Linearizations of matrix polynomials via Rosenbrock polynomial system matrices

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Outline

1. Rosenbrock Polynomial System Matrices
2. Gohberg-Lancaster-Rodman linearizations of polynomial matrices
3. Frobenius companion linearization and Rosenbrock
4. Comrade companion linearizations and Rosenbrock
5. Block Kronecker linearizations and Rosenbrock
6. Extended block Kronecker linearizations and Rosenbrock
7. Two advantages of Rosenbrock's point on view
8. Conclusions

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Minimal polynomial system matrices of rational matrices

Definition (Rosenbrock, 1970)

Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be a rational matrix. The polynomial matrix

$$S(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$$

is a **polynomial system matrix** of $G(\lambda)$ if

$$G(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda).$$

If, in addition, $\begin{bmatrix} A(\lambda) \\ -C(\lambda) \end{bmatrix}$ and $\begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix}$ have respectively full column and row ranks when evaluated in any $\lambda_0 \in \overline{\mathbb{F}}$, then $S(\lambda)$ is a **minimal polynomial system matrix** of $G(\lambda)$.

Theorem (Rosenbrock, 1970)

Each rational matrix has infinitely many minimal polynomial system matrices.

The position of the state matrix $A(\lambda)$ is not important: it may be anywhere, the point is to take **the Schur complement with respect to it.**

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Example of (minimal) polynomial system matrix

Consider the rational matrix

$$G(\lambda) = -B_0 + \lambda A_0 + \frac{B_1}{\lambda - \sigma_1} + \cdots + \frac{B_s}{\lambda - \sigma_s} \in \mathbb{F}(\lambda)^{p\times p},$$

$A_0, B_i \in \mathbb{F}^{p\times p}$ and $\sigma_i \neq \sigma_j$ if $i \neq j$, from El-Guide, Miedlar, Saad, 2020.
Then, these authors introduce the pencil,

$$S(\lambda) = \left[\begin{array}{cccc|c} (\lambda-\sigma_1)I & & & & I \\ & (\lambda-\sigma_2)I & & & I \\ & & \ddots & & \vdots \\ & & & (\lambda-\sigma_s)I & I \\ \hline -B_1 & -B_2 & \cdots & -B_s & \lambda A_0 - B_0 \end{array}\right]$$

which **is a polynomial system matrix of** $G(\lambda)$ of degree 1.

Moreover, $S(\lambda)$ **is minimal if and only if all the matrices** $B_1, \ldots, B_s$ **are nonsingular**.

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Minimal polynomial system matrices contain the whole finite structure

Theorem (Rosenbrock, 1970)

If

$$S(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$$

is a **minimal** **polynomial system matrix** *of* $G(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$, *then:*

1. *The finite eigenvalue structure of* $S(\lambda)$ *(including all types of multiplicities, geometric, algebraic, partial) coincides exactly with the finite zero structure of* $G(\lambda)$.
2. *The finite eigenvalue structure of* $A(\lambda)$ *(including all types of multiplicities, geometric, algebraic, partial) coincides exactly with the finite pole structure of* $G(\lambda)$.

Polynomial system matrices with unimodular state matrix $A(\lambda)$

- **They are automatically minimal.**
- **Their associated rational matrices** $G(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$ are **polynomial matrices.**
- They satisfy the following

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$$S(\lambda) \quad \textbf{is unimodularly equivalent to} \quad \begin{bmatrix} I_n & \\ & G(\lambda) \end{bmatrix}$$

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GLR linearizations and strong linearizations of polynomial matrices

GLR $\equiv$ Gohberg-Lancaster-Rodman

Definition (Gohberg-Lancaster-Rodman, 1982)

Let

$$P(\lambda) = \lambda^k A_k + \cdots + \lambda A_1 + A_0 \in \mathbb{F}[\lambda]^{p \times m}$$

be a polynomial matrix of degree at most k. A **linearization** for $P(\lambda)$ is a **linear polynomial matrix (or pencil)** $L(\lambda)$ such that

$$L(\lambda) = \lambda L_1 + L_0 \quad \textbf{is unimodularly equivalent to} \quad \begin{bmatrix} I_n & \\ & P(\lambda) \end{bmatrix}.$$

Definition (Gohberg-Kaashoek-Lancaster, 1988)

$L(\lambda)$ is a **strong linearization** of $P(\lambda)$ if, in addition,

$$\operatorname{rev}_1 L(\lambda) := \lambda L_0 + L_1 \quad \text{is a linearization for} \quad \operatorname{rev}_k P(\lambda),$$

where $\operatorname{rev}_k P(\lambda) := \lambda^k A_0 + \cdots + \lambda A_{k-1} + A_k$.

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Key observation for this talk

Corollary

A **linear polynomial system matrix**

$$S(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$$

with **unimodular state-matrix** $A(\lambda)$ *is a* **GLR-linearization** *of its associated* **polynomial** *matrix* $G(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$.

- **We emphasize:** linear polynomial system matrices with unimodular state-matrix are **particular cases** of GLR-linearizations.
- But, we will see that **they include many famous GLR-linearizations** available in the literature, which connects Rosenbrock (previous) and GLR approaches for polynomial matrices.
- **Recall: We have to identify unimodular submatrices.**

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GLR-Linearizations and eigenvalues

Theorem

A GLR-linearization $L(\lambda)$ *of a polynomial matrix* $P(\lambda)$ *satisfies*

- $L(\lambda)$ *and* $P(\lambda)$ **have the same finite eigenvalues** *with the same partial multiplicities.*

A GLR-strong-linearization $L(\lambda)$ *of a polynomial matrix* $P(\lambda)$ *satisfies*

- $L(\lambda)$ *and* $P(\lambda)$ **have the same finite and infinite eigenvalues** *with the same partial multiplicities.*

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The most famous (strong) linearization: Frobenius

The **Frobenius companion form** of $P(\lambda) = P_k\lambda^k + \cdots + P_1\lambda + P_0 \in \mathbb{F}[\lambda]^{p\times m}$ is

$$C_1(\lambda) := \begin{bmatrix} \lambda P_k + P_{k-1} & P_{k-2} & \cdots & P_1 & P_0 \\ -I_m & \lambda I_m & & & \\ & \ddots & \ddots & & \\ & & \ddots & \lambda I_m & \\ & & & -I_m & \lambda I_m \end{bmatrix}$$

Theorem (Frobenius companion is a Rosenbrock system matrix)

The Frobenius companion form of $P(\lambda)$ is a linear polynomial system matrix

- *with unimodular state-matrix $A(\lambda)$, i.e., the submatrix obtained by removing the first block row and last block column, and*
- *associated polynomial matrix equal to $P(\lambda)$, i.e.,*

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The proof is easy: the presence of $A(\lambda)^{-1}$ does not create a mess

$$\left[\begin{array}{cccc|c} -I_m & \lambda I_m & & & \\ & \ddots & \ddots & & \\ & & \ddots & \lambda I_m & \\ & & & -I_m & \lambda I_m \end{array}\right] \begin{bmatrix} \lambda^{k-1} I_m \\ \lambda^{k-2} I_m \\ \vdots \\ \lambda I_m \\ I_m \end{bmatrix} = \begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix} \begin{bmatrix} \lambda^{k-1} I_m \\ \lambda^{k-2} I_m \\ \vdots \\ \lambda I_m \\ I_m \end{bmatrix} = 0$$

$$\Longrightarrow \quad A(\lambda) \begin{bmatrix} \lambda^{k-1} I_m \\ \lambda^{k-2} I_m \\ \vdots \\ \lambda I_m \end{bmatrix} + B(\lambda) = 0 \quad \Longrightarrow \quad A(\lambda)^{-1} B(\lambda) = - \begin{bmatrix} \lambda^{k-1} I_m \\ \lambda^{k-2} I_m \\ \vdots \\ \lambda I_m \end{bmatrix}$$

- **Similar tricks can be used for the rest of linearizations in the talk.**

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$$\left[\begin{array}{ccccc|c} -I_m & \lambda I_m & & & & \\ & \ddots & \ddots & & & \\ & & \ddots & \lambda I_m & & \\ & & & -I_m & & \lambda I_m \end{array}\right] \begin{bmatrix} \lambda^{k-1} I_m \\ \lambda^{k-2} I_m \\ \vdots \\ \lambda I_m \\ I_m \end{bmatrix} = \begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix} \begin{bmatrix} \lambda^{k-1} I_m \\ \lambda^{k-2} I_m \\ \vdots \\ \lambda I_m \\ I_m \end{bmatrix} = 0$$

$$\Longrightarrow \quad A(\lambda) \begin{bmatrix} \lambda^{k-1} I_m \\ \lambda^{k-2} I_m \\ \vdots \\ \lambda I_m \end{bmatrix} + B(\lambda) = 0 \quad \Longrightarrow \quad A(\lambda)^{-1} B(\lambda) = - \begin{bmatrix} \lambda^{k-1} I_m \\ \lambda^{k-2} I_m \\ \vdots \\ \lambda I_m \end{bmatrix}$$

- **Similar tricks can be used for the rest of linearizations in the talk.**

Similar arguments for the reversal of the Frobenius companion form

$$\operatorname{rev}_1 C_1(\lambda) := \begin{bmatrix} P_k + \lambda P_{k-1} & \lambda P_{k-2} & \cdots & \lambda P_1 & \lambda P_0 \\ -\lambda I_m & I_m & & & \\ & \ddots & \ddots & & \\ & & \ddots & I_m & \\ & & & -\lambda I_m & I_m \end{bmatrix}$$

Theorem (Reversal Frobenius is a Rosenbrock system matrix)

The **reversal** *of the Frobenius companion form of $P(\lambda)$ is a linear polynomial system matrix*

- *with unimodular state-matrix $A_r(\lambda)$, i.e., the submatrix obtained by removing the first block row and first block column, and*
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$$\operatorname{rev}_k P(\lambda) = D_r(\lambda) + C_r(\lambda) A_r(\lambda)^{-1} B_r(\lambda).$$

*Therefore, $\operatorname{rev}_1 C_1(\lambda)$ is a GLR-linearization of $\operatorname{rev}_k P(\lambda)$ and $C_1(\lambda)$ is a GLR-***strong***-linearization of $P(\lambda)$.*

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2. Gohberg-Lancaster-Rodman linearizations of polynomial matrices
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For polynomial matrices expressed in “orthogonal” bases

Scalar polynomial basis satisfying a three-term recurrence relation

$$\alpha_j\phi_{j+1}(\lambda) = (\lambda - \beta_j)\phi_j(\lambda) - \gamma_j\phi_{j-1}(\lambda) \quad j \geq 0$$

where $\alpha_j, \beta_j, \gamma_j \in \mathbb{F}$, $\alpha_j \neq 0$, $\phi_{-1}(\lambda) = 0$, and $\phi_0(\lambda) = 1$.

Matrix polynomial expressed in such a basis

$P(\lambda) = P_k\phi_k(\lambda) + P_{k-1}\phi_{k-1}(\lambda) + \cdots + P_1\phi_1(\lambda) + P_0\phi_0(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$

- It is “well-known” that the **“comrade” companion matrix** in the next slide is a GLR-strong-linearization of $P(\lambda)$.

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Comrade is a Rosenbrock polynomial system matrix

$$C_\phi(\lambda) = \begin{bmatrix} \frac{(\lambda-\beta_{k-1})}{\alpha_{k-1}}P_k + P_{k-1} & P_{k-2} - \frac{\gamma_{k-1}}{\alpha_{k-1}}P_k & P_{k-3} & \cdots & P_1 & P_0 \\ -\alpha_{k-2}I & (\lambda-\beta_{k-2})I & -\gamma_{k-2}I & & & \\ & -\alpha_{k-3}I & (\lambda-\beta_{k-3})I & -\gamma_{k-3}I & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\alpha_1 I & (\lambda-\beta_1)I & -\gamma_1 I \\ & & & & -\alpha_0 I & (\lambda-\beta_0)I \end{bmatrix}$$

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An example of block Kronecker linearization

A strong linearization of

$$P(\lambda) = \lambda^5 P_5 + \lambda^4 P_4 + \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0 \in \mathbb{F}[\lambda]^{p\times m}$$

is the following block Kronecker pencil

$$C_K(\lambda) := \left[\begin{array}{ccc|cc} \lambda P_5 & \lambda P_4 & \lambda P_3 & -I_p & 0 \\ 0 & 0 & \lambda P_2 & \lambda I_p & -I_p \\ 0 & 0 & \lambda P_1 + P_0 & 0 & \lambda I_p \\ \hline -I_m & \lambda I_m & 0 & 0 & 0 \\ 0 & -I_m & \lambda I_m & 0 & 0 \end{array}\right]$$

The “yellow” submatrix is unimodular. This is general!!

- **Block Kronecker linearizations** (D, Lawrence, Pérez, Van Dooren, 2018) are a wide infinite family of strong linearizations of polynomial matrices that **include among many others the Fiedler linearizations** (Fiedler, 2003) modulo permutations.
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Two fundamental auxiliary polynomial matrices in the rest of the talk

$$L_k(\lambda) := \begin{bmatrix} -1 & \lambda & & & \\ & -1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & -1 & \lambda \end{bmatrix} \in \mathbb{F}[\lambda]^{k\times(k+1)},$$

$$\Lambda_k(\lambda)^T := \left[\ \lambda^k \quad \lambda^{k-1} \quad \cdots \quad \lambda \quad 1\ \right] \in \mathbb{F}[\lambda]^{1\times(k+1)},$$

and their Kronecker products by identities

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Definition and key property of Block Kronecker Pencils

Definition

Let $\lambda M_1 + M_0$ be an arbitrary pencil. Then any pencil of the form

$$C_K(\lambda) = \left[\begin{array}{c|c} \lambda M_1 + M_0 & L_\eta(\lambda)^T \otimes I_p \\ \hline L_\varepsilon(\lambda) \otimes I_m & 0 \end{array}\right] \begin{array}{l} \}\,(\eta+1)p \\ \}\,\varepsilon m \end{array} \quad ,$$

$$\underbrace{\qquad\qquad}_{(\varepsilon+1)m} \quad \underbrace{\qquad\qquad}_{\eta p}$$

is called a block Kronecker pencil (one-block row and column cases included).

Theorem (Key Theorem of Block Kronecker Pencils)

Any block Kronecker pencil $C_K(\lambda)$ is a GLR-strong-linearization of the matrix polynomial

$$P(\lambda) := (\Lambda_\eta(\lambda)^T \otimes I_p)(\lambda M_1 + M_0)(\Lambda_\varepsilon(\lambda) \otimes I_m) \in \mathbb{F}[\lambda]^{p\times m} .$$

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Block Kronecker Linearizations are Rosenbrock poly system matrices

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Theorem (Block Kronecker are Rosenbrock system matrices)

- **The submatrix** $A(\lambda)$ **of** $C_K(\lambda)$ *obtained by removing the block-column corresponding to the last block-column of* $L_\varepsilon(\lambda) \otimes I_m$ *and the block-row corresponding to the last block-row of* $L_\eta(\lambda)^T \otimes I_p$ **is unimodular**.
- **The Schur complement of** $A(\lambda)$ **in** $C_K(\lambda)$ *is the polynomial matrix*

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Block Kronecker Linearizations are Rosenbrock poly system matrices

$$C_K(\lambda) = \begin{array}{cc} \left[\begin{array}{c|c} \lambda M_1 + M_0 & L_\eta(\lambda)^T \otimes I_p \\ \hline L_\varepsilon(\lambda) \otimes I_m & 0 \end{array}\right] & \begin{array}{l} \}\,(\eta+1)p \\ \}\,\varepsilon m \end{array} \\ \begin{array}{cc} \underbrace{\hphantom{L_\varepsilon(\lambda) \otimes I_m}}_{(\varepsilon+1)m} & \underbrace{\hphantom{L_\eta(\lambda)^T \otimes I_p}}_{\eta p} \end{array} & \end{array}$$

Theorem (Block Kronecker are Rosenbrock system matrices)

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1. Rosenbrock Polynomial System Matrices
2. Gohberg-Lancaster-Rodman linearizations of polynomial matrices
3. Frobenius companion linearization and Rosenbrock
4. Comrade companion linearizations and Rosenbrock
5. Block Kronecker linearizations and Rosenbrock
6. **Extended block Kronecker linearizations and Rosenbrock**
7. Two advantages of Rosenbrock's point on view
8. Conclusions

A very wide family of linearizations

Extended block Kronecker linearizations were introduced by Bueno, D, Pérez, Saavedra, and Zykoski **in 2018 and they include, among many others,**

- all block Kronecker linearizations,
- all Fiedler linearizations (modulo permutations),
- all Generalized Fiedler linearizations (modulo permutations),
- all Generalized Fiedler linearizations with repetition (modulo permutations),
- all the pencils in the canonical basis of the vector space $\mathbb{DL}(P)$ (modulo permutations), since they are Fiedler pencils with repetition (Bueno, Curlett, Furtado, 2014),
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They can be (apparently) quite complicated and they are linearizations under some conditions.

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An example of Extended block Kronecker linearization

Given

$$P(\lambda) = \lambda^4 P_4 + \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0 \in \mathbb{F}[\lambda]^{p\times p},$$

the following extended block Kronecker pencil is, **under some conditions**, a GLR-strong-linearization of $P(\lambda)$

$$C_{EK}(\lambda) := \begin{bmatrix} \lambda P_4 + P_3 & P_2 & P_1 & -P_4 \\ P_2 & -\lambda P_2 + P_1 & -\lambda P_1 + P_0 & \lambda P_4 \\ -P_2 & \lambda P_2 - P_3 & \lambda P_3 & 0 \\ -P_1 & \lambda P_1 & 0 & 0 \end{bmatrix}$$

The "yellow" submatrix is unimodular if P_1, P_3 and P_4 are invertible.

To see this note

$$\begin{bmatrix} -P_2 & \lambda P_2 - P_3 \\ -P_1 & \lambda P_1 \end{bmatrix} = \begin{bmatrix} P_2 & P_3 \\ P_1 & 0 \end{bmatrix} \begin{bmatrix} -I & \lambda I \\ 0 & -I \end{bmatrix}$$

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Definition and key property of Extended Block Kronecker Pencils

Definition

Let $\lambda M_1 + M_0$ be an arbitrary pencil and $Y \in \mathbb{F}^{\varepsilon m \times \varepsilon m}$, $Z \in \mathbb{F}^{\eta p \times \eta p}$ be arbitrary constant matrices. Then any pencil of the form

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is called a Extended block Kronecker pencil (one-block row and column cases included).

Theorem (Key Theorem of Extended Block Kronecker Pencils)

Any Extended block Kronecker pencil $C_{EK}(\lambda)$ with Y and Z invertible is a GLR-strong-linearization of the matrix polynomial

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Extended Block Kronecker Lins are Rosenbrock poly system matrices

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Universal recovery of eigenvectors

Once a linearization of a regular polynomial matrix $P(\lambda)$ is viewed as a linear polynomial system matrix

$$L(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix},$$

with $A(\lambda)$ **unimodular** and

$$P(\lambda) = D(\lambda) + C(\lambda) A(\lambda)^{-1} B(\lambda),$$

the eigenvectors of an eigenvalue λ_0 can be recovered always in the same way.

Theorem (Universal recovery of eigenvectors)

$$\begin{bmatrix} A(\lambda_0) & B(\lambda_0) \\ -C(\lambda_0) & D(\lambda_0) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \iff \begin{cases} x_1 = -A(\lambda_0)^{-1} B(\lambda_0) x_2, \\ P(\lambda_0)\, x_2 = 0 \end{cases}$$

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Universal construction of linearizations of rational matrices

- Suppose that a rational matrix $R(\lambda)$ is expressed as

$$R(\lambda) = P(\lambda) + R_{sp}(\lambda)$$

with $P(\lambda)$ its polynomial part and $R_{sp}(\lambda)$ its strictly proper part.

- Assume that we have a linearization of $P(\lambda)$ that is a polynomial system matrix

$$L(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix},$$

with $A(\lambda)$ **unimodular**,

- and a $R_{sp}(\lambda) = C_s(\lambda E_s - A_s)^{-1} B_s$ minimal state-space realization of the strictly proper part.
- Then,

$$\left[\begin{array}{cc|c} (\lambda E_s - A_s) & 0 & B_s \\ 0 & A(\lambda) & B(\lambda) \\ \hline -C_s & -C(\lambda) & D(\lambda) \end{array}\right]$$

is a linear minimal polynomial system matrix of $R(\lambda)$ and, so, a linearization of $P(\lambda)$.

Universal construction of linearizations of rational matrices

- Suppose that a rational matrix $R(\lambda)$ is expressed as

$$R(\lambda) = P(\lambda) + R_{sp}(\lambda)$$

with $P(\lambda)$ its polynomial part and $R_{sp}(\lambda)$ its strictly proper part.

- Assume that we have a linearization of $P(\lambda)$ that is a polynomial system matrix

$$L(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix},$$

with $A(\lambda)$ **unimodular**,

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Outline

1. Rosenbrock Polynomial System Matrices
2. Gohberg-Lancaster-Rodman linearizations of polynomial matrices
3. Frobenius companion linearization and Rosenbrock
4. Comrade companion linearizations and Rosenbrock
5. Block Kronecker linearizations and Rosenbrock
6. Extended block Kronecker linearizations and Rosenbrock
7. Two advantages of Rosenbrock's point on view
8. **Conclusions**

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- When working with a pencil that might be a linearization of a polynomial matrix, **one should look for unimodular submatrices** of this pencil.
- This may lead to easy proofs that such a pencil is a linearization, as well as to other advantages.
- This idea **links Rosenbrock's Polynomial system matrices** (introduced in 1970) to the more modern definition of **GLR-linearizations** of polynomial matrices and to many specific families of such linearizations.
- Rosenbrock's results included in his classical book "State-Space and Multivariable Theory" (1970) have received very limited attention recently by the Linear Algebra community.
- I hope this talk will attract more attention on Rosenbrock's work and that it serves as a small tribute to Rosenbrock from the Linear Algebra community.
- The results in this talk can be easily extended to many families of modern **ℓ-ifications of polynomial matrices**.

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