Strongly minimal self-conjugate linearizations for polynomial and rational matrices

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joint work with **María C. Quintana** (Aalto University, Finland) and **Paul Van Dooren** (UC Louvain, Belgium)

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Different classes of matrix eigenvalue problems (I)

From a *simplified* point of view, we can consider the following matrix eigenvalue problems:

- **The basic eigenvalue problem (BEP).** Given $A \in \mathbb{C}^{n\times n}$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that

$$Av = \lambda v \iff \boxed{(\lambda I_n - A)\, v = 0}$$

- **The GENERALIZED eigenvalue problem (GEP).** Given $A, B \in \mathbb{C}^{m\times n}$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that

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- **The POLYNOMIAL eigenvalue problem (PEP).** Given $P_0, P_1, \ldots, P_d \in \mathbb{C}^{m\times n}$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that

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- **The RATIONAL eigenvalue problem (REP).** Given a rational matrix $G(z) \in \mathbb{C}(z)^{m\times n}$, i.e., such that $G(z)_{ij}$ is a scalar rational function of $z \in \mathbb{C}$, for $1 \leq i, j \leq n$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that λ is not a pole of any $G(z)_{ij}$ and

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We focus in this talk on PEPs and REPs, which are important by themselves but also as approximations of more general nonlinear eigenvalue problems.

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A key idea on matrix eigenvalue problems

1. **BEP**: $\boxed{(\lambda I_n - A)\, v = 0}$
2. **GEP**: $\boxed{(\lambda B - A)\, v = 0}$!!!!
3. **PEP**: $\boxed{(P_d \lambda^d + \cdots + P_1 \lambda + P_0) v = 0}$
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- **Key idea: PEPs and REPs can be solved by transforming the problem into a GEP** via a process known as **LINEARIZATION**.
- This transformation is **exact**, i.e., the obtained **GEP** contains (or allows us to easily extract) exactly all the eigen-information of the original **PEP** or **REP**.
- The use of **linearizations** is one of the **most reliable** approaches for solving numerically PEPs and REPs, because **there exist very reliable algorithms for solving GEPs**.
- This approach has been studied by many researchers in the last two decades.

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- So far, the linearizations used in the literature for PEPs fit into the classical definition of **Gohberg-Lancaster-Rodman (GLR)**,
- and the ones for REPs fit into combining the GLR-approach with Rosenbrock's polynomial system matrices (Alam, Behera (SIMAX, 2016); Amparan, D, Marcaida, Zaballa (SIMAX, 2018)).
- We will introduce a new unified definition of linearizations of PEPs and REPs (strongly minimal linearizations) that guarantee stronger properties than those of GLR-linearizarions.
- Moreover, we will show how to construct such linearizations for any polynomial or rational matrix in such a way that
- for structured PEPs and REPs (Hermitian, skew-Hermitian, alternating odd and even) **always preserve such structures**,
- which is not always possible for GLR-linearizations,
- in particular for polynomial matrices of even-degree (Mackey, Mackey, Mehl, Mehrmann, De Terán, D).

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Outline

1. Brief reminder of “Eigenstructures” of PEPs and REPs
2. Gohberg-Lancaster-Rodman linearizations of PEPs (and REPs)
3. Strongly minimal linearizations of polynomial and rational matrices
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GEPs-PEPs-REPs have more spectral “structural” data than BEPs

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- So far, we have only considered informally **finite eigenvalues**, but
- **GEPs, PEPs, REPs** may have also **infinite eigenvalues**.
- **GEPs, PEPs, REPs** may be **singular**, i.e., rectangular or square with identically zero determinant, (BEPs are always regular) and to have, in addition to eigenvalues, **minimal indices**.
- Moreover, **REPs** have **poles**.
- We define quickly these concepts.

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Given $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0 \in \mathbb{C}[\lambda]^{m\times n}$,

- $\lambda_0 \in \mathbb{C}$ is a **finite eigenvalue** of $P(\lambda)$ if

$$\operatorname{rank} P(\lambda_0) < \max_{\lambda\in\mathbb{C}} \operatorname{rank} P(\lambda)$$

- The infinite eigenvalue of $P(\lambda)$ is defined through **the reversal polynomial**.
- The reversal of $P(\lambda)$ is

$$\operatorname{rev}P(\lambda) := \lambda^d P\left(\frac{1}{\lambda}\right) = P_0\lambda^d + \cdots + P_{d-1}\lambda + P_d \quad .$$

- Then the **infinite eigenvalue** (and its mutiplicities) of $P(\lambda)$ correspond to the **zero eigenvalue** (and its mutiplicities) of $\operatorname{rev}P(\lambda)$.

Finite and infinite eigenvalues of PEPs

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- PEPs are **singular** when $\boxed{P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0}$ is either **rectangular or square with** $\det P(\lambda) \equiv 0$.
- **Singular PEPs appear in applications**, in particular in Multivariable System Theory and Control Theory.
- In addition to eigenvalues, **singular matrix polynomials have** other "interesting numbers" called **minimal indices**,
- which are related to the fact that a singular $m \times n$ matrix polynomial $P(\lambda)$ has non-trivial left and/or right null-spaces over the field $\mathbb{C}(\lambda)$ of rational functions:

$$\begin{aligned} \mathcal{N}_\ell(P) &:= \left\{ y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times m} : y(\lambda)^T P(\lambda) \equiv 0^T \right\}, \\ \mathcal{N}_r(P) &:= \left\{ x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} : P(\lambda)x(\lambda) \equiv 0 \right\}. \end{aligned}$$

- They have bases consisting entirely of vector polynomials.
- The polynomial bases with "minimal sum of the degrees" of their vectors are the **minimal bases** of $P(\lambda)$. The **minimal indices** of $P(\lambda)$ are the degrees of the vectors of any minimal basis.

Minimal indices of singular PEPs

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$$\begin{aligned} \mathcal{N}_\ell(P) &:= \left\{ y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times m} : y(\lambda)^T P(\lambda) \equiv 0^T \right\}, \\ \mathcal{N}_r(P) &:= \left\{ x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} : P(\lambda)x(\lambda) \equiv 0 \right\}. \end{aligned}$$

- They have bases consisting entirely of vector polynomials.
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- PEPs are **singular** when $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0$ is either **rectangular or square with** $\det P(\lambda) \equiv 0$.
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As a consequence of the previous discussion, we define:

Definition

The **complete “eigenstructure”** of a polynomial matrix $P(\lambda)$ is comprised of:

- its finite eigenvalues, together with their partial multiplicities,
- its infinite eigenvalue, together with its partial multiplicities,
- its right minimal indices, and
- its left minimal indices.

Remarks

- The partial multiplicities are rigorously defined through the Smith form of $P(\lambda)$ and for matrices and pencils they are just the sizes of the Jordan blocks associated to each eigenvalue.

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Analogously, we define:

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The **complete “eigenstructure”** of a rational matrix $G(\lambda)$ is comprised of:

- its finite zeros and **poles**, together with their partial multiplicities,
- its infinite zeros and **poles**, together with its partial multiplicities,
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- The partial multiplicities are rigorously defined through the Smith-McMillan form of $G(\lambda)$.
- The **eigenvalues** of $G(\lambda)$ are those zeros that are not poles.
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3 Strongly minimal linearizations of polynomial and rational matrices

4 Constructing strongly minimal linearizations of polynomial matrices

5 Constructing strongly minimal linearizations of rational matrices

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Definition

- A **linear polynomial matrix (or matrix pencil)** $L(\lambda)$ is a **(GLR) linearization** of $P(\lambda) = P_d\,\lambda^d + \cdots + P_1\lambda + P_0$ if there exist **unimodular** polynomial matrices $U(\lambda), V(\lambda)$ such that

$$\boxed{U(\lambda)\,L(\lambda)\,V(\lambda) = \begin{bmatrix} I_s & 0 \\ 0 & P(\lambda) \end{bmatrix}}.$$

- $L(\lambda)$ is a **(GLR) strong linearization** of $P(\lambda)$ if, **in addition**, $\mathrm{rev}\, L(\lambda)$ is a linearization for $\mathrm{rev}\, P(\lambda)$, i.e.,

$$\boxed{\widetilde{U}(\lambda)\,(\mathrm{rev}\, L(\lambda))\,\widetilde{V}(\lambda) = \begin{bmatrix} I_s & 0 \\ 0 & \mathrm{rev}\, P(\lambda) \end{bmatrix}},$$

with $\widetilde{U}(\lambda)$ and $\widetilde{V}(\lambda)$ unimodular.

Theorem

A matrix pencil $L(\lambda)$ is a (GLR) linearization *of a polynomial matrix $P(\lambda)$* **if and only if**

(1) $L(\lambda)$ *and* $P(\lambda)$ **have the same number of right minimal indices**.

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(3) $L(\lambda)$ *and* $P(\lambda)$ **have the same finite eigenvalues** *with the same partial multiplicities.*

$L(\lambda)$ **is a (GLR) strong linearization** *of* $P(\lambda)$ **if and only if** *(1), (2), (3) and*

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Remark: The **minimal indices** of $L(\lambda)$ **may have arbitrarily different values** from those of $P(\lambda)$, though in the most important classes of (GLR) linearizations they are easily related.

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The most famous strong linearization

The classical **Frobenius companion form** of the $m \times n$ matrix polynomial

$$P(\lambda) = P_d \lambda^d + \cdots + P_1 \lambda + P_0$$

is

$$C_1(\lambda) := \begin{bmatrix} \lambda P_d + P_{d-1} & P_{d-2} & \cdots & P_1 & P_0 \\ -I_n & \lambda I_n & & & \\ & \ddots & \ddots & & \\ & & \ddots & \lambda I_n & \\ & & & -I_n & \lambda I_n \end{bmatrix} \in \mathbb{C}[\lambda]^{(m+n(d-1))\times nd}$$

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Since polynomial matrices are also rational matrices the next definition applies to both. $R(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ denotes that $R(\lambda)$ is a $m \times n$ rational matrix.

Definition (D, Quintana, Van Dooren, SIMAX, 2022)

A **strongly minimal linearization** of $R(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ is a matrix pencil

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & -(B_1\lambda + B_0) \\ C_1\lambda + C_0 & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{C}[\lambda]^{(p+m)\times(p+n)}$$

such that:

(a) $R(\lambda) = (D_1\lambda + D_0) + (C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0)$,

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is a strongly minimal linearization of $R(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ then:

- *The finite eigenvalue structure of $L(\lambda)$ coincides exactly with the finite zero structure of $R(\lambda)$.*
- *The finite eigenvalue structure of $A_1\lambda + A_0$ coincides exactly with the finite pole structure of $R(\lambda)$.*
- *The infinite eigenvalue structure of $L(\lambda)$ and $A(\lambda)$ allows us to recover exactly the infinite zero/pole structure of $R(\lambda)$ (next slide).*
- *$L(\lambda)$ and $R(\lambda)$* **have the same left and right minimal indices**.

Theorem (Recovery at infinity)

If $R(\lambda)$ has normal rank r, $0 < e_1 \leq \cdots \leq e_s$ are the partial multiplicities of $\operatorname{rev} A(\lambda)$ at 0, and $0 < \widetilde{e}_1 \leq \cdots \leq \widetilde{e}_u$ are the partial multiplicities of $\operatorname{rev} L(\lambda)$ at 0, then the structural indices at infinity of $R(\lambda)$ are

$$(d_1, d_2, \ldots, d_r) = (-e_s, -e_{s-1}, \ldots, -e_1, \underbrace{0, \ldots, 0}_{r-s-u}, \widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_u) - (1, 1, \ldots, 1).$$

Recovery of eigenvectors and minimal bases

The eigenvectors and minimal bases of $R(\lambda)$ can be recovered from those of $L(\lambda)$ simply by removing the first p entries.

Relation with GLR linearizations

- Strongly minimal linearizations are GLR-linearizations.
- Strongly minimal linearizations are NOT strong GLR-linearizations.
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For any

$$P(\lambda) = P_d \lambda^d + \cdots + P_1 \lambda + P_0 \in \mathbb{C}[\lambda]^{m\times n}$$

we define

$$L_s(\lambda) = \left[\begin{array}{cccc|c} & & & -P_d & \lambda P_d \\ & & \iddots & \lambda P_d - P_{d-1} & \vdots \\ & -P_d & \iddots & \vdots & \vdots \\ -P_d & \lambda P_d - P_{d-1} & \dots & \lambda P_3 - P_2 & \lambda P_2 \\ \hline \lambda P_d & \dots & \dots & \lambda P_2 & \lambda P_1 + P_0 \end{array}\right]$$

- It was proposed by Lancaster for regular polynomial matrices with P_d invertible in 1966!!
- If P_d is invertible, then $L_s(\lambda)$ is a GLR strong linearization of $P(\lambda)$. Otherwise, **it is not a GLR-linearization**.
- $L_s(\lambda)$ is one of the famous $\mathbb{DL}(P)$ pencils introduced by Mackey, Mackey, Mehl and Mehrmann (SIMAX, 2006) and further studied by Nakatsukasa, Noferini and Townsend (SIMAX, 2017). The one with ansatz vector e_d.

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we define

$$T = \begin{bmatrix} & & & P_d \\ & & \iddots & P_{d-1} \\ & P_d & \iddots & \vdots \\ P_d & P_{d-1} & \dots & P_2 \end{bmatrix}$$

and consider a rank-revealing factorization of T, for instance a SVD,

$$U^* T V = \begin{bmatrix} 0 & 0 \\ 0 & \widehat{T} \end{bmatrix},$$

where U, V, and $\widehat{T} \in \mathbb{C}^{r \times r}$ are invertible.

Based on

$$L_s(\lambda) = \left[\begin{array}{cccc|c} & & & -P_d & \lambda P_d \\ & & \iddots & \lambda P_d - P_{d-1} & \vdots \\ & -P_d & \iddots & \vdots & \vdots \\ -P_d & \lambda P_d - P_{d-1} & \dots & \lambda P_3 - P_2 & \lambda P_2 \\ \hline \lambda P_d & \dots & \dots & \lambda P_2 & \lambda P_1 + P_0 \end{array}\right],$$

we define

$$T = \begin{bmatrix} & & & P_d \\ & & \iddots & P_{d-1} \\ & P_d & \iddots & \vdots \\ P_d & P_{d-1} & \dots & P_2 \end{bmatrix}$$

and consider a rank-revealing factorization of T, for instance a SVD,

$$U^* T V = \begin{bmatrix} 0 & 0 \\ 0 & \widehat{T} \end{bmatrix},$$

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A rank revealing factorization of a constant matrix associated to $L_s(\lambda)$

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A strongly minimal linearization for $P(\lambda)$

Theorem (D, Quintana, Van Dooren, SIMAX, 2022)

$$\left[\begin{array}{c|c} U^* & \\ \hline & I_m \end{array}\right]
\left[\begin{array}{cccc|c}
 & & & -P_d & \lambda P_d \\
 & & \iddots & \lambda P_d - P_{d-1} & \vdots \\
 & -P_d & \iddots & \vdots & \vdots \\
-P_d & \lambda P_d - P_{d-1} & \dots & \lambda P_3 - P_2 & \lambda P_2 \\ \hline
\lambda P_d & \dots & \dots & \lambda P_2 & \lambda P_1 + P_0
\end{array}\right]
\left[\begin{array}{c|c} V & \\ \hline & I_n \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & \widehat{A}_s(\lambda) & -\widehat{B}_s(\lambda) \\ \hline 0 & \widehat{C}_s(\lambda) & \widehat{D}_s(\lambda) \end{array}\right],$$

where $\widehat{A}_s(\lambda) \in \mathbb{C}[\lambda]^{r\times r}$ *is regular*

and

$$\widehat{L}_s(\lambda) = \left[\begin{array}{c|c} \widehat{A}_s(\lambda) & -\widehat{B}_s(\lambda) \\ \hline \widehat{C}_s(\lambda) & \widehat{D}_s(\lambda) \end{array}\right]$$

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$$

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$$

and

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Comments on preservation of structures

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- is Hermitian (resp. skew-Hermitian) if $P(\lambda)$ is.
- Moreover, the rank-revealing factorization of T can be chosen to preserve the Hermitian (resp. skew-Hermitian) structure and, so, to get a
- Hermitian (resp. skew-Hermitian) strongly minimal linearization of $P(\lambda)$.
- Using appropriate block diagonal scalings $S := \mathrm{diag}((-1)^{(d-1)} I_m, \dots, (-1)^2 I_m, -I_m)$ in the factors of the rank-revealing factorization of T, the process above can be easily adapted to preserve alternating structures of $P(\lambda)$.

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$$P(\lambda) = P_0 + \lambda P_1 + \lambda^2 P_2 \in \mathbb{C}[\lambda]^{m\times n}$$

- The Lancaster pencil is very simple in the quadratic case

$$L_s(\lambda) = \left[\begin{array}{c|c} -P_2 & \lambda P_2 \\ \hline \lambda P_2 & \lambda P_1 + P_0 \end{array}\right] \in \mathbb{C}[\lambda]^{2m\times 2n} \quad \text{and} \quad T = P_2.$$

- If $P_2 = U_2 \widehat{T} V_2^*$, with $\widehat{T} \in \mathbb{C}^{r_2\times r_2}$ invertible and $U_2 \in \mathbb{C}^{m\times r_2}, V_2 \in \mathbb{C}^{n\times r_2}$ with orthornormal columns. Then

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  is a strongly minimal linearization of $P(\lambda)$.
- In important applications, the leading coefficient P_2 has low rank r_2.
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Outline

1. Brief reminder of “Eigenstructures” of PEPs and REPs
2. Gohberg-Lancaster-Rodman linearizations of PEPs (and REPs)
3. Strongly minimal linearizations of polynomial and rational matrices
4. Constructing strongly minimal linearizations of polynomial matrices
5. **Constructing strongly minimal linearizations of rational matrices**
6. Conclusions

Polynomial and strictly proper parts of a rational matrix

Any rational matrix $R(\lambda)$ can be uniquely expressed as

$$R(\lambda)=P(\lambda)+R_{sp}(\lambda),$$

where

1. $P(\lambda)$ is a polynomial matrix (polynomial part of $R(\lambda)$), and
2. the rational matrix $R_{sp}(\lambda)$ is **strictly proper** (strictly proper part of $R(\lambda)$), i.e., $\lim_{\lambda\to\infty} R_{sp}(\lambda)=0$.

For strictly proper rational matrices $R_{sp}(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$, we represent them via a Laurent expansion around the point at infinity

$$R_{sp}(\lambda) := R_{-1}\lambda^{-1} + R_{-2}\lambda^{-2} + R_{-3}\lambda^{-3} + \dots$$

and consider the block Hankel matrix H and shifted block Hankel matrix H_σ:

$$H := \begin{bmatrix} R_{-1} & R_{-2} & \dots & R_{-k} \\ R_{-2} & & \iddots & R_{-k-1} \\ \vdots & \iddots & \iddots & \vdots \\ R_{-k} & R_{-k-1} & \dots & R_{-2k+1} \end{bmatrix}, \quad H_\sigma := \begin{bmatrix} R_{-2} & R_{-3} & \dots & R_{-k-1} \\ R_{-3} & & \iddots & R_{-k-2} \\ \vdots & \iddots & \iddots & \vdots \\ R_{-k-1} & R_{-k-2} & \dots & R_{-2k} \end{bmatrix}.$$

For sufficiently large k the rank r_f of H equals the total polar degree of the finite poles, i.e., the sum of the degrees of the denominators in the Smith McMillan form of $R_{sp}(\lambda)$ and does not increase more with k.

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Strongly minimal linearizations for strictly proper rational matrices (II)

Theorem (D, Quintana, Van Dooren, SIMAX, 2022)

Let $R_{sp}(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ be a strictly proper rational matrix. Let H and H_σ be the block Hankel matrices and $r_f := \operatorname{rank} H$. Let $U := \begin{bmatrix} U_1 & U_2 \end{bmatrix}$ and $V := \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ be unitary matrices such that

$$U^* H V = \begin{bmatrix} \widehat{H} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} U_1^* H V_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where $\widehat{H}$ is $r_f \times r_f$ and invertible. Partition the matrices U_1 and V_1 as

$$U_1 = \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix}, \quad \text{and} \quad V_1 = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix},$$

where the matrices U_{11} and V_{11} have dimension $m \times r_f$ and $n \times r_f$. Then

$$L_{sp}(\lambda) := \left[\begin{array}{c|c} U_1^* H_\sigma V_1 - \lambda \widehat{H} & \widehat{H} V_{11}^* \\ \hline U_{11} \widehat{H} & 0 \end{array}\right]$$

is a strongly minimal linearization for $R_{sp}(\lambda)$. Consider $\boldsymbol{U} = \boldsymbol{V}$ if $R_{sp}(\lambda)$ is Hermitian or skew-Hermitian.

Theorem (D, Quintana, Van Dooren, SIMAX, 2022)

Let $R(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ be an arbitrary (resp. structured) rational matrix. Let

$$R(\lambda) = P(\lambda) + R_{sp}(\lambda),$$

with $P(\lambda)$ polynomial and $R_{sp}(\lambda)$ strictly proper. Let

$$\widehat{L}_s(\lambda) := \left[\begin{array}{c|c} \widehat{A}_s(\lambda) & \widehat{B}_s(\lambda) \\ \hline -\widehat{C}_s(\lambda) & \widehat{D}_s(\lambda) \end{array}\right] \quad \text{and} \quad L_{sp}(\lambda) := \left[\begin{array}{c|c} A_{sp}(\lambda) & B_{sp}(\lambda) \\ \hline -C_{sp}(\lambda) & 0 \end{array}\right]$$

be (resp. structured) strongly minimal linearizations of $P(\lambda)$ and $R_{sp}(\lambda)$, respectively. Then

$$L(\lambda) := \left[\begin{array}{cc|c} \widehat{A}_s(\lambda) & 0 & \widehat{B}_s(\lambda) \\ 0 & A_{sp}(\lambda) & B_{sp}(\lambda) \\ \hline -\widehat{C}_s(\lambda) & -C_{sp}(\lambda) & \widehat{D}_s(\lambda) \end{array}\right]$$

is a (structured) strongly minimal linearization of $R(\lambda)$.

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- We have introduced the new definition of strongly minimal linearizations.
- It is simultaneously valid for polynomial and rational matrices.
- We have proved that they have excellent recovery properties.
- We showed that they exist for any rational matrix
- and we have shown how to construct them via stable rank-revealing decompositions of constant matrices (SVD, for instance).
- Our constructions always preserve the Hermitian, skew-Hermitian, and alternating structures,
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