Topological properties of bundles of matrix pencils under strict equivalence

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Outline

1. Orbits closures of matrix pencils under strict equivalence
2. Bundles of matrix pencils come into play
3. Conclusions and open questions

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1. **Orbits closures of matrix pencils under strict equivalence**

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3. Conclusions and open questions

Matrix pencil: $A + \lambda B$, with $A, B \in \mathbb{C}^{m \times n}$ (or matrix pair (A, B)).

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Definition (strict equivalence)

Two $m \times n$ pencils $A + \lambda B$ and $A' + \lambda B'$ are strictly equivalent if

$$A' = PAQ, \; B' = PBQ, \qquad \text{for some invertible matrices } P, Q,$$

or $A' + \lambda B' = P(A + \lambda B)Q$.

Basic definitions: pencil, strict equivalence, orbit

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Definition (orbit under strict equivalence)

Given an $m \times n$ pencil $A + \lambda B$, its orbit (under strict equivalence) is the set

$$\mathcal{O}(A + \lambda B) := \{P(A + \lambda B)Q : \; P, Q \text{ invertible}\},$$

i.e., it is the set of $m \times n$ pencils which are strictly equivalent to $A + \lambda B$.

Orbits and the Kronecker Canonical Form

Theorem (Kronecker Canonical Form = KCF)

Every pencil is strictly equivalent to a unique (up to permutation) direct sum of blocks of the following types:

- **Blocks associated with finite evals** *(μ):*

$$J_k(\mu) := \begin{bmatrix} \lambda-\mu & 1 & & \\ & \ddots & \ddots & \\ & & \lambda-\mu & 1 \\ & & & \lambda-\mu \end{bmatrix}_{k\times k} \qquad (k \geq 1).$$

- **Blocks associated with the ∞ eval**: $J_k(\infty) := \begin{bmatrix} 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \\ & & & 1 \end{bmatrix}_{k\times k} \qquad (k \geq 1).$

- **Right singular blocks**: $R_k(\lambda) =: \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \end{bmatrix}_{k\times(k+1)} \qquad (k \geq 0).$

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Remark

- All the pencils in an orbit have the same KCF.
- Every orbit is uniquely determined by such KCF.

Orbit closures and related problems

Let L, L_1, L_2 be $m \times n$ pencils.

$\overline{\mathcal{O}}(L)$: closure of $\mathcal{O}(L)$ (in the standard topology of $\mathbb{C}^{m\times n} \times \mathbb{C}^{m\times n} \simeq \mathbb{C}^{2mn}$).

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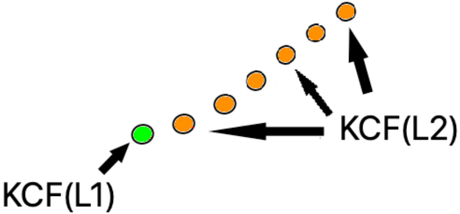
One practical motivation: When computing KCF(L_1), if $L_1 \in \overline{\mathcal{O}}(L_2)$, there are arbitrarily small perturbations, $L_1 + L_\varepsilon$, s.t. KCF($L_1 + L_\varepsilon$) = KCF(L_2).



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Lemma

$L_1 \in \overline{\mathcal{O}}(L_2) \iff \mathcal{O}(L_1) \subseteq \overline{\mathcal{O}}(L_2) \iff \overline{\mathcal{O}}(L_1) \subseteq \overline{\mathcal{O}}(L_2)$.

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Proof: If $P_n L_2 Q_n \to L_1$, then $(PP_n)L_2(Q_nQ) \to PL_1Q$. □

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Remark

The inclusion relationships between orbit closures allows us to classify the KCFs according to their “genericity".

Inclusion of orbit closures: domination rules via majorization

Definition (Weyr characteristic of a sequence of nonnegative integers)

The Weyr characteristic of a sequence of nonnegative integers $N = (n_1, n_2, \ldots)$ is

$$W(N) := (w_1(N), w_2(N), \ldots), \quad \text{where } w_i(N) = \#\{n_j : \ n_j \geq i\}.$$

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Definition (Weyr characteristics of a pencil L)

$r(L)$: Weyr characteristic of the sizes of the right singular blocks in KCF(L).

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$W(\mu, L)$: Weyr characteristic of the sizes of the Jordan blocks of μ in KCF(L).

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Definition (Majorization of two lists of non-increasing integers)

$(m_1, m_2, \ldots) \prec (n_1, n_2, \ldots)$ if $\sum_{i=1}^{k} m_i \leq \sum_{i=1}^{k} n_i$, for all $k \geq 1$.

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Theorem (Pokrzywa, LAA, 1986)

Let L_1, L_2 be two $m \times n$ pencils and $h := \mathrm{rank} L_2 - \mathrm{rank} L_1$. Then $\overline{\mathcal{O}}(L_1) \subseteq \overline{\mathcal{O}}(L_2)$ iff

(i) $r(L_1) \prec r(L_2) + (h, h, \ldots)$,

(ii) $\ell(L_1) \prec \ell(L_2) + (h, h, \ldots)$,

(iii) $W(\mu, L_2) \prec W(\mu, L_1) + (h, h, \ldots)$, $\forall \mu \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

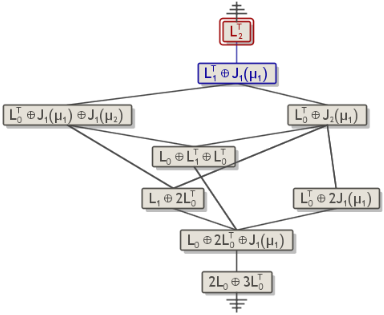
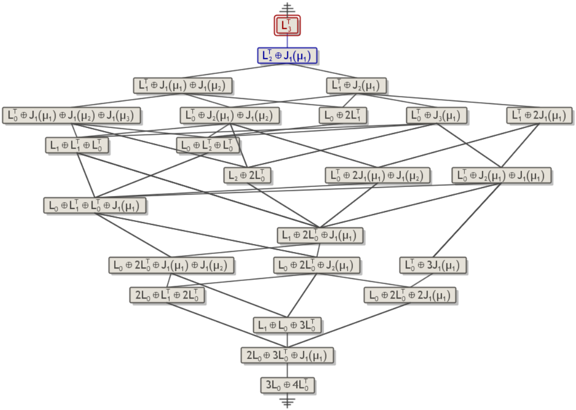


Table: *

Stratification of closure orbits of 3×2 pencils

Domination rules: Visualization



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Stratification of closure orbits of 4×3 pencils

Made with `Stratigraph` (Dmytryshyn, Elmroth, Johansson, Johansson, Kågström, Umeå University) `https://www.umu.se/en/research/projects/stratigraph-and-mcs-toolbox/`

Orbits are open in their closures

It is well-known that orbits of varieties under the action of a group are open in their closures (see, for instance: Humphreys. *Linear Algebraic Groups*. Springer, 1975). As a corollary, we have:

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Remark on genericity

Thus, $\mathcal{O}(L)$ is open and dense in $\overline{\mathcal{O}}(L)$ and we can state in the standard topological sense that KCF(L) is *generic* among the KCFs of all the pencils in $\overline{\mathcal{O}}(L)$.

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- This allows us to know whether a given KCF can be obtained after an arbitrarily small perturbation of another one
- and to classify the KCFs according to their “genericity",
- since the orbits are open (and dense) in their closures.

However, the eigenvalues of all the pencils in an orbit are the same!, which is not convenient in many applications concerning perturbations. (For instance, if L is regular, then all the regular pencils in $\overline{\mathcal{O}}(L)$ have the same eigenvalues!!)

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2. **Bundles of matrix pencils come into play**
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Definition (bundle)

The bundle of an $m \times n$ matrix pencil L, $\mathcal{B}(L)$, is the set of matrix pencils with the same KCF as L, up to the specific values of the distinct eigenvalues.

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Example: If

$$L = R_1(\lambda) \oplus J_1(5) = \left[\begin{array}{cc|c} \lambda & 1 & 0 \\ \hline 0 & 0 & \lambda - 5 \end{array}\right],$$

then

$$\begin{aligned} \mathcal{B}(L) \quad = \quad & \left\{ P \left[\begin{array}{cc|c} \lambda & 1 & 0 \\ \hline 0 & 0 & \lambda - \alpha \end{array}\right] Q : \; P, Q \text{ invertible}, \; \alpha \in \mathbb{C} \right\} \\ & \cup \left\{ P \left[\begin{array}{cc|c} \lambda & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] Q : \; P, Q \text{ invertible} \right\}. \end{aligned}$$

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(Bundles of matrices under similarity were introduced by Arnold (1971) and of pencils by Edelman, Elmroth and Kågström, 1997)

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Example: If

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then

$$\mathcal{B}(L) = \{P(J_2(\alpha) \oplus J_1(\beta))Q \,:\, P, Q \text{ invertible and } \alpha \neq \beta\}$$

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Therefore:

$$J_2(0) \oplus J_1(0) = \left[\begin{array}{cc|c} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ \hline 0 & 0 & \lambda \end{array}\right] \notin \mathcal{B}(L).$$

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Definition (Coalescence of eigenvalues)

We say that some eigenvalues $\lambda_1, \ldots, \lambda_p$ of a pencil **coalesce** to only one eigenvalue μ of another pencil if the Weyr characteristic of μ is the union of the Weyr characteristics of $\lambda_1, \ldots, \lambda_p$ (i.e.: add up the sizes of the largest Jordan blocks of each eigenvalue, then of the second largest, ...).

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Example:

$$\mathrm{KCF}(L) = J_3(1) \oplus J_2(1) \oplus J_2(0) \oplus J_1(0) \oplus J_4(2)$$

$$= \left[\begin{array}{ccc|cc|cc|c|cccc}
\lambda-1 & 1 & & & & & & & & & & \\
 & \lambda-1 & 1 & & & & & & & & & \\
 & & \lambda-1 & & & & & & & & & \\ \hline
 & & & \lambda-1 & 1 & & & & & & & \\
 & & & & \lambda-1 & & & & & & & \\ \hline
 & & & & & \lambda & 1 & & & & & \\
 & & & & & & \lambda & & & & & \\ \hline
 & & & & & & & \lambda & & & & \\ \hline
 & & & & & & & & \lambda-2 & 1 & & \\
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 & & & & & & & & & & \lambda-2 & 1 \\
 & & & & & & & & & & & \lambda-2
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$$\mathrm{KCF}(L) = J_3(1) \oplus J_2(1) \oplus J_2(0) \oplus J_1(0) \oplus J_4(2)$$

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\lambda-1 & 1 & & & & & & & & & & \\
 & \lambda-1 & 1 & & & & & & & & & \\
 & & \lambda-1 & & & & & & & & & \\
\hline
 & & & \lambda-1 & 1 & & & & & & & \\
 & & & & \lambda-1 & & & & & & & \\
\hline
 & & & & & \lambda & 1 & & & & & \\
 & & & & & & \lambda & & & & & \\
\hline
 & & & & & & & \lambda & & & & \\
\hline
 & & & & & & & & \lambda-2 & 1 & & \\
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Rules for the inclusion of bundle closures

Theorem (domination rules for bundle closures)

Let $\widetilde{L}$ and L be two $m \times n$ pencils. Then $\overline{\mathcal{B}}(\widetilde{L}) \subseteq \overline{\mathcal{B}}(L)$ if and only if $\mathrm{KCF}(\widetilde{L})$ is obtained from $\mathrm{KCF}(L)$ after coalescing eigenvalues and applying the dominance rules for closure orbit inclusion.

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- However, no formal definition of coalescence is provided in this reference.

Let's compare bundles and orbits:

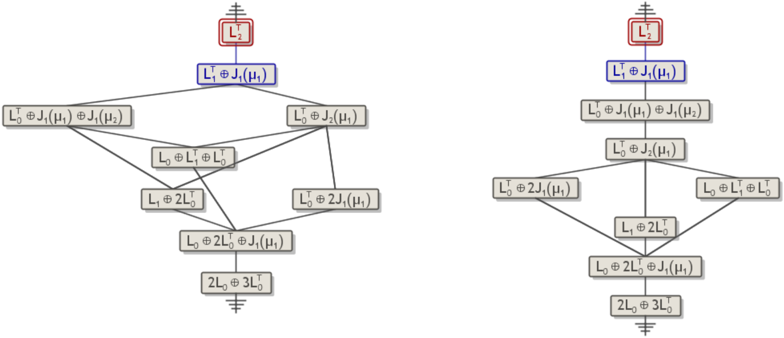

Figure: *

Orbits (left) and bundles (right) of 3×2 pencils

Domination rules for pencils visualized with stratigraph

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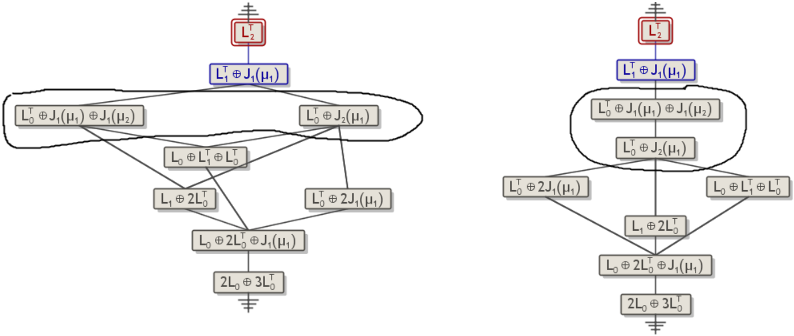

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Other interesting results we have obtained

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Let $\widetilde{L}$ *and* L *be two* $m \times n$ *pencils. Then* $\widetilde{L} \in \overline{\mathcal{B}}(L) \Rightarrow \mathcal{B}(\widetilde{L}) \subseteq \overline{\mathcal{B}}(L)$.

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Theorem

The closure of a bundle is a "stratified manifold" (namely, the union of the bundle itself with a finite number of other bundles of smaller dimension).

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- The same result holds for bundles of matrices under similarity.
- **The same result holds for bundles of matrix polynomials of arbitrary degree.**

Outline

1. Orbits closures of matrix pencils under strict equivalence
2. Bundles of matrix pencils come into play
3. **Conclusions and open questions**

Our contribution

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- We have proved that bundles of matrix polynomials are open in their closures.

Some open questions

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- For structured pencils and structured matrix polynomials: Are bundles open in their closure?