Minimal rank factorizations of low rank polynomial matrices

Froilán M. Dopico

joint work with **Andrii Dmytryshyn** (Örebro University, Sweden) and **Paul Van Dooren** (UC Louvain, Belgium)

Departamento de Matemáticas
Universidad Carlos III de Madrid, Spain

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- We will consider the vector space

$$\mathrm{POL}_d^{m\times n} := \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \end{array} \right\}.$$

- The Euclidean distance in $\mathrm{POL}_d^{m\times n}$ is defined as follows. Given

$$P(\lambda) = \lambda^d P_d + \cdots + \lambda P_1 + P_0 \in \mathrm{POL}_d^{m\times n}, \quad (P_i \in \mathbb{C}^{m\times n}),$$
$$Q(\lambda) = \lambda^d Q_d + \cdots + \lambda Q_1 + Q_0 \in \mathrm{POL}_d^{m\times n}, \quad (Q_i \in \mathbb{C}^{m\times n}),$$

$$\rho(P,Q) := \sqrt{\sum_{i=0}^{d} \|P_i - Q_i\|_F^2}.$$

- It makes $\mathrm{POL}_d^{m\times n}$ a metric space and we can consider closures of subsets of $\mathrm{POL}_d^{m\times n}$, as well as any other topological concept.
- The closure of any set $\mathcal{A}$ is denoted by $\overline{\mathcal{A}}$.

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Setting (II): The subsets of $\mathrm{POL}_d^{m\times n}$ studied in this talk

- **Our main goal** is to describe the elements $P(\lambda)$ in the sets of singular polynomials

$$\mathrm{POL}_{d,r}^{m\times n} := \left\{ \begin{array}{c} m\times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \\ \text{and (normal) rank at most } r < \min\{m,n\} \end{array} \right\} \subset \mathrm{POL}_d^{m\times n}$$

- as products of **two polynomial factors** $L(\lambda) \in \mathbb{C}[\lambda]^{m\times r}$, $R(\lambda) \in \mathbb{C}[\lambda]^{r\times n}$

$$P(\lambda) = L(\lambda)R(\lambda) =$$

- **with certain matching properties for the degrees of the columns of $L(\lambda)$ and the rows of $R(\lambda)$.**
- Moreover, we will connect the new factor description and the one of $\mathrm{POL}_{d,r}^{m\times n}$ in terms **of generic eigenstructures**.

A. Dmytryshyn and F.M. Dopico, Generic complete eigenstructures for sets of matrix polynomials with bounded rank and degree, Linear Algebra Appl., 535 (2017) 213–230.

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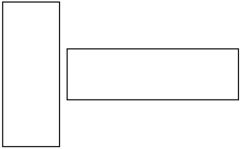

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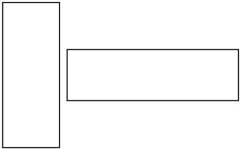

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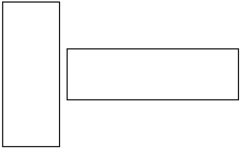

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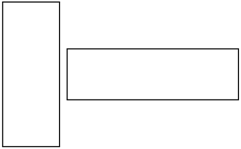

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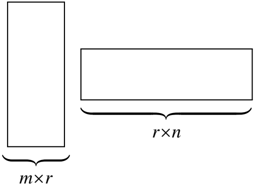

in such a way that

- the degrees of the columns of $L(\lambda)$ differ at most by one (they try to be as equal as possible),
- the degrees of the rows of $R(\lambda)$ differ at most by one (they try to be as equal as possible), and
- $\deg \text{col}_i(L) + \deg \text{row}_i(R) = d, \quad$ for $i = 1, \dots, r$.
- We refer to these properties as "the column degrees of $L(\lambda)$ and the row degrees of $R(\lambda)$ are generically almost homogeneous and are paired-up to sum d."

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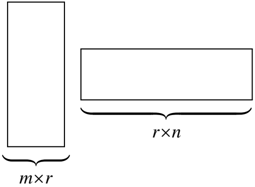

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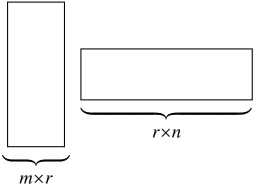

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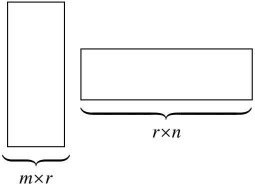

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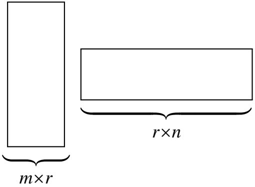

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Example illustrating the main result

$$P(\lambda) = \begin{bmatrix} 0 & \lambda^2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \lambda^2 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 1 & \lambda^2 & 1 \\ 0 & \lambda^2 & 1 \end{bmatrix} \in \mathrm{POL}_{2,2}^{3\times 3},$$

cannot be factorized with "almost homogeneous column and row degrees paired up to sum 2".

But if we perturb $P(\lambda)$ as follows

$$P_\epsilon(\lambda) = \begin{bmatrix} \lambda^2 & 0 & -\epsilon\lambda \\ 1 & \lambda^2 + \epsilon\lambda & 1 \\ 0 & \lambda^2 & 1 \end{bmatrix} \in \mathrm{POL}_{2,2}^{3\times 3},$$

then $P_\epsilon(\lambda)$ can be factorized as

$$P_\epsilon(\lambda) = \begin{bmatrix} -\epsilon\lambda & 0 \\ 1 & \frac{1}{\epsilon}\lambda + 1 \\ 1 & \frac{1}{\epsilon}\lambda \end{bmatrix} \begin{bmatrix} -\frac{1}{\epsilon}\lambda & 0 & 1 \\ 1 & \epsilon\lambda & 0 \end{bmatrix}.$$

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Remark: the pairing of the degrees to sum d is essential

If not, one can do essentially "everything" with the degrees of the factors by cancelling high degree terms. For instance:

$$P(\lambda) = \begin{bmatrix} \lambda^2 & \lambda^2 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \lambda^2 & 1 \\ 1 & -\lambda^2 & -1 \end{bmatrix} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 1 & \lambda^2 & 1 \\ 0 & \lambda^2 & 1 \end{bmatrix} \in \mathrm{POL}_{2,2}^{3\times 3},$$

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- In the case of matrix pencils, matrix polynomials of degree at most one,
- describing the set $\mathrm{POL}_{1,r}^{m\times n} =: \mathrm{PENCIL}_{r}^{m\times n}$ in terms of factors

> F. De Terán, F.M. Dopico, J.M. Landsberg, An explicit description of the irreducible components of the set of matrix pencils with bounded normal rank, Linear Algebra Appl., 520 (2017) 80–103

- has been fundamental for determining the generic change of the partial multiplicities of regular matrix pencils under **low-rank perturbations** for unstructured pencils

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- and for structured pencils

> F. De Terán, C. Mehl, V. Mehrmann, Low-rank perturbation of regular matrix pencils with symmetry structures, Foundations of Computational Mathematics, 22 (2022) 257–311

- since such descriptions allow to parameterize $\mathrm{PENCIL}_{r}^{m\times n}$.
- We hope that a "similar" description of $\mathrm{POL}_{d,r}^{m\times n}$ for arbitrary degrees d may help to solve the corresponding generic low-rank perturbation problem for matrix polynomials.

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- describing the set $\mathrm{POL}_{1,r}^{m\times n} =: \mathrm{PENCIL}_r^{m\times n}$ in terms of factors

F. De Terán, F.M. Dopico, J.M. Landsberg, An explicit description of the irreducible components of the set of matrix pencils with bounded normal rank, Linear Algebra Appl., 520 (2017) 80–103

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Outline

1. **A review of the results for pencils**
2. **The set of matrix polynomials with bounded rank and degree in terms of eigenstructures**
3. **The set of matrix polynomials with bounded rank and degree in terms of factors**

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1. A review of the results for pencils

2. The set of matrix polynomials with bounded rank and degree in terms of eigenstructures

3. The set of matrix polynomials with bounded rank and degree in terms of factors

- **All the $m \times n$ pencils with the same complete eigenstructure form an orbit** under strict equivalence:

$$\mathrm{O}(\lambda A + B) := \{P(\lambda A + B)Q \mid \det P \cdot \det Q \neq 0\}.$$

- The complete eigenstructure of a pencil is determined by its **Kronecker canonical form (KCF)** under strict equivalence, which is a direct sum of four types of canonical matrix pencils:

- the regular $k \times k$ Jordan blocks for **finite and infinite eigenvalues**

$$\mathcal{J}_k(\mu) := \begin{bmatrix} \lambda - \mu & 1 & & \\ & \lambda - \mu & \ddots & \\ & & \ddots & 1 \\ & & & \lambda - \mu \end{bmatrix}, \quad \mathcal{J}_k(\infty) := \begin{bmatrix} 1 & \lambda & & \\ & 1 & \ddots & \\ & & \ddots & \lambda \\ & & & 1 \end{bmatrix} \quad k = 1, 2, 3, \ldots$$

- the singular $k \times (k+1)$ and $(k+1) \times k$ blocks for right and left **minimal indices** of value k

$$\mathcal{L}_k := \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \end{bmatrix}, \quad \mathcal{L}_k^T, \quad k = 0, 1, 2, \ldots$$

Matrix pencils and Kronecker Canonical Form

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$$\text{PENCIL}_r^{m\times n} = \left\{ \begin{array}{c} m\times n \text{ complex matrix pencils} \\ \text{with rank at most } r < \min\{m,n\} \end{array} \right\} = \bigcup_{0\le a\le r} \overline{\text{O}}(\mathcal{K}_a),$$

where the $m\times n$ complex matrix pencils $\mathcal{K}_a, a = 0, 1, \ldots, r$, have rank r and the KCF

$$\mathcal{K}_a = \text{diag}\left(\overbrace{\underbrace{\underbrace{\mathcal{L}_{\alpha+1}, \ldots, \mathcal{L}_{\alpha+1}}_{s}, \underbrace{\mathcal{L}_{\alpha}, \ldots, \mathcal{L}_{\alpha}}_{n-r-s}}_{\text{rank}=a}}^{\text{right minimal indices}}, \overbrace{\underbrace{\underbrace{\mathcal{L}^T_{\beta+1}, \ldots, \mathcal{L}^T_{\beta+1}}_{t}, \underbrace{\mathcal{L}^T_{\beta}, \ldots, \mathcal{L}^T_{\beta}}_{m-r-t}}_{\text{rank}=r-a}}^{\text{left minimal indices}} \right)$$

with $\alpha = \lfloor a/(n-r) \rfloor$ and $s = a \mod (n-r)$,
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$\bigcup_{0\le a\le r} \mathrm{O}(\mathcal{K}_a)$ is an open dense subset of $\mathrm{PENCIL}_r^{m\times n}$. So, **generically, the $m \times n$ pencils with rank at most r have only $r+1$ possible KCFs given by $\mathcal{K}_a$ for $a = 0, 1, \ldots, r$.**

The previous result in simple words

The pencils in $\mathrm{O}(\mathcal{K}_a) \subset \mathrm{PENCIL}_r^{m\times n}$

- do not have eigenvalues (finite or infinite),
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- The parameter $a = 0, 1, \ldots, r$ determines how much rank is attached to the right minimal indices and, then, the rank attached to the left minimal indices is determined by the index sum theorem.
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where, for $a = 0, 1, \ldots, r$,

$$\mathcal{C}_a := \left\{ L(\lambda)R(\lambda) \;:\; \begin{array}{l} L(\lambda) \in \mathrm{PENCIL}_r^{m\times r},\; R(\lambda) \in \mathrm{PENCIL}_r^{r\times n}, \\ \deg \mathit{row}_i(R) = 0, \quad \text{for } i = a+1, \ldots, r, \\ \deg \mathit{col}_i(L) = 0, \quad \text{for } i = 1, \ldots, a \end{array} \right\}.$$

Moreover,

$$\mathcal{C}_a = \overline{\mathrm{O}}(\mathcal{K}_a),$$

where $\mathcal{K}_a$ *are the* $m\times n$ *pencils with rank exactly* r *and with the generic eigenstructures defined in the previous slides.*

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Some comments on the previous theorem

- There are no closures involved in $\mathrm{PENCIL}_r^{m\times n} = \bigcup_{0\le a\le r} \mathcal{C}_a$.
- The conditions of the theorem guarantee the pairing $\deg \mathrm{col}_i(L) + \deg \mathrm{col}_i(R) \le 1$ for $i = 1, \ldots, r$ (generically we will have equality) of the column degrees of $L(\lambda)$ and of the row degrees of $R(\lambda)$.
- Since $L(\lambda)$ and $R(\lambda)$ are pencils the degrees of its columns and rows, respectively, differ automatically at most by 1.
- The factors $L(\lambda)$ and $R(\lambda)$ can be easily parameterized.
- The parameter $a = 0, 1, \ldots, r$ determines the (maximal) sum of the degrees of the rows of $R(\lambda)$.
- In addition, it was proved that $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_r$ are the irreducible components of the closed set $\mathrm{PENCIL}_r^{m\times n}$ (in the Zariski topology).

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Outline

1. A review of the results for pencils

2. **The set of matrix polynomials with bounded rank and degree in terms of eigenstructures**

3. The set of matrix polynomials with bounded rank and degree in terms of factors

Complete eigenstructure of matrix polynomials

$$P(\lambda) = \lambda^d P_d + \cdots + \lambda P_1 + P_0, \qquad P_i \in \mathbb{C}^{m\times n}$$

- **Essentially the same as in pencils** but definitions more complicated since there is NO KCF.
- **Finite and infinite eigenvalues and their elementary divisors** defined with Smith Form under unimodular equivalence of $P(\lambda)$ and rev$P(\lambda)$:

$$U(\lambda)P(\lambda)V(\lambda) = \text{diag}\left(g_1(\lambda), \ldots, g_r(\lambda)\right) \oplus 0_{(m-r)\times(n-r)}, \quad g_j(\lambda) \mid g_{j+1}(\lambda).$$

Invariant polynomials: $g_j(\lambda) = (\lambda - \alpha_1)^{\delta_{j1}} \cdot (\lambda - \alpha_2)^{\delta_{j2}} \cdot \ldots \cdot (\lambda - \alpha_{l_j})^{\delta_{jl_j}}$.
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$$\mathcal{N}_{\text{left}}(P) := \{y(\lambda)^T \in \mathbb{C}(\lambda)^{1\times m} : y(\lambda)^T P(\lambda) = 0_{1\times n}\},$$
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$$\mathrm{POL}_{d,r}^{m\times n} = \left\{ \begin{array}{c} m\times n \text{ complex matrix polynomials} \\ \text{with degree at most } d\text{, with rank at most } r < \min\{m,n\} \end{array} \right\} = \bigcup_{0\le a\le rd} \overline{\mathrm{O}}(\mathcal{K}_a),$$

where the $m \times n$ complex matrix polynomial $\mathcal{K}_a$, $a = 0, 1, \ldots, rd$, has

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A. Dmytryshyn and F.M. Dopico, Generic complete eigenstructures for sets of matrix polynomials with bounded rank and degree, Linear Algebra Appl., 535 (2017) 213–230

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$\bigcup_{0\le a\le rd} \mathrm{O}(K_a)$ is an open dense subset of $\mathrm{POL}_{d,r}^{m\times n}$. So, **generically, the $m\times n$ matrix polys with degree at most d and with rank at most r have only $rd+1$ possible complete eigenstructures given by $\mathbf{K}_a$ for $a = 0, 1, \ldots, rd$.**

The previous result in simple words

The matrix polynomials in $\mathrm{O}(K_a) \subset \mathrm{POL}_{d,r}^{m\times n}$ of degree exactly d and rank exactly r

- do not have eigenvalues (finite or infinite),
- have $n - r$ right minimal indices differing at most by 1 (almost homogeneous) and summing up a, and
- have $m - r$ left minimal indices differing at most by 1 (almost homogeneous) and summing up $rd - a$.
- The parameter $a = 0, 1, \ldots, rd$ determines in the index sum theorem

$$\left(\sum \text{right minimal indices}\right) + \left(\sum \text{left minimal indices}\right) = rd$$

how much of the total sum corresponds to the right minimal indices.

- These are the generic eigenstructures in $\mathrm{POL}_{d,r}^{m\times n}$.

This description in terms of eigenstructures is very similar to the result for pencils, but the description of $\mathrm{POL}_{d,r}^{m\times n}$ in terms of factors is missing.

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- Any $m \times n$ constant matrix A of rank r can be written as

$$A = LR, \quad \text{where} \begin{cases} L \text{ is } m \times r \text{ and } \operatorname{rank} L = r, \\ R \text{ is } r \times n \text{ and } \operatorname{rank} R = r. \end{cases}$$

- The idea is to get a similar description of $\mathrm{POL}_{d,r}^{m\times n}$ but the degree of the factors makes the problem not trivial: it might be cancellations of "high degrees", how to distribute degrees between the factors, etc.
- Nevertheless, generically, i.e., using closures of open dense sets, we can prove that if $P(\lambda) \in \mathrm{POL}_{d,r}^{m\times n}$

$$P(\lambda) = L(\lambda)R(\lambda),$$

where

1. $L(\lambda)$ is an $m \times r$ matrix polynomial, $\operatorname{rank} L(\lambda) = r$, and degrees of its columns differ at most by one,
2. $R(\lambda)$ is an $r \times n$ matrix polynomial, $\operatorname{rank} R(\lambda) = r$, and degrees of its rows differ at most by one, and
3. $\deg \mathrm{col}_i(L(\lambda)) + \deg \mathrm{row}_i(R(\lambda)) = d$, for $i = 1, \ldots, r$.

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- The idea is to get a similar description of $\mathrm{POL}_{d,r}^{m\times n}$ but the degree of the factors makes the problem not trivial: it might be cancellations of "high degrees", how to distribute degrees between the factors, etc.

- Nevertheless, generically, i.e., using closures of open dense sets, we can prove that if $P(\lambda) \in \mathrm{POL}_{d,r}^{m\times n}$

$$P(\lambda) = L(\lambda)R(\lambda),$$

where

1. $L(\lambda)$ is an $m \times r$ matrix polynomial, $\operatorname{rank} L(\lambda) = r$, and degrees of its columns differ at most by one,
2. $R(\lambda)$ is an $r \times n$ matrix polynomial, $\operatorname{rank} R(\lambda) = r$, and degrees of its rows differ at most by one, and
3. $\deg \mathrm{col}_i(L(\lambda)) + \deg \mathrm{row}_i(R(\lambda)) = d$, for $i = 1, \ldots, r$.

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Theorem (Dmytryshyn, D., Van Dooren)

$$\mathrm{POL}_{d,r}^{m\times n} = \left\{ \begin{array}{c} m \times n \textit{ complex matrix polynomials} \\ \textit{with degree at most } d, \textit{ with rank at most } r < \min\{m,n\} \end{array} \right\} = \bigcup_{0\le a\le rd} \overline{\mathcal{B}_a},$$

where, for $a = 0, 1, \ldots, rd$,

$$\mathcal{B}_a := \left\{ L(\lambda)R(\lambda) \; : \; \begin{array}{l} L(\lambda) \in \mathbb{C}[\lambda]^{m\times r},\; R(\lambda) \in \mathbb{C}[\lambda]^{r\times n}, \\ \deg \textit{row}_i(R) = d_R + 1, \quad \textit{for } i = 1, \ldots, t_R, \\ \deg \textit{row}_i(R) = d_R, \quad \textit{for } i = t_R + 1, \ldots, r, \\ \deg \textit{col}_i(L) = d - \deg \textit{row}_i(R), \quad \textit{for } i = 1, \ldots, r \end{array} \right\},$$

with $d_R = \lfloor a/r \rfloor$ *and* $t_R = a \bmod r$. *Moreover,*

$$\overline{\mathcal{B}_a} = \overline{\mathrm{O}}(K_a),$$

where K_a *are the* $m \times n$ *matrix polynomials of degree exactly* d *and rank exactly* r *with the generic eigenstructures defined in the previous section.*

The precise main new result

Theorem (Dmytryshyn, D., Van Dooren)

$$\mathrm{POL}_{d,r}^{m\times n} = \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d, \text{ with rank at most } r < \min\{m,n\} \end{array} \right\} = \bigcup_{0\le a\le rd} \overline{\mathcal{B}_a},$$

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Some comments

- The factors $L(\lambda)$ and $R(\lambda)$ can be easily parameterized.
- The parameter $a = 0, 1, \ldots, rd$ determines the sum of the degrees of the rows of $R(\lambda)$.
- Though $\overline{\mathcal{B}_a} = \overline{\mathcal{O}}(K_a)$, it is easy to see that, in general, $\mathcal{B}_a \neq \mathcal{O}(K_a)$, even more $\mathcal{B}_a \not\subseteq \mathcal{O}(K_a)$ and $\mathcal{O}(K_a) \not\subseteq \mathcal{B}_a$.
- The proof of the main theorem is rather technical and needs several preliminary results but a key idea is the fact that $r \times n$ $(r < n)$ rectangular matrix polynomials with given row degrees are generically minimal bases with almost homogeneous right minimal indices.

F.M. Dopico and P. Van Dooren, Robustness and perturbations of minimal bases II: The case with given row degrees, Linear Algebra and its Applications, 576 (2019) 268-300

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THANK YOU VERY MUCH FOR YOUR ATTENTION!!