rank pencils may be linearizations at infinity

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- As far as I know, all the linearizations of rational matrices available in the "modern" literature can be seen as block full rank pencils.
- Sometimes, some preliminary permutations are needed to identify correctly the full rank blocks.
- The results I have just presented allow to prove very easily and fully rigurously,
- that block full rank pencils contain the complete zero structure (finite and infinite) of the corresponding rational matrices in adequate sets,
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4. Block full rank pencils: linearizations with empty state matrices
5. **The NLEIGS “linearizations” as block full rank pencils**
6. Conclusions

NLEIGS approximation

In the influential paper,

- Güttel, Van Beeumen, Meerbergen, Michiels, NLEIGS: a class of fully rational Krylov methods for nonlinear eigenvalue problems, SISC (2014),

a NEP

$$T(\lambda_0)v = 0, \quad \lambda_0 \in \mathbb{C},\ v \in \mathbb{C}^m$$

is approximated in a certain region via Hermite's rational interpolation by a rational matrix of the type

$$Q_N(\lambda) = b_0(\lambda)D_0 + b_1(\lambda)D_1 + \cdots + b_N(\lambda)D_N,$$

with $D_j \in \mathbb{C}^{m\times m}$ and

$$b_0(\lambda) = \frac{1}{\beta_0}, \quad b_j(\lambda) = \frac{1}{\beta_0}\prod_{k=1}^{j}\frac{\lambda - \sigma_{k-1}}{\beta_k(1-\lambda/\xi_k)}, \quad j = 1,\dots,N,$$

a sequence of rational scalar functions. The poles ξ_i are all distinct from the nodes σ_j, some poles ξ_i can be infinite, and β_i are nonzero scaling parameters.

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Once the rational matrix

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is obtained, Güttel, Van Beeumen, Meerbergen, Michiels (2014) construct the following pencil associated to $Q_N(\lambda)$

$$L_N(\lambda) = \begin{bmatrix} \left(1 - \frac{\lambda}{\xi_N}\right) D_0 & \left(1 - \frac{\lambda}{\xi_N}\right) D_1 & \cdots & \left(1 - \frac{\lambda}{\xi_N}\right) D_{N-2} & \left(1 - \frac{\lambda}{\xi_N}\right) D_{N-1} + \frac{\lambda - \sigma_{N-1}}{\beta_N} D_N \\ (\sigma_0 - \lambda) I_m & \beta_1 (1 - \frac{\lambda}{\xi_1}) I_m & & & \\ & \ddots & \ddots & & \\ & & & (\sigma_{N-2} - \lambda) I_m & \beta_{N-1} (1 - \frac{\lambda}{\xi_{N-1}}) I_m \end{bmatrix}$$

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corresponding, in the block full rank notation, to $L(\lambda) = \begin{bmatrix} M_N(\lambda) \\ K_N(\lambda) \end{bmatrix}$.

Moreover,

- $K_N(\lambda)$ **has full row rank in** $\mathbb{C}$, since $\xi_i \neq \sigma_j$,
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As a consequence, the NLEIGS "linearizations" satisfy

Theorem (D., Marcaida, Quintana, Van Dooren, 2019)

- $L_N(\lambda)$ *is a linearization with empty state matrix of*

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Other partitions of the NLEIGS "linearizations"

give information on the finite poles of $Q_N(\lambda)$, but require much more effort and additional assumptions. Thus, if one considers the partition

$$L_N(\lambda) = \left[\begin{array}{c|ccccc} \left(1-\frac{\lambda}{\xi_N}\right)D_0 & \left(1-\frac{\lambda}{\xi_N}\right)D_1 & \dots & \left(1-\frac{\lambda}{\xi_N}\right)D_{N-2} & \left(1-\frac{\lambda}{\xi_N}\right)D_{N-1} + \frac{\lambda-\sigma_{N-1}}{\beta_N}D_N \\ \hline (\sigma_0-\lambda)I_m & \beta_1(1-\frac{\lambda}{\xi_1})I_m & & & \\ & \ddots & \ddots & & \\ & & & (\sigma_{N-2}-\lambda)I_m & \beta_{N-1}(1-\frac{\lambda}{\xi_{N-1}})I_m \end{array}\right]$$

$$=: \left[\begin{array}{c|c} D(\lambda) & -C(\lambda) \\ \hline B(\lambda) & A(\lambda) \end{array}\right],$$

then, it can be proved that $L_N(\lambda)$ is a polynomial system matrix with state matrix $A(\lambda)$ of

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NLEIGS "linearizations" with another partition

Theorem (D., Marcaida, Quintana, Van Dooren, 2019)

If $L_N(\lambda)$ is viewed with the partition in the previous page and the rational matrix

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is such that the constant matrix $R_N(\xi_i)$ is nonsingular for every finite $\xi_i \in \{\xi_1, \xi_2, \ldots, \xi_{N-1}\}$, then

$L_N(\lambda)$ is a linearization with state matrix $A(\lambda)$ of $Q_N(\lambda)$ in $\mathbb{C}$, if $\xi_N = \infty$, or in $\mathbb{C} \setminus \{\xi_N\}$ otherwise.

Remark: Thus, under these assumptions, all the information about the poles of $Q_N(\lambda)$ is in the eigenvalue structure of the state matrix $A(\lambda)$.

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- A new theory of “local”, i.e., in certain sets, polynomial system matrices of rational matrices has been presented, extending classical global results by Rosenbrock.
- This theory has been applied to present new definitions of “local” linearizations of rational matrices, and to prove that such linearizations are meaningful.
- These new definitions and theory have been applied to give a complete theoretical foundation of some “linearizations” of rational matrices that have been used recently by different authors in the numerical solution of NEPs.
- The new definitions and theory can be applied to all the “linearizations” that have been published in the “modern” literature and that we know.

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