Beyond matrix eigenvalues

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- Universidad Católica del Norte (Chile): Javier González-Pizarro.
- University of Montana (USA): Javier Pérez.
- Örebro University (Sweden): Andrii Dmytryshyn.
- Universidad del País Vasco/Euskal Herriko Unibertsitatea: Agurtzane Amparan, Silvia Marcaida, Ion Zaballa.
- Universidad Pontificia Comillas (Spain): Luis M. Anguas.
- Universidade do Porto (Portugal): Susana Furtado.
- Western Michigan University (USA): Steve Mackey.

The basic eigenvalue problem (BEP). Given $A \in \mathbb{C}^{n\times n}$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that

$$Av = \lambda v \iff \boxed{(\lambda I_n - A)\,v = 0}$$

- It arises in many applications. For instance, if one looks for solutions of the form $y(t) = e^{\lambda t} v$ in the system of first order ODEs

$$\frac{dy(t)}{dt} = Ay(t) \Longrightarrow \lambda v = Av$$

- There are stable algorithms for its numerical solution.
- QR algorithm (Francis-Kublanovskaya 1961) for small to medium size dense matrices.
- Arnoldi method (1951) equipped with automatic implicit re-starting techniques (Sorensen 1992, Stewart 2002) for large-scale problems and sparse matrices.
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Different classes of regular matrix eigenvalue problems (I)

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The POLYNOMIAL eigenvalue problem (PEP). Given $P_0, \ldots, P_d \in \mathbb{C}^{n\times n}$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that

$$(P_d\lambda^d + \cdots + P_1\lambda + P_0)v = 0,$$

under the regularity assumption $\det(P_d z^d + \cdots + P_1 z + P_0) \not\equiv 0$.

- It arises in many applications. For instance, if one looks for solutions $y(t) = e^{\lambda t}v$ in the system of dth order ALGEBRAIC-ODEs

$$P_d \frac{d^d y(t)}{dt^d} + \cdots + P_1 \frac{dy(t)}{dt} + P_0 y(t) = 0 \Longrightarrow (P_d\lambda^d + \cdots + P_1\lambda + P_0)v = 0$$

- There are stable (? debatable) algorithms for its numerical solution.
- Easy to use software for small to medium size dense matrices: MATLAB's commands `polyeig(P0,P1,...,Pd)` (Van Dooren, 1979).
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It is important to emphasize that the theories of GEPs and PEPs are very different:

- The complete eigenstructure of the linear matrix polynomial $\lambda A - B$ is revealed (even in the singular case) by the Kronecker canonical form (1890) obtained by multiplications by constant invertible matrices:

$$\lambda A - B \longrightarrow U(\lambda A - B)V = \lambda UAV - UBV\,.$$

- In addition, the complete eigenstructure can be determined by using unitary matrices via the staircase form (Van Dooren, 1979).
- The use of constant matrices on a polynomial matrix of degree larger than one is not sufficient for reveling its complete eigenstructure,
- unless we transform the problem into a larger one,
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Intrinsic differences between GEPs and PEPs of degree larger than 1

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The RATIONAL eigenvalue problem (REP). Given a rational matrix

$$G(z) \in \mathbb{C}(z)^{n \times n},$$

i.e., such that $G(z)_{ij}$ is a scalar rational function of $z \in \mathbb{C}$, for $1 \leq i, j \leq n$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that λ is not a pole of any $G(z)_{ij}$ and

$$\boxed{G(\lambda)v = 0},$$

under the regularity assumption $\det(G(z)) \not\equiv 0$.

- It arises in applications either directly (multivariable system theory and control theory) or as an approximation.
- There are algorithms for its numerical solution (stability analysis open).
- For small to medium size dense matrices via linearizations (Van Dooren, 1979-1981 - Su-Bai, 2011).
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The NONLINEAR eigenvalue problem (NEP). Given a non-empty open set $\Omega \subseteq \mathbb{C}$ and a holomorphic matrix-valued function

$$\begin{array}{rccc} F: & \Omega & \to & \mathbb{C}^{n\times n} \\ & z & \mapsto & F(z), \end{array}$$

compute scalars $\lambda \in \Omega$ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that

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- It arises in applications. For instance, if one looks for solutions $y(t) = e^{\lambda t}v$ in the system of first order DELAYED differential equations

$$\frac{dy(t)}{dt} + Ay(t) + By(t-1) = 0 \Longrightarrow (\lambda I_n + A + Be^{-\lambda})v = 0$$

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compute scalars $\lambda \in \Omega$ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that

$$\boxed{F(\lambda)v = 0},$$

under the regularity assumption $\det(F(z)) \not\equiv 0$.

- It arises in applications. For instance, if one looks for solutions $y(t) = e^{\lambda t}v$ in the system of first order DELAYED differential equations

$$\frac{dy(t)}{dt} + Ay(t) + By(t-1) = 0 \Longrightarrow (\lambda I_n + A + Be^{-\lambda})v = 0$$

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- There are different algorithms for the numerical solution of NEP.
- One of the most important family of algorithms is based on the following two step strategy
  1. Approximate $F(z)$ by a rational matrix $G(z)$ with poles outside Ω.
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- There is software available for NEPs developed by the authors of some key papers that follow the previous strategy:
  1. NLEIGS (Güttel, Van Beeumen, Meerbergen, Michiels, 2014) (not easy to use),
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Different classes of regular matrix eigenvalue problems (V) (continued)

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1. **BEP**: $(\lambda I_n - A)\, v = 0$
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- **1st KEY IDEA: ALL THESE PROBLEMS CAN BE SOLVED BY TRANSFORMING THE PROBLEM INTO A (much) LARGER GEP** $\rightarrow$ **LINEARIZATION**.
- For **PEPs** and **REPs**, this transformation is mathematically **exact!!!!!**.
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4. Rational vector spaces: minimal bases and indices
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4th KEY IDEA: current applications often lead to very short "split forms"

- Every matrix $F(z)$ defining an $n \times n$ PEP, REP or NEP can be written in **"split form"** with at most n^2 terms, i.e.,

$$F(z) = f_1(z)\, C_1 + f_2(z)\, C_2 + \cdots + f_\ell(z)\, C_\ell,$$

where $f_i : \mathbb{C} \to \mathbb{C}$, $C_i \in \mathbb{C}^{n \times n}$, and $\ell \leq n^2$.

- This result is, of course, a triviality,

$$\begin{bmatrix} e^z & z^2+1 \\ \frac{1}{z+1} & \sin(z) \end{bmatrix} = e^z \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (z^2+1) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \frac{1}{z+1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \sin(z) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

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How large is the degree of $P(z) = P_d z^d + \cdots + P_1 z + P_0$ in practical PEPs?

- In most direct applications coming from vibrational problems in mechanics $\mathbf{d = 2}$: **the quadratic eigenvalue problem (QEP)**

$$(z^2 M + zC + K)v = 0,$$

while $M, C, K \in \mathbb{C}^{n \times n}$ with $\mathbf{n = 10^2, 10^3, 10^4, 10^5, 10^6, \ldots}$.

- Betcke, Higham, Mehrmann, Schröder, Tisseur, *"NLEVP: A Collection of Nonlinear Eigenvalue Problems"*, (2013) reports on applications with
  - $\mathbf{d = 4}$: Hamiltonian control problems, homography-based method for calibrating a central cadioptric vision system, spatial stability analysis of the Orr-Sommerfeld equation, and finite element solution of the equation for the modes of a planar waveguide using piecewise linear basis functions.
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Change of notation $z \to \lambda$

- Loaded elastic string (Betcke et al., NLEVP-collection, (2013)):

$$G(\lambda) = (A + E) - \lambda B + \frac{\sigma}{\lambda - \sigma} E.$$

Only 3 functions (terms) in split form, $A, B, E \in \mathbb{R}^{n \times n}$. $n \geq 10^2$ large.

- Damped vibration of a viscoelastic structure (Mehrmann & Voss, (2004)):

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Only $k + 2$ functions in split form, M, K positive definite, $n = 10704$ large, $k \approx 10$.

A few examples of "direct" applied REPs

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- NLEIGS-REPs coming from linear rational interpolation of NEPs (Güttel, Van Beeumen, Meerbergen, Michiels (2014)):

$$Q_N(\lambda) = b_0(\lambda)D_0 + b_1(\lambda)D_1 + \cdots + b_N(\lambda)D_N,$$

with $D_j \in \mathbb{C}^{n\times n}$,

$$b_0(\lambda) = \frac{1}{\beta_0}, \qquad b_j(\lambda) = \frac{1}{\beta_0}\prod_{k=1}^{j} \frac{\lambda - \sigma_{k-1}}{\beta_k(1-\lambda/\xi_k)},$$

$j = 1, \ldots, N$, rational scalar functions, with the "poles" ξ_i different from zero and all distinct from the nodes σ_j. $N \leq 140$, $n = 16281$.

“Approximating” REPs have been used to approximate...

among many others, **the following NEPs**:

- **The radio-frequency gun cavity problem**:

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  where M, K, W_1, W_2 are real sparse symmetric 9956×9956 matrices (only 4 scalar functions involved in split form).

- **Bound states in semiconductor devices problems**:

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  where $H, S_j \in \mathbb{R}^{16281 \times 16281}$, H symmetric and the matrices S_j have low rank (only 83 scalar functions involved in split form).

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1. The “flavor” of applied PEPs, REPs, NEPs: examples
2. **Additional “difficulties” of GEPs, PEPs, and REPs over BEPs**
3. Linearizations of matrix polynomials
4. Rational vector spaces: minimal bases and indices
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GEPs-PEPs-REPs have more spectral “structural” data than BEPs

1. **BEP**: $(\lambda I_n - A)\,v = 0$
2. **GEP**: $(\lambda B - A)\,v = 0$
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- So far, we have only considered **finite eigenvalues**, but
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Example: Infinite eigenvalues in regular PEPs

Let ϵ be a small parameter and consider the quadratic matrix polynomial

$$P(\lambda) = \begin{bmatrix} (\lambda-1)(\lambda-2) & 0 \\ 0 & \lambda(\epsilon\lambda - 1) \end{bmatrix} = \lambda^2 \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix} + \lambda \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

- If $\epsilon \neq 0$, then the eigenvalues are $\{1, 2, 0, 1/\epsilon\}$, (very large if $|\epsilon| \ll 1$).
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Remarks:

- Infinite eigenvalues are related to the presence of **algebraic constraints in ALGEBRAIC-ODES**, i.e., singularity or rank deficiency of the highest degree matrix coefficient.
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Singular PEPs (I)

- An additional step of difficulty is that PEPs can be **singular**, which happens when

$$P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0$$

is either **rectangular or square with** $\det P(\lambda) \equiv 0$, i.e., zero for all λ.

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- **In addition to finite and infinite eigenvalues, singular matrix polynomials have** other "interesting numbers" attached to them called **minimal indices**.
- Recall that eigenvalues are related to the existence of nontrivial **null spaces**. For instance, $\mathcal{N}_r(\lambda_0 I_n - A) \neq \{0\}$ in BEPs.
- **Minimal indices** are related to the fact that a singular $m \times n$ matrix polynomial $P(\lambda)$ has non-trivial left and/or right null-spaces over the field $\mathbb{C}(\lambda)$ of rational functions:

$$\begin{aligned} \mathcal{N}_\ell(P) &:= \left\{ y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times m} \,:\, y(\lambda)^T P(\lambda) \equiv 0^T \right\}, \\ \mathcal{N}_r(P) &:= \left\{ x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} \,:\, P(\lambda) x(\lambda) \equiv 0 \right\}. \end{aligned}$$

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- rank$_{\mathbb{C}} P(0) = 3 \Longrightarrow \lambda = 0$ is an eigenvalue (partial multiplicities $0, 0, 0, 1$).
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Example about the complete eigenstructure of matrix polynomials (2)

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- Bases of **right and left rational null spaces** of $P(\lambda)$:

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- There are many other polynomial bases but each of these ones have **minimal sum of the degrees of its vectors**.
- Thus, **right minimal indices of $P(\lambda)$ are $\{3, 2\}$** and **left minimal indices of $P(\lambda)$ are $\{2, 0\}$**.

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As said before, the most reliable methods for solving numerically PEPs are based on the concept of linearization.

Definition

- A **linear polynomial matrix (or matrix pencil)** $L(\lambda)$ is a **linearization** of $P(\lambda) = P_d \lambda^d + \cdots + P_1 \lambda + P_0$ if there exist **unimodular** polynomial matrices $U(\lambda), V(\lambda)$ such that

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A matrix pencil $L(\lambda)$ is a linearization *of a polynomial matrix $P(\lambda)$* **if and only if**

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The most famous strong linearization (I)

The classical **Frobenius companion form** of the $m \times n$ matrix polynomial

$$P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0$$

is

$$C_1(\lambda) := \begin{bmatrix} \lambda P_d + P_{d-1} & P_{d-2} & \cdots & P_1 & P_0 \\ -I_n & \lambda I_n & & & \\ & \ddots & \ddots & & \\ & & \ddots & \lambda I_n & \\ & & & -I_n & \lambda I_n \end{bmatrix} \in \mathbb{C}[\lambda]^{(m+n(d-1))\times nd}$$

Additional property of $C_1(\lambda)$: Example of strong linearization whose right (resp. left) minimal indices allow us to recover the ones of the polynomial via addition of a constant.

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The most famous strong linearization (II)

Theorem (recovery of eigenvectors from $C_1(\lambda)$)

Let $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0$ *be a* **regular** *matrix polynomial,* $\lambda_0 \in \mathbb{C}$ **be a finite eigenvalue of** $P(\lambda)$*, and* $C_1(\lambda)$ *be the Frobenius companion form of* $P(\lambda)$*. Then,* **any eigenvector** v **of** $C_1(\lambda)$ **associated to** λ_0 **has the form**

$$v = \begin{bmatrix} \lambda_0^{d-1}\, x \\ \vdots \\ \lambda_0\, x \\ x \end{bmatrix}$$

with x an eigenvector of $P(\lambda)$ associated to λ_0.

- $C_1(\lambda)$ is one (among many others) strong linearization of $P(\lambda)$ that allows us to recover without computational cost the eigenvectors of the polynomial from those of the linearization.

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There are many other strong linearizations of PEPs (I)

- Since 2006 (Mackey, Mackey, Mehl, Mehrmann), many "new" strong linearizations of matrix polynomials have been developed by many authors all around the world
- which also allow us to recover minimal indices and eigenvectors of PEPs without any computational cost. Explosion of new linearizations.
- One relevant motivation for developing new classes of linearizations is to preserve structures appearing in applications, which is important for saving operations in algorithms and for preserving properties of the eigenvalues in floating point arithmetic.
- For instance, if $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0$ is Hermitian, i.e., it has Hermitian coefficients, **the Frobenius companion form is not!!**

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$$\widetilde{L}(\lambda) = \begin{bmatrix} \lambda P_1 + P_0 & \lambda I_n & & & & & 0 \\ \lambda I_n & 0 & I_n & & & & \\ & I_n & \lambda P_3 + P_2 & \lambda I_n & & & \\ & & \lambda I_n & 0 & I_n & & \\ & & & I_n & \lambda P_5 + P_4 & \lambda I_n & \\ & & & & \lambda I_n & 0 & I_n \\ 0 & & & & & I_n & \lambda P_7 + P_6 \end{bmatrix},$$

  is a **Hermitian strong linearization** of the $n \times n$ Hermitian matrix polynomial $P(\lambda) = P_7\lambda^7 + \cdots + P_1\lambda + P_0$ (Antoniou-Vologiannidis 2004; De Terán-D-Mackey 2010; Mackey-Mackey-Mehl-Mehrmann 2010).

Linearizations transform PEPs into GEPs ($P(\lambda) \longrightarrow \lambda B - A$)

- "Good" strong linearizations of a matrix polynomial $P(\lambda)$ are linear matrix polynomials (matrix pencils) that have the same eigenvalues as $P(\lambda)$ and that allow us to recover the eigenvectors when $P(\lambda)$ is regular, and the minimal indices/bases when $P(\lambda)$ is singular.
- **They allow to solve numerically PEPs because there exist excellent algorithms for solving linear PEPs, i.e., GEPs.**
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Outline

1. The "flavor" of applied PEPs, REPs, NEPs: examples
2. Additional "difficulties" of GEPs, PEPs, and REPs over BEPs
3. Linearizations of matrix polynomials
4. **Rational vector spaces: minimal bases and indices**
5. Unifying theory of linearizations of polynomial matrices
6. Global backward stability of PEPs solved with linearizations
7. Conclusions

In this section:

- $\mathbb{C}[\lambda]$ is the ring of polynomials with coefficients in $\mathbb{C}$.
- $\mathbb{C}(\lambda)$ is the field of rational functions over $\mathbb{C}$.
- $\mathbb{C}(\lambda)^n$ is the vector space over the field $\mathbb{C}(\lambda)$ of n-tuples with entries in $\mathbb{C}(\lambda)$.
- **Example:**

$$\begin{bmatrix} \dfrac{\lambda+2}{\lambda^2} \\ \dfrac{1}{(\lambda+1)^3} \end{bmatrix} \in \mathbb{C}(\lambda)^2$$

- $\mathbb{C}(\lambda)^n$ **is known as a rational vector space and its subspaces as rational vector subspaces.** (Wolovich-1974, Forney-1975)

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Minimal bases of rational vector subspaces

- Any rational subspace $\mathcal{V} \subseteq \mathbb{C}(\lambda)^n$ has bases consisting entirely of vector polynomials.
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$$\begin{bmatrix} \frac{\lambda+2}{\lambda^2} \\ \frac{1}{(\lambda+1)^3} \end{bmatrix} \in \mathcal{V} \Longrightarrow \lambda^2(\lambda+1)^3 \begin{bmatrix} \frac{\lambda+2}{\lambda^2} \\ \frac{1}{(\lambda+1)^3} \end{bmatrix} = \begin{bmatrix} (\lambda+2)(\lambda+1)^3 \\ \lambda^2 \end{bmatrix} \in \mathcal{V}$$

Definition (Minimal basis)

A **minimal basis** of a rational subspace $\mathcal{V} \subseteq \mathbb{C}(\lambda)^n$ is a basis of $\mathcal{V}$

1. consisting of vector polynomials
2. whose sum of degrees is minimal among all bases of $\mathcal{V}$ consisting of vector polynomials.

- Introduced by Dedekind and Weber-1882, Plemelj-1908, Muskhelishvili and Vekua-1943, but **Forney-1975 simplified this concept and made it very important in Multivariable Linear System Theory and in Code Theory.**

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The ordered list of degrees of the vector polynomials in any minimal basis of $\mathcal{V} \subseteq \mathbb{C}(\lambda)^n$ is always the same.

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Definition

Let $P(\lambda) \in \mathbb{C}[\lambda]^{m\times n}$ be a matrix polynomial. Then:

- The **right minimal indices/bases** of $P(\lambda)$ are the **minimal indices/bases of the rational right NULL space** of $P(\lambda)$, when it is nontrivial
- The **left minimal indices/bases** of $P(\lambda)$ are the **minimal indices/bases of the rational left NULL space** of $P(\lambda)$, when it is nontrivial.

REMARK: In the rest of the talk, we arrange minimal bases as the rows of matrices and often call “basis” to the matrix.

Theorem (Forney 1975)

The rows of a polynomial matrix $N(\lambda)$ over $\mathbb{C}$ are a minimal basis of the subspace they span if and only if

(a) $N(\lambda_0)$ *has full row rank for all* $\lambda_0 \in \mathbb{C}$*, and*

(b) *the highest-row-degree coefficient matrix of* $N(\lambda)$ *has also full row rank.*

Example (of minimal basis)

$$N(\lambda) = \begin{bmatrix} -\lambda^3 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2 & -\lambda & 1 \end{bmatrix}$$

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Definition (Dual Minimal Bases)

Two minimal bases $M(\lambda) \in \mathbb{C}[\lambda]^{m\times n}$ and $N(\lambda) \in \mathbb{C}[\lambda]^{k\times n}$ are **dual** if

(a) $m + k = n$,

(b) and $M(\lambda)\, N(\lambda)^T = 0$.

Remarks

- Dual minimal bases have classical applications in Linear System Theory
- and **we have used them intensively in our research in the last decade**
  1. for constructing (and unifying) strong linearizations and ℓ-ifications of polynomial and rational matrices,
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- We have obtained a criterion to check numerically whether a polynomial matrix is a minimal basis in terms of a finite number of rank conditions in contrast with Forney's classical "practical" characterization.
- We have proved that "most matrix polynomials are minimal bases" and, moreover, that "are minimal bases with the especial property that their dual minimal bases have row degrees as equal as possible" (almost homogeneity of such row degrees).
- More precisely: in the vector space of complex matrix polynomials of size $m \times n$ $(m < n)$ and with degree at most d, the minimal bases with such particular properties form an open and dense set (its complement is a proper algebraic set).
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- More precisely: in the vector space of complex matrix polynomials of size $m \times n$ $(m < n)$ and with degree at most d, the minimal bases with such particular properties form an open and dense set (its complement is a proper algebraic set).
- These properties have allowed us to prove that these minimal bases and their duals are robust under perturbations, which has been fundamental for proving backward error results of PEPs solved numerically through linearizations.

Outline

1 The “flavor” of applied PEPs, REPs, NEPs: examples

2 Additional “difficulties” of GEPs, PEPs, and REPs over BEPs

3 Linearizations of matrix polynomials

4 Rational vector spaces: minimal bases and indices

5 **Unifying theory of linearizations of polynomial matrices**

6 Global backward stability of PEPs solved with linearizations

7 Conclusions

Most of the (many) linearizations of polynomial matrices in the literature are inside (or very closely connected to) the following class of pencils.

Definition (D, Lawrence, Pérez, Van Dooren, Numer. Math., 2018)

A matrix pencil (linear polynomial matrix)

$$\mathcal{L}(\lambda) = \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}$$

is a block minimal bases pencil (BMBP) if $K_1(\lambda)$ and $K_2(\lambda)$ are minimal bases. If, in addition, the row degrees of $K_1(\lambda)$ and $K_2(\lambda)$ are all one, and the row degrees of each of their dual minimal bases $N_1(\lambda)$ and $N_2(\lambda)$ are all equal, then $\mathcal{L}(\lambda)$ is a strong block minimal bases pencil (SBMBP).

Theorem (D, Lawrence, Pérez, Van Dooren, Numer. Math., 2018)

If $\mathcal{L}(\lambda)$ is a BMBP (resp. SBMBP), then it is a linearization (resp. strong linearization) of the matrix polynomial

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Examples of SBMBP: block-Kronecker pencils (I)

Two fundamental auxiliary matrix polynomials in the rest of the talk are the pair of dual minimal bases

$$L_k(\lambda) := \begin{bmatrix} -1 & \lambda & & & \\ & -1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & -1 & \lambda \end{bmatrix} \in \mathbb{C}[\lambda]^{k\times(k+1)},$$

$$\Lambda_k(\lambda)^T := \begin{bmatrix} \lambda^k & \lambda^{k-1} & \cdots & \lambda & 1 \end{bmatrix} \in \mathbb{C}[\lambda]^{1\times(k+1)},$$

and their Kronecker products by identities

$$L_k(\lambda) \otimes I_n := \begin{bmatrix} -I_n & \lambda I_n & & & \\ & -I_n & \lambda I_n & & \\ & & \ddots & \ddots & \\ & & & -I_n & \lambda I_n \end{bmatrix} \in \mathbb{C}[\lambda]^{nk\times n(k+1)},$$

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which are also dual minimal bases.

The Frobenius companion form of the $m \times n$ matrix polynomial $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0$ is

$$C_1(\lambda) := \begin{bmatrix} \lambda P_d + P_{d-1} & P_{d-2} & \cdots & P_1 & P_0 \\ -I_n & \lambda I_n & & & \\ & \ddots & \ddots & & \\ & & \ddots & \lambda I_n & \\ & & & -I_n & \lambda I_n \end{bmatrix},$$

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$$C_1(\lambda) := \left[\begin{array}{ccccc} \lambda P_d + P_{d-1} & P_{d-2} & \cdots & P_1 & P_0 \\ \hline \multicolumn{5}{c}{L_{d-1}(\lambda) \otimes I_n} \end{array} \right].$$

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Let $M(\lambda)$ be an arbitrary pencil. Then any pencil of the form

$$\mathcal{L}(\lambda) = \underbrace{\left[\begin{array}{c|c} M(\lambda) & L_\eta(\lambda)^T \otimes I_m \\ \hline L_\varepsilon(\lambda) \otimes I_n & 0 \end{array}\right]}_{(\varepsilon+1)n \quad\quad \eta m} \begin{array}{l} \}\, (\eta+1)m \\ \}\, \varepsilon n \end{array} ,$$

is called a block Kronecker pencil (one-block row and column cases included).

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Any block Kronecker pencil $\mathcal{L}(\lambda)$ is a SBMBP and, so, a strong linearization of the matrix polynomial

$$Q(\lambda) := (\Lambda_\eta(\lambda)^T \otimes I_m) M(\lambda) (\Lambda_\varepsilon(\lambda) \otimes I_n) \in \mathbb{C}[\lambda]^{m\times n} .$$

Examples of SBMBP: block-Kronecker pencils (III)

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$$P(\lambda) = \lambda^5 P_5 + \lambda^4 P_4 + \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0 \in \mathbb{C}[\lambda]^{m\times n}$$

$$\left[\begin{array}{ccc|cc} \lambda P_5 + P_4 & 0 & 0 & -I_m & 0 \\ 0 & \lambda P_3 + P_2 & 0 & \lambda I_m & -I_m \\ 0 & 0 & \lambda P_1 + P_0 & 0 & \lambda I_m \\ \hline -I_n & \lambda I_n & 0 & 0 & 0 \\ 0 & -I_n & \lambda I_n & 0 & 0 \end{array}\right]$$

Examples of block Kronecker pencils (II)

$$P(\lambda) = \lambda^5 P_5 + \lambda^4 P_4 + \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0 \in \mathbb{C}[\lambda]^{m\times n}$$

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- We consider a **general** $m \times n$ **polynomial matrix** of degree d

$$P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0 \quad , \qquad P_i \in \mathbb{C}^{m\times n},$$

- and we assume that its **complete eigenstructure**
- has been computed by applying a **backward stable algorithm,** QZ for regular (Moler-Stewart, 1973), Staircase for singular (Van Dooren, 1979),
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- *The computed complete eigenstructure of $\mathcal{L}(\lambda)$ is the exact complete eigenstructure of a matrix pencil $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ such that*

$$\frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F} = O(\mathbf{u}),$$

  where $\mathbf{u} \approx 10^{-16}$ is the unit roundoff and

- $\|\cdot\|_F$ is the Frobenius norm, i.e., for any matrix polynomial

$$\|Q_k\lambda^k + \cdots + Q_1\lambda + Q_0\|_F = \sqrt{\|Q_k\|_F^2 + \cdots + \|Q_1\|_F^2 + \|Q_0\|_F^2}\,.$$

- But, does this imply that the computed complete eigenstructure of $P(\lambda)$ is the exact complete eigenstructure of a polynomial matrix of the same degree $P(\lambda) + \Delta P(\lambda)$ such that

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because block Kronecker linearizations are highly structured pencils and perturbations destroy the structure!!

Example: The Frobenius Companion Form

$$C_1(\lambda) := \begin{bmatrix} \lambda P_d + P_{d-1} & P_{d-2} & \cdots & P_1 & P_0 \\ -I_n & \lambda I_n & & & \\ & \ddots & \ddots & & \\ & & \ddots & \lambda I_n & \\ & & & -I_n & \lambda I_n \end{bmatrix}$$

$C_1(\lambda) + \Delta\mathcal{L}(\lambda) =$

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The matrix perturbation problems to be solved

- **Problem 1:** To establish conditions on $\|\Delta\mathcal{L}(\lambda)\|_F$ such that $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ is a strong linearization for some polynomial matrix $P(\lambda) + \Delta P(\lambda)$ of degree d.
- **Problem 2:** To prove a perturbation bound

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq C_{P,\mathcal{L}} \frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F},$$

  with $C_{P,\mathcal{L}}$ a number depending on $P(\lambda)$ and $\mathcal{L}(\lambda)$.
- For those $P(\lambda)$ and $\mathcal{L}(\lambda)$ s.t. $C_{P,\mathcal{L}}$ is moderate, to use global backward stable algorithms on $\mathcal{L}(\lambda)$ gives global backward stability for $P(\lambda)$.

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Previous works on this type of backward error analyses

1. There were just a few previous analyses of this type when we worked on this problem:
   - Van Dooren & De Wilde (LAA 1983).
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2. **Our analysis improved considerably these analyses**, because
3. previous analyses are only valid to first order for infinitesimal perturbations, are very particular since are only valid for classical Frobenius linearizations or closely connected linearizations, and often are only valid for regular polynomial matrices.

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The main perturbation theorem

Theorem (D, Lawrence, Pérez, Van Dooren, Numer. Math., 2018)

Let $\mathcal{L}(\lambda)$ be a block Kronecker pencil for $P(\lambda) = \sum_{i=0}^{d} P_i \lambda^i \in \mathbb{C}[\lambda]^{m\times n}$, i.e.,

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M(\lambda) & L_\eta(\lambda)^T \otimes I_m \\ \hline L_\varepsilon(\lambda) \otimes I_n & 0 \end{array}\right].$$

If $\Delta\mathcal{L}(\lambda)$ is any pencil with the same size as $\mathcal{L}(\lambda)$ and such that

$$\|\Delta\mathcal{L}(\lambda)\|_F < \frac{(\sqrt{2}-1)^2}{d^{5/2}} \frac{1}{1+\|M(\lambda)\|_F},$$

then $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ is a strong linearization of a polynomial matrix $P(\lambda) + \Delta P(\lambda)$ with degree d and such that

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq 14\, d^{5/2} \frac{\|\mathcal{L}(\lambda)\|_F}{\|P(\lambda)\|_F} (1 + \|M(\lambda)\|_F + \|M(\lambda)\|_F^2) \frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}.$$

Discussion of the perturbation bounds for block Kronecker pencils

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M(\lambda) & L_\eta(\lambda)^T \otimes I_m \\ \hline L_\varepsilon(\lambda) \otimes I_n & 0 \end{array}\right].$$

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- From this bound, it is possible to show that for getting "backward stability" from Block Kronecker linearizations, one needs to normalize the matrix poly $\|P(\lambda)\|_F = 1$ and to use pencils such that $\|M(\lambda)\|_F \approx \|P(\lambda)\|_F$. Then

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \lesssim d^3 \sqrt{m+n}\, \frac{\|\Delta \mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}.$$

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$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq \underbrace{14\, d^{5/2} \frac{\|\mathcal{L}(\lambda)\|_F}{\|P(\lambda)\|_F} \left(1 + \|M(\lambda)\|_F + \|M(\lambda)\|_F^2\right)}_{C_{P,\mathcal{L}}} \frac{\|\Delta \mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}.$$

- From this bound, it is possible to show that for getting "backward stability" from Block Kronecker linearizations, one needs to normalize the matrix poly $\|P(\lambda)\|_F = 1$ and to use pencils such that $\|M(\lambda)\|_F \approx \|P(\lambda)\|_F$. Then

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \lesssim d^3 \sqrt{m+n}\, \frac{\|\Delta \mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}.$$

For Fiedler, Frobenius, etc linearizations $\|M(\lambda)\|_F = \|P(\lambda)\|_F$.

Discussion of the perturbation bounds for block Kronecker pencils

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Outline

1. The “flavor” of applied PEPs, REPs, NEPs: examples
2. Additional “difficulties” of GEPs, PEPs, and REPs over BEPs
3. Linearizations of matrix polynomials
4. Rational vector spaces: minimal bases and indices
5. Unifying theory of linearizations of polynomial matrices
6. Global backward stability of PEPs solved with linearizations
7. **Conclusions**

Conclusions

- There are many matrix eigenvalue problems in addition to the basic one that are attracting considerable attention.
- There are still many open problems in this area.
- We have developed new classes of linearizations of PEPs that unify and extend the previous (many) ones and a theory of local and strong linearizations of REPs.
- We have performed a rather general and rigorous backward stability analysis of PEPs solved with linearizations, but more analyses, including PEPs represented in other bases, are necessary.
- We have performed for the first time in the literature a backward stability analysis of REPs solved with linearizations, but this is just the beginning of these analyses.
- The abstract algebraic concept of **minimal bases and indices** has played a fundamental role in these developments.