# Minimal rank factorizations of low rank polynomial matrices

## Froilán M. Dopico

joint work with **Andrii Dmytryshyn** (Örebro University, Sweden) and **Paul Van Dooren** (UC Louvain, Belgium)

Depto de Matemáticas, Universidad Carlos III de Madrid, Spain Part of "Proyecto de I+D+i PID2019-106362GB-I00 financiado por MCIN/AEI/10.13039/501100011033"

> Minisymposium "Bounded rank perturbations in matrix theory and related problems" 25th ILAS Conference. Madrid. June 12, 2023





uc3m Universidad Carlos III de Madrid

F. M. Dopico (U. Carlos III, Madrid)

 $\operatorname{POL}_d^{m \times n} \coloneqq \begin{cases} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \end{cases}.$ 

The Euclidean distance in POL<sup>m×n</sup> is defined as follows. Given

$$P(\lambda) = \lambda^d P_d + \dots + \lambda P_1 + P_0 \in \text{POL}_d^{m \times n}, \quad (P_i \in \mathbb{C}^{m \times n}),$$
  
$$Q(\lambda) = \lambda^d Q_d + \dots + \lambda Q_1 + Q_0 \in \text{POL}_d^{m \times n}, \quad (Q_i \in \mathbb{C}^{m \times n}),$$

$$\rho(P, Q) \coloneqq \sqrt{\sum_{i=0}^d ||P_i - Q_i||_F^2} \,.$$

 It makes POL<sup>m×n</sup> a metric space and we can consider closures of subsets of POL<sup>m×n</sup>, as well as any other topological concept.

The closure of any set A is denoted by A.

$$\operatorname{POL}_{d}^{m \times n} \coloneqq \begin{cases} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \end{cases}$$

• The Euclidean distance in  $POL_d^{m \times n}$  is defined as follows. Given

$$\begin{split} P(\lambda) &= \lambda^d P_d + \dots + \lambda P_1 + P_0 \in \mathrm{POL}_d^{m \times n}, \quad (P_i \in \mathbb{C}^{m \times n}), \\ Q(\lambda) &= \lambda^d Q_d + \dots + \lambda Q_1 + Q_0 \in \mathrm{POL}_d^{m \times n}, \quad (Q_i \in \mathbb{C}^{m \times n}), \end{split}$$

$$\rho(P,Q) \coloneqq \sqrt{\sum_{i=0}^d ||P_i - Q_i||_F^2}.$$

 It makes POL<sup>m×n</sup> a metric space and we can consider closures of subsets of POL<sup>m×n</sup>, as well as any other topological concept.

The closure of any set A is denoted by A.

$$\operatorname{POL}_{d}^{m \times n} \coloneqq \begin{cases} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \end{cases}$$

• The Euclidean distance in  $POL_d^{m \times n}$  is defined as follows. Given

$$\begin{split} P(\lambda) &= \lambda^d P_d + \dots + \lambda P_1 + P_0 \in \mathrm{POL}_d^{m \times n}, \quad (P_i \in \mathbb{C}^{m \times n}), \\ Q(\lambda) &= \lambda^d Q_d + \dots + \lambda Q_1 + Q_0 \in \mathrm{POL}_d^{m \times n}, \quad (Q_i \in \mathbb{C}^{m \times n}), \end{split}$$

$$\rho(P,Q) \coloneqq \sqrt{\sum_{i=0}^d ||P_i - Q_i||_F^2}.$$

- It makes POL<sup>m×n</sup> a metric space and we can consider closures of subsets of POL<sup>m×n</sup>, as well as any other topological concept.
- The closure of any set A is denoted by  $\overline{A}$ .

.

$$\operatorname{POL}_{d}^{m \times n} \coloneqq \begin{cases} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \end{cases}$$

• The Euclidean distance in  $POL_d^{m \times n}$  is defined as follows. Given

$$\begin{split} P(\lambda) &= \lambda^d P_d + \dots + \lambda P_1 + P_0 \in \mathrm{POL}_d^{m \times n}, \quad (P_i \in \mathbb{C}^{m \times n}), \\ Q(\lambda) &= \lambda^d Q_d + \dots + \lambda Q_1 + Q_0 \in \mathrm{POL}_d^{m \times n}, \quad (Q_i \in \mathbb{C}^{m \times n}), \end{split}$$

$$\rho(P,Q) \coloneqq \sqrt{\sum_{i=0}^d ||P_i - Q_i||_F^2}.$$

- It makes POL<sup>m×n</sup> a metric space and we can consider closures of subsets of POL<sup>m×n</sup>, as well as any other topological concept.
- The closure of any set A is denoted by  $\overline{A}$ .

$$\operatorname{POL}_{d}^{m \times n} \coloneqq \begin{cases} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \end{cases}$$

• The Euclidean distance in  $POL_d^{m \times n}$  is defined as follows. Given

$$\begin{split} P(\lambda) &= \lambda^d P_d + \dots + \lambda P_1 + P_0 \in \mathrm{POL}_d^{m \times n}, \quad (P_i \in \mathbb{C}^{m \times n}), \\ Q(\lambda) &= \lambda^d Q_d + \dots + \lambda Q_1 + Q_0 \in \mathrm{POL}_d^{m \times n}, \quad (Q_i \in \mathbb{C}^{m \times n}), \end{split}$$

$$\rho(P,Q) \coloneqq \sqrt{\sum_{i=0}^d ||P_i - Q_i||_F^2}.$$

- It makes POL<sup>m×n</sup> a metric space and we can consider closures of subsets of POL<sup>m×n</sup>, as well as any other topological concept.
- The closure of any set A is denoted by  $\overline{A}$ .

Our main goal is to describe the elements P(λ) in the sets of singular polynomials

$$\operatorname{POL}_{d,r}^{m \times n} := \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \\ \text{and (normal) rank at most } r < \min\{m, n\} \end{array} \right\} \subset \operatorname{POL}_{d}^{m \times n}$$

• as products of two polynomial factors  $L(\lambda) \in \mathbb{C}[\lambda]^{m \times r}$ ,  $R(\lambda) \in \mathbb{C}[\lambda]^{r \times n}$ 

$$P(\lambda) = L(\lambda)R(\lambda) =$$

• with certain matching properties for the degrees of the columns of  $L(\lambda)$  and the rows of  $R(\lambda)$ .

• Moreover, we will connect the new factor description and the one of POL<sup>mxn</sup><sub>d,r</sub> in terms of generic eigenstructures.

A. Dmytryshyn and F.M. Dopico, Generic complete eigenstructures for sets of matrix polynomials with bounded rank and degree, inear Algebra Appl., 535 (2017) 213–230

Our main goal is to describe the elements P(λ) in the sets of singular polynomials

$$\operatorname{POL}_{d,r}^{m \times n} := \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \\ \text{and (normal) rank at most } r < \min\{m, n\} \end{array} \right\} \subset \operatorname{POL}_{d}^{m \times n}$$

• as products of two polynomial factors  $L(\lambda) \in \mathbb{C}[\lambda]^{m \times r}$ ,  $R(\lambda) \in \mathbb{C}[\lambda]^{r \times n}$ 

$$P(\lambda) = L(\lambda)R(\lambda) =$$

• with certain matching properties for the degrees of the columns of  $L(\lambda)$  and the rows of  $R(\lambda)$ .

• Moreover, we will connect the new factor description and the one of POL<sup>*m×n*</sup><sub>*d,r*</sub> in terms of generic eigenstructures.

A. Dmytryshyn and F.M. Dopico, Generic complete eigenstructures for sets of matrix polynomials with bounded rank and degree, inear Algebra Appl., 535 (2017) 213–230

Our main goal is to describe the elements P(λ) in the sets of singular polynomials

$$\operatorname{POL}_{d,r}^{m \times n} := \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \\ \text{and (normal) rank at most } r < \min\{m, n\} \end{array} \right\} \subset \operatorname{POL}_{d}^{m \times n}$$

• as products of two polynomial factors  $L(\lambda) \in \mathbb{C}[\lambda]^{m \times r}$ ,  $R(\lambda) \in \mathbb{C}[\lambda]^{r \times n}$ 

$$P(\lambda) = L(\lambda)R(\lambda) =$$

• with certain matching properties for the degrees of the columns of  $L(\lambda)$  and the rows of  $R(\lambda)$ .

 Moreover, we will connect the new factor description and the one of POL<sup>mxn</sup><sub>d,r</sub> in terms of generic eigenstructures.

A. Dmytryshyn and F.M. Dopico, Generic complete eigenstructures for sets of matrix polynomials with bounded rank and degree, inear Algebra Appl., 535 (2017) 213–230

Our main goal is to describe the elements P(λ) in the sets of singular polynomials

$$\operatorname{POL}_{d,r}^{m \times n} := \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \\ \text{and (normal) rank at most } r < \min\{m, n\} \end{array} \right\} \subset \operatorname{POL}_{d}^{m \times n}$$

• as products of two polynomial factors  $L(\lambda) \in \mathbb{C}[\lambda]^{m \times r}$ ,  $R(\lambda) \in \mathbb{C}[\lambda]^{r \times n}$ 

$$P(\lambda) = L(\lambda)R(\lambda) =$$

• with certain matching properties for the degrees of the columns of  $L(\lambda)$  and the rows of  $R(\lambda)$ .

Moreover, we will connect the new factor description and the one of POL<sup>m×n</sup><sub>d,r</sub> in terms of generic eigenstructures.

A. Dmytryshyn and F.M. Dopico, Generic complete eigenstructures for sets of matrix polynomials with bounded rank and degree, Linear Algebra Appl., 535 (2017) 213–230

Our main goal is to describe the elements P(λ) in the sets of singular polynomials

$$\operatorname{POL}_{d,r}^{m \times n} := \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \\ \text{and (normal) rank at most } r < \min\{m, n\} \end{array} \right\} \subset \operatorname{POL}_{d}^{m \times n}$$

• as products of two polynomial factors  $L(\lambda) \in \mathbb{C}[\lambda]^{m \times r}$ ,  $R(\lambda) \in \mathbb{C}[\lambda]^{r \times n}$ 

$$P(\lambda) = L(\lambda)R(\lambda) =$$

• with certain matching properties for the degrees of the columns of  $L(\lambda)$  and the rows of  $R(\lambda)$ .

 Moreover, we will connect the new factor description and the one of POL<sup>m×n</sup><sub>d,r</sub> in terms of generic eigenstructures.

A. Dmytryshyn and F.M. Dopico, Generic complete eigenstructures for sets of matrix polynomials with bounded rank and degree, Linear Algebra Appl., 535 (2017) 213–230

#### **Generically** a matrix polynomial $P(\lambda) \in \text{POL}_{d,r}^{m \times n}$ can be factorized as



#### in such a way that

- the degrees of the columns of L(λ) differ at most by one (they try to be as equal as possible),
- the degrees of the rows of *R*(λ) differ at most by one (they try to be as equal as possible), and
- $\deg \operatorname{col}_i(L) + \deg \operatorname{row}_i(R) = d$ , for  $i = 1, \dots, r$ .

We refer to these properties as "the column degrees of L(λ) and the row degrees of R(λ) are generically almost homogeneous and are paired-up to sum d."

#### **Generically** a matrix polynomial $P(\lambda) \in \text{POL}_{d,r}^{m \times n}$ can be factorized as



in such a way that

- the degrees of the columns of L(λ) differ at most by one (they try to be as equal as possible),
- the degrees of the rows of R(λ) differ at most by one (they try to be as equal as possible), and
- $\deg \operatorname{col}_i(L) + \deg \operatorname{row}_i(R) = d$ , for  $i = 1, \dots, r$ .

We refer to these properties as "the column degrees of L(λ) and the row degrees of R(λ) are generically almost homogeneous and are paired-up to sum d."

#### **Generically** a matrix polynomial $P(\lambda) \in \text{POL}_{d,r}^{m \times n}$ can be factorized as



in such a way that

- the degrees of the columns of L(λ) differ at most by one (they try to be as equal as possible),
- the degrees of the rows of *R*(λ) differ at most by one (they try to be as equal as possible), and
- $\deg \operatorname{col}_i(L) + \deg \operatorname{row}_i(R) = d$ , for  $i = 1, \dots, r$ .

• We refer to these properties as "the column degrees of  $L(\lambda)$  and the row degrees of  $R(\lambda)$  are generically almost homogeneous and are paired-up to sum *d*."

#### **Generically** a matrix polynomial $P(\lambda) \in \text{POL}_{d,r}^{m \times n}$ can be factorized as



in such a way that

- the degrees of the columns of L(λ) differ at most by one (they try to be as equal as possible),
- the degrees of the rows of R(λ) differ at most by one (they try to be as equal as possible), and
- $\deg \operatorname{col}_i(L) + \deg \operatorname{row}_i(R) = d$ , for  $i = 1, \ldots, r$ .

• We refer to these properties as "the column degrees of  $L(\lambda)$  and the row degrees of  $R(\lambda)$  are generically almost homogeneous and are paired-up to sum *d*."

#### **Generically** a matrix polynomial $P(\lambda) \in \text{POL}_{d,r}^{m \times n}$ can be factorized as



in such a way that

- the degrees of the columns of L(λ) differ at most by one (they try to be as equal as possible),
- the degrees of the rows of *R*(λ) differ at most by one (they try to be as equal as possible), and
- $\deg \operatorname{col}_i(L) + \deg \operatorname{row}_i(R) = d$ , for  $i = 1, \ldots, r$ .
- We refer to these properties as "the column degrees of L(λ) and the row degrees of R(λ) are generically almost homogeneous and are paired-up to sum d."

$$P(\lambda) = \begin{bmatrix} 0 & \lambda^2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \lambda^2 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 1 & \lambda^2 & 1 \\ 0 & \lambda^2 & 1 \end{bmatrix} \in \text{POL}_{2,2}^{3 \times 3},$$

cannot be factorized with "almost homogeneous column and row degrees paired up to sum 2".

But if we perturb  $P(\lambda)$  as follows

$$P_{\epsilon}(\lambda) = \begin{bmatrix} \lambda^2 & 0 & -\epsilon\lambda \\ 1 & \lambda^2 + \epsilon\lambda & 1 \\ 0 & \lambda^2 & 1 \end{bmatrix} \in \text{POL}_{2,2}^{3\times3}$$

then  $P_{\epsilon}(\lambda)$  can be factorized as

$$P_{\epsilon}(\lambda) = \begin{bmatrix} -\epsilon\lambda & 0\\ 1 & \frac{1}{\epsilon}\lambda + 1\\ 1 & \frac{1}{\epsilon}\lambda \end{bmatrix} \begin{bmatrix} -\frac{1}{\epsilon}\lambda & 0 & 1\\ 1 & \epsilon\lambda & 0 \end{bmatrix}$$

イロト イポト イヨト イヨト

$$P(\lambda) = \begin{bmatrix} 0 & \lambda^2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \lambda^2 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 1 & \lambda^2 & 1 \\ 0 & \lambda^2 & 1 \end{bmatrix} \in \text{POL}_{2,2}^{3 \times 3},$$

cannot be factorized with "almost homogeneous column and row degrees paired up to sum 2". But if we perturb  $P(\lambda)$  as follows

$$P_{\epsilon}(\lambda) = \begin{bmatrix} \lambda^2 & 0 & -\epsilon\lambda \\ 1 & \lambda^2 + \epsilon\lambda & 1 \\ 0 & \lambda^2 & 1 \end{bmatrix} \in \text{POL}_{2,2}^{3\times3},$$

then  $P_{\epsilon}(\lambda)$  can be factorized as

$$P_{\epsilon}(\lambda) = \begin{bmatrix} -\epsilon\lambda & 0\\ 1 & \frac{1}{\epsilon}\lambda + 1\\ 1 & \frac{1}{\epsilon}\lambda \end{bmatrix} \begin{bmatrix} -\frac{1}{\epsilon}\lambda & 0 & 1\\ 1 & \epsilon\lambda & 0 \end{bmatrix}$$

$$P(\lambda) = \begin{bmatrix} 0 & \lambda^2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \lambda^2 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 1 & \lambda^2 & 1 \\ 0 & \lambda^2 & 1 \end{bmatrix} \in \text{POL}_{2,2}^{3 \times 3},$$

cannot be factorized with "almost homogeneous column and row degrees paired up to sum 2".

But if we perturb  $P(\lambda)$  as follows

$$P_{\epsilon}(\lambda) = \begin{bmatrix} \lambda^2 & 0 & -\epsilon\lambda \\ 1 & \lambda^2 + \epsilon\lambda & 1 \\ 0 & \lambda^2 & 1 \end{bmatrix} \in \text{POL}_{2,2}^{3\times 3},$$

then  $P_{\epsilon}(\lambda)$  can be factorized as

$$P_{\epsilon}(\lambda) = \begin{bmatrix} -\epsilon\lambda & 0\\ 1 & \frac{1}{\epsilon}\lambda + 1\\ 1 & \frac{1}{\epsilon}\lambda \end{bmatrix} \begin{bmatrix} -\frac{1}{\epsilon}\lambda & 0 & 1\\ 1 & \epsilon\lambda & 0 \end{bmatrix}.$$

< 口 > < 同 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ >

If not, one can do essentially "everything" with the degrees of the factors by cancelling high degree terms. For instance:

$$P(\lambda) = \begin{bmatrix} \lambda^2 & \lambda^2 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \lambda^2 & 1 \\ 1 & -\lambda^2 & -1 \end{bmatrix} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 1 & \lambda^2 & 1 \\ 0 & \lambda^2 & 1 \end{bmatrix} \in \text{POL}_{2,2}^{3 \times 3},$$

but at the cost of not "reading" the degree of the product from the degrees of the columns and rows, respectively, of the factors.

< 口 > < 同 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ >

6/23

If not, one can do essentially "everything" with the degrees of the factors by cancelling high degree terms. For instance:

$$P(\lambda) = \begin{bmatrix} \lambda^2 & \lambda^2 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \lambda^2 & 1 \\ 1 & -\lambda^2 & -1 \end{bmatrix} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 1 & \lambda^2 & 1 \\ 0 & \lambda^2 & 1 \end{bmatrix} \in \text{POL}_{2,2}^{3 \times 3},$$

but at the cost of not "reading" the degree of the product from the degrees of the columns and rows, respectively, of the factors.

< 口 > < 同 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ >

If not, one can do essentially "everything" with the degrees of the factors by cancelling high degree terms. For instance:

$$P(\lambda) = \begin{bmatrix} \lambda^2 & \lambda^2 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \lambda^2 & 1 \\ 1 & -\lambda^2 & -1 \end{bmatrix} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 1 & \lambda^2 & 1 \\ 0 & \lambda^2 & 1 \end{bmatrix} \in \text{POL}_{2,2}^{3 \times 3},$$

but at the cost of not "reading" the degree of the product from the degrees of the columns and rows, respectively, of the factors.

- In the case of matrix pencils, matrix polynomials of degree at most one,
- describing the set  $POL_{1,r}^{m \times n} =: PENCIL_{r}^{m \times n}$  in terms of factors

F. De Terán, F.M. Dopico, J.M. Landsberg, An explicit description of the irreducible components of the set of matrix pencils with bounded normal rank, Linear Algebra Appl., 520 (2017) 80–103

 has been fundamental for determining the generic change of the partial multiplicities of regular matrix pencils under low-rank perturbations for unstructured pencils

F. De Terán, F.M. Dopico, Generic change of the partial multiplicities of regular matrix pencils under low-rank perturbations, SIAM J. Matrix Anal. Appl., 37 (2016) 823–835

#### and for structured pencils

F. De Terán, C. Mehl, V. Mehrmann, Low-rank perturbation of regular matrix pencils with symmetry structures, Foundations of Computational Mathematics, 22 (2022) 257–311

• since such descriptions allow to parameterize  $PENCIL_r^{m \times n}$ .

 We hope that a "similar" description of POL<sup>m×n</sup><sub>d,r</sub> for arbitrary degrees d may help to solve the corresponding generic low-rank perturbation problem for matrix polynomials.

F. M. Dopico (U. Carlos III, Madrid)

Minimal rank factorizations matrix polys

- In the case of matrix pencils, matrix polynomials of degree at most one,
- describing the set  $POL_{1,r}^{m \times n} =: PENCIL_{r}^{m \times n}$  in terms of factors

F. De Terán, F.M. Dopico, J.M. Landsberg, An explicit description of the irreducible components of the set of matrix pencils with bounded normal rank, Linear Algebra Appl., 520 (2017) 80–103

 has been fundamental for determining the generic change of the partial multiplicities of regular matrix pencils under low-rank perturbations for unstructured pencils

F. De Terán, F.M. Dopico, Generic change of the partial multiplicities of regular matrix pencils under low-rank perturbations, SIAM J. Matrix Anal. Appl., 37 (2016) 823–835

#### and for structured pencils

F. De Terán, C. Mehl, V. Mehrmann, Low-rank perturbation of regular matrix pencils with symmetry structures, Foundations of Computational Mathematics, 22 (2022) 257–311

• since such descriptions allow to parameterize  $PENCIL_r^{m \times n}$ .

We hope that a "similar" description of POL<sup>m×n</sup><sub>d,r</sub> for arbitrary degrees d may help to solve the corresponding generic low-rank perturbation problem for matrix polynomials.

F. M. Dopico (U. Carlos III, Madrid)

Minimal rank factorizations matrix polys

- In the case of matrix pencils, matrix polynomials of degree at most one,
- describing the set  $POL_{1,r}^{m \times n} =: PENCIL_r^{m \times n}$  in terms of factors

F. De Terán, F.M. Dopico, J.M. Landsberg, An explicit description of the irreducible components of the set of matrix pencils with bounded normal rank, Linear Algebra Appl., 520 (2017) 80–103

 has been fundamental for determining the generic change of the partial multiplicities of regular matrix pencils under low-rank perturbations for unstructured pencils

F. De Terán, F.M. Dopico, Generic change of the partial multiplicities of regular matrix pencils under low-rank perturbations, SIAM J. Matrix Anal. Appl., 37 (2016) 823–835

#### and for structured pencils

F. De Terán, C. Mehl, V. Mehrmann, Low-rank perturbation of regular matrix pencils with symmetry structures, Foundations of Computational Mathematics, 22 (2022) 257–311

• since such descriptions allow to parameterize  $PENCIL_r^{m \times n}$ .

 We hope that a "similar" description of POL<sup>m×n</sup><sub>d,r</sub> for arbitrary degrees d may help to solve the corresponding generic low-rank perturbation problem for matrix polynomials.

F. M. Dopico (U. Carlos III, Madrid)

Minimal rank factorizations matrix polys

June 12, 2023

- In the case of matrix pencils, matrix polynomials of degree at most one,
- describing the set  $POL_{1,r}^{m \times n} =: PENCIL_{r}^{m \times n}$  in terms of factors

F. De Terán, F.M. Dopico, J.M. Landsberg, An explicit description of the irreducible components of the set of matrix pencils with bounded normal rank, Linear Algebra Appl., 520 (2017) 80–103

#### has been fundamental for determining the generic change of the partial multiplicities of regular matrix pencils under low-rank perturbations for unstructured pencils

F. De Terán, F.M. Dopico, Generic change of the partial multiplicities of regular matrix pencils under low-rank perturbations, SIAM J. Matrix Anal. Appl., 37 (2016) 823–835

#### and for structured pencils

F. De Terán, C. Mehl, V. Mehrmann, Low-rank perturbation of regular matrix pencils with symmetry structures, Foundations of Computational Mathematics, 22 (2022) 257–311

• since such descriptions allow to parameterize  $PENCIL_r^{m \times n}$ .

 We hope that a "similar" description of POL<sup>m×n</sup><sub>d,r</sub> for arbitrary degrees d may help to solve the corresponding generic low-rank perturbation problem for matrix polynomials.

F. M. Dopico (U. Carlos III, Madrid)

Minimal rank factorizations matrix polys

- In the case of matrix pencils, matrix polynomials of degree at most one,
- describing the set  $POL_{1,r}^{m \times n} =: PENCIL_{r}^{m \times n}$  in terms of factors

F. De Terán, F.M. Dopico, J.M. Landsberg, An explicit description of the irreducible components of the set of matrix pencils with bounded normal rank, Linear Algebra Appl., 520 (2017) 80–103

 has been fundamental for determining the generic change of the partial multiplicities of regular matrix pencils under low-rank perturbations for unstructured pencils

F. De Terán, F.M. Dopico, Generic change of the partial multiplicities of regular matrix pencils under low-rank perturbations, SIAM J. Matrix Anal. Appl., 37 (2016) 823–835

#### and for structured pencils

F. De Terán, C. Mehl, V. Mehrmann, Low-rank perturbation of regular matrix pencils with symmetry structures, Foundations of Computational Mathematics, 22 (2022) 257–311

• since such descriptions allow to parameterize  $PENCIL_r^{m \times n}$ .

We hope that a "similar" description of POL<sup>m×n</sup><sub>d,r</sub> for arbitrary degrees d may help to solve the corresponding generic low-rank perturbation problem for matrix polynomials.

F. M. Dopico (U. Carlos III, Madrid)

Minimal rank factorizations matrix polys

June 12, 2023

- In the case of matrix pencils, matrix polynomials of degree at most one,
- describing the set  $POL_{1,r}^{m \times n} =: PENCIL_{r}^{m \times n}$  in terms of factors

F. De Terán, F.M. Dopico, J.M. Landsberg, An explicit description of the irreducible components of the set of matrix pencils with bounded normal rank, Linear Algebra Appl., 520 (2017) 80–103

 has been fundamental for determining the generic change of the partial multiplicities of regular matrix pencils under low-rank perturbations for unstructured pencils

F. De Terán, F.M. Dopico, Generic change of the partial multiplicities of regular matrix pencils under low-rank perturbations, SIAM J. Matrix Anal. Appl., 37 (2016) 823–835

#### and for structured pencils

F. De Terán, C. Mehl, V. Mehrmann, Low-rank perturbation of regular matrix pencils with symmetry structures, Foundations of Computational Mathematics, 22 (2022) 257–311

• since such descriptions allow to parameterize  $\operatorname{PENCIL}_{r}^{m \times n}$ .

We hope that a "similar" description of POL<sup>m×n</sup><sub>d,r</sub> for arbitrary degrees d may help to solve the corresponding generic low-rank perturbation problem for matrix polynomials.

F. M. Dopico (U. Carlos III, Madrid)

Minimal rank factorizations matrix polys

- In the case of matrix pencils, matrix polynomials of degree at most one,
- describing the set  $POL_{1,r}^{m \times n} =: PENCIL_{r}^{m \times n}$  in terms of factors

F. De Terán, F.M. Dopico, J.M. Landsberg, An explicit description of the irreducible components of the set of matrix pencils with bounded normal rank, Linear Algebra Appl., 520 (2017) 80–103

 has been fundamental for determining the generic change of the partial multiplicities of regular matrix pencils under low-rank perturbations for unstructured pencils

F. De Terán, F.M. Dopico, Generic change of the partial multiplicities of regular matrix pencils under low-rank perturbations, SIAM J. Matrix Anal. Appl., 37 (2016) 823–835

#### and for structured pencils

F. De Terán, C. Mehl, V. Mehrmann, Low-rank perturbation of regular matrix pencils with symmetry structures, Foundations of Computational Mathematics, 22 (2022) 257–311

• since such descriptions allow to parameterize  $\operatorname{PENCIL}_r^{m \times n}$ .

We hope that a "similar" description of POL<sup>mxn</sup> for arbitrary degrees d may help to solve the corresponding generic low-rank perturbation problem for matrix polynomials.

F. M. Dopico (U. Carlos III, Madrid)

Minimal rank factorizations matrix polys

June 12, 2023

- In the case of matrix pencils, matrix polynomials of degree at most one,
- describing the set  $POL_{1,r}^{m \times n} =: PENCIL_{r}^{m \times n}$  in terms of factors

F. De Terán, F.M. Dopico, J.M. Landsberg, An explicit description of the irreducible components of the set of matrix pencils with bounded normal rank, Linear Algebra Appl., 520 (2017) 80–103

 has been fundamental for determining the generic change of the partial multiplicities of regular matrix pencils under low-rank perturbations for unstructured pencils

F. De Terán, F.M. Dopico, Generic change of the partial multiplicities of regular matrix pencils under low-rank perturbations, SIAM J. Matrix Anal. Appl., 37 (2016) 823–835

#### and for structured pencils

F. De Terán, C. Mehl, V. Mehrmann, Low-rank perturbation of regular matrix pencils with symmetry structures, Foundations of Computational Mathematics, 22 (2022) 257–311

• since such descriptions allow to parameterize  $PENCIL_r^{m \times n}$ .

 We hope that a "similar" description of POL<sup>m×n</sup><sub>d,r</sub> for arbitrary degrees d may help to solve the corresponding generic low-rank perturbation problem for matrix polynomials.

F. M. Dopico (U. Carlos III, Madrid)

Minimal rank factorizations matrix polys

- In the case of matrix pencils, matrix polynomials of degree at most one,
- describing the set  $POL_{1,r}^{m \times n} =: PENCIL_{r}^{m \times n}$  in terms of factors

F. De Terán, F.M. Dopico, J.M. Landsberg, An explicit description of the irreducible components of the set of matrix pencils with bounded normal rank, Linear Algebra Appl., 520 (2017) 80–103

 has been fundamental for determining the generic change of the partial multiplicities of regular matrix pencils under low-rank perturbations for unstructured pencils

F. De Terán, F.M. Dopico, Generic change of the partial multiplicities of regular matrix pencils under low-rank perturbations, SIAM J. Matrix Anal. Appl., 37 (2016) 823–835

#### and for structured pencils

F. De Terán, C. Mehl, V. Mehrmann, Low-rank perturbation of regular matrix pencils with symmetry structures, Foundations of Computational Mathematics, 22 (2022) 257–311

- since such descriptions allow to parameterize  $PENCIL_r^{m \times n}$ .
- We hope that a "similar" description of POL<sup>mxn</sup><sub>d,r</sub> for arbitrary degrees d may help to solve the corresponding generic low-rank perturbation problem for matrix polynomials.

F. M. Dopico (U. Carlos III, Madrid)

Minimal rank factorizations matrix polys

A review of the results for pencils

2 The set of matrix polynomials with bounded rank and degree in terms of eigenstructures

3 The set of matrix polynomials with bounded rank and degree in terms of factors

< 同 ト < 三 ト < 三 ト

## A review of the results for pencils

2 The set of matrix polynomials with bounded rank and degree in terms of eigenstructures

3 The set of matrix polynomials with bounded rank and degree in terms of factors

#### Matrix pencils and Kronecker Canonical Form

# • All the m × n pencils with the same complete eigenstructure form an orbit under strict equivalence:

 $O(\lambda A + B) := \{P(\lambda A + B)Q \mid \det P \cdot \det Q \neq 0\}.$ 

 The complete eigenstructure of a pencil is determined by its Kronecker canonical form (KCF) under strict equivalence, which is a direct sum of four types of canonical matrix pencils:

• the regular *k* × *k* Jordan blocks for **finite and infinite eigenvalues** 

$$\mathcal{J}_{k}(\mu) := \begin{bmatrix} \lambda - \mu & 1 & & \\ & \lambda - \mu & \ddots & \\ & & \ddots & 1 \\ & & & \lambda - \mu \end{bmatrix}, \quad \mathcal{J}_{k}(\infty) := \begin{bmatrix} 1 & \lambda & & \\ & 1 & \ddots & \\ & & \ddots & \lambda \\ & & & 1 \end{bmatrix} \quad k = 1, 2, 3, \dots$$

the singular k × (k + 1) and (k + 1) × k blocks for right and left minimal indices of value k

$$\mathcal{L}_k \coloneqq \begin{bmatrix} \lambda & 1 & \\ & \ddots & \ddots & \\ & & \lambda & 1 \end{bmatrix}, \quad \mathcal{L}_k^T, \quad k = 0, 1, 2, \dots$$

(日)

#### Matrix pencils and Kronecker Canonical Form

• All the m × n pencils with the same complete eigenstructure form an orbit under strict equivalence:

 $O(\lambda A + B) \coloneqq \{P(\lambda A + B)Q \mid \det P \cdot \det Q \neq 0\}.$ 

- The complete eigenstructure of a pencil is determined by its Kronecker canonical form (KCF) under strict equivalence, which is a direct sum of four types of canonical matrix pencils:
- the regular *k* × *k* Jordan blocks for **finite and infinite eigenvalues**

$$\mathcal{J}_{k}(\mu) := \begin{bmatrix} \lambda - \mu & 1 & & \\ & \lambda - \mu & \ddots & \\ & & \ddots & 1 \\ & & & \lambda - \mu \end{bmatrix}, \quad \mathcal{J}_{k}(\infty) := \begin{bmatrix} 1 & \lambda & & \\ & 1 & \ddots & \\ & & \ddots & \lambda \\ & & & 1 \end{bmatrix} \quad k = 1, 2, 3, \dots$$

• the singular k × (k + 1) and (k + 1) × k blocks for right and left minimal indices of value k

$$\mathcal{L}_k := \begin{bmatrix} \lambda & 1 & \\ & \ddots & \ddots & \\ & & \lambda & 1 \end{bmatrix}, \quad \mathcal{L}_k^T, \quad k = 0, 1, 2, \dots$$

(日)

#### Matrix pencils and Kronecker Canonical Form

• All the m × n pencils with the same complete eigenstructure form an orbit under strict equivalence:

 $O(\lambda A + B) \coloneqq \{P(\lambda A + B)Q \mid \det P \cdot \det Q \neq 0\}.$ 

- The complete eigenstructure of a pencil is determined by its Kronecker canonical form (KCF) under strict equivalence, which is a direct sum of four types of canonical matrix pencils:
- the regular  $k \times k$  Jordan blocks for finite and infinite eigenvalues

$$\mathcal{J}_{k}(\mu) \coloneqq \begin{bmatrix} \lambda - \mu & 1 & & \\ & \lambda - \mu & \ddots & \\ & & \ddots & 1 \\ & & & \lambda - \mu \end{bmatrix}, \quad \mathcal{J}_{k}(\infty) \coloneqq \begin{bmatrix} 1 & \lambda & & \\ & 1 & \ddots & \\ & & \ddots & \lambda \\ & & & 1 \end{bmatrix} \quad k = 1, 2, 3, \dots$$

• the singular k × (k + 1) and (k + 1) × k blocks for right and left minimal indices of value k

$$\mathcal{L}_k \coloneqq \begin{bmatrix} \lambda & 1 \\ & \ddots & \ddots \\ & & \lambda & 1 \end{bmatrix}, \quad \mathcal{L}_k^T, \quad k = 0, 1, 2, \dots$$

(日)
## Matrix pencils and Kronecker Canonical Form

• All the m × n pencils with the same complete eigenstructure form an orbit under strict equivalence:

 $O(\lambda A + B) \coloneqq \{P(\lambda A + B)Q \mid \det P \cdot \det Q \neq 0\}.$ 

- The complete eigenstructure of a pencil is determined by its Kronecker canonical form (KCF) under strict equivalence, which is a direct sum of four types of canonical matrix pencils:
- the regular  $k \times k$  Jordan blocks for finite and infinite eigenvalues

$$\mathcal{J}_{k}(\mu) := \begin{bmatrix} \lambda - \mu & 1 & & \\ & \lambda - \mu & \ddots & \\ & & \ddots & 1 \\ & & & \lambda - \mu \end{bmatrix}, \quad \mathcal{J}_{k}(\infty) := \begin{bmatrix} 1 & \lambda & & \\ & 1 & \ddots & \\ & & \ddots & \lambda \\ & & & 1 \end{bmatrix} \quad k = 1, 2, 3, \dots$$

the singular k × (k + 1) and (k + 1) × k blocks for right and left minimal indices of value k

$$\mathcal{L}_{k} := \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \end{bmatrix}, \quad \mathcal{L}_{k}^{T}, \quad k = 0, 1, 2, \dots$$

 $\operatorname{PENCIL}_{r}^{m \times n} = \left\{ \begin{array}{l} m \times n \text{ complex matrix pencils} \\ \text{with rank at most } r < \min\{m, n\} \end{array} \right\} = \bigcup_{0 \le a \le r} \overline{O}(\mathcal{K}_{a}),$ 

where the  $m \times n$  complex matrix pencils  $\mathcal{K}_a, a = 0, 1, \dots, r$ , have rank r and the KCF



with 
$$\alpha = \lfloor a/(n-r) \rfloor$$
 and  $s = a \mod (n-r)$ ,  
 $\beta = \lfloor (r-a)/(m-r) \rfloor$  and  $t = (r-a) \mod (m-r)$ .

Moreover,  $\overline{O}(\mathcal{K}_a) \cap O(\mathcal{K}_{a'}) = \emptyset$  whenever  $a \neq a'$  (but  $\overline{O}(\mathcal{K}_a) \cap \overline{O}(\mathcal{K}_{a'}) \neq \emptyset$ ).

F. De Terán and F.M. Dopico, A note on generic Kronecker orbits of matrix pencils with fixed rank, SIAM J. Matrix Anal. Appl., 30 (2008) 491–496

F. M. Dopico (U. Carlos III, Madrid)

Minimal rank factorizations matrix polys

June 12, 2023

・ロト ・四ト ・ヨト ・ヨト

11/23

 $\operatorname{PENCIL}_{r}^{m \times n} = \left\{ \begin{array}{c} m \times n \text{ complex matrix pencils} \\ \text{with rank at most } r < \min\{m, n\} \end{array} \right\} = \bigcup_{0 \le a \le r} \overline{O}(\mathcal{K}_{a}),$ 

where the  $m \times n$  complex matrix pencils  $\mathcal{K}_a, a = 0, 1, \dots, r$ , have rank r and the KCF



with 
$$\alpha = \lfloor a/(n-r) \rfloor$$
 and  $s = a \mod (n-r)$ ,  
 $\beta = \lfloor (r-a)/(m-r) \rfloor$  and  $t = (r-a) \mod (m-r)$ .

Moreover,  $\overline{O}(\mathcal{K}_a) \cap O(\mathcal{K}_{a'}) = \emptyset$  whenever  $a \neq a'$  (but  $\overline{O}(\mathcal{K}_a) \cap \overline{O}(\mathcal{K}_{a'}) \neq \emptyset$ ).

F. De Terán and F.M. Dopico, A note on generic Kronecker orbits of matrix pencils with fixed rank, SIAM J. Matrix Anal. Appl., 30 (2008) 491–496

F. M. Dopico (U. Carlos III, Madrid)

Minimal rank factorizations matrix polys

June 12, 2023

イロト イヨト イヨト イヨト

11/23

 $\operatorname{PENCIL}_{r}^{m \times n} = \left\{ \begin{array}{c} m \times n \text{ complex matrix pencils} \\ \text{with rank at most } r < \min\{m, n\} \end{array} \right\} = \bigcup_{0 \le a \le r} \overline{O}(\mathcal{K}_{a}),$ 

where the  $m \times n$  complex matrix pencils  $\mathcal{K}_a, a = 0, 1, \dots, r$ , have rank r and the KCF



with 
$$\alpha = \lfloor a/(n-r) \rfloor$$
 and  $s = a \mod (n-r)$ ,  
 $\beta = \lfloor (r-a)/(m-r) \rfloor$  and  $t = (r-a) \mod (m-r)$ .

Moreover,  $\overline{O}(\mathcal{K}_a) \cap O(\mathcal{K}_{a'}) = \emptyset$  whenever  $a \neq a'$  (but  $\overline{O}(\mathcal{K}_a) \cap \overline{O}(\mathcal{K}_{a'}) \neq \emptyset$ ).

F. De Terán and F.M. Dopico, A note on generic Kronecker orbits of matrix pencils with fixed rank, SIAM J. Matrix Anal. Appl., 30 (2008) 491–496

F. M. Dopico (U. Carlos III, Madrid)

Minimal rank factorizations matrix polys

June 12, 2023

<ロト < 回 > < 回 > < 三 > < 三 > 三 三

 $\operatorname{PENCIL}_{r}^{m \times n} = \left\{ \begin{array}{c} m \times n \text{ complex matrix pencils} \\ \text{with rank at most } r < \min\{m, n\} \end{array} \right\} = \bigcup_{0 \le a \le r} \overline{O}(\mathcal{K}_{a}),$ 

where the  $m \times n$  complex matrix pencils  $\mathcal{K}_a, a = 0, 1, \dots, r$ , have rank r and the KCF



with 
$$\alpha = \lfloor a/(n-r) \rfloor$$
 and  $s = a \mod (n-r)$ ,  
 $\beta = \lfloor (r-a)/(m-r) \rfloor$  and  $t = (r-a) \mod (m-r)$ .

Moreover,  $\overline{O}(\mathcal{K}_a) \cap O(\mathcal{K}_{a'}) = \emptyset$  whenever  $a \neq a'$  (but  $\overline{O}(\mathcal{K}_a) \cap \overline{O}(\mathcal{K}_{a'}) \neq \emptyset$ ).

F. De Terán and F.M. Dopico, A note on generic Kronecker orbits of matrix pencils with fixed rank, SIAM J. Matrix Anal. Appl., 30 (2008) 491–496

F. M. Dopico (U. Carlos III, Madrid)

Minimal rank factorizations matrix polys

June 12, 2023

< □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > <

 $\operatorname{PENCIL}_{r}^{m \times n} = \left\{ \begin{array}{l} m \times n \text{ complex matrix pencils} \\ \text{with rank at most } r < \min\{m, n\} \end{array} \right\} = \overline{\bigcup_{0 \le a \le r} O(\mathcal{K}_a)},$ 

where the  $m \times n$  complex matrix pencils  $\mathcal{K}_a, a = 0, 1, \dots, r$  have rank r and the KCF

$$\mathcal{K}_{a} = \operatorname{diag}\left(\underbrace{\underbrace{\mathcal{L}_{\alpha+1}, \ldots, \mathcal{L}_{\alpha+1}}_{s}, \underbrace{\mathcal{L}_{\alpha}, \ldots, \mathcal{L}_{\alpha}}_{n-r-s}, \underbrace{\underbrace{\mathcal{L}_{\beta+1}^{T}, \ldots, \mathcal{L}_{\beta+1}^{T}}_{t}, \underbrace{\mathcal{L}_{\beta}^{T}, \ldots, \mathcal{L}_{\beta}^{T}}_{m-r-t}, \underbrace{\mathcal{L}_{\beta-1}^{T}, \ldots, \underbrace{\mathcal{L}_{\beta-1}^{T}}_{m-r-t}, \underbrace{\mathcal{L}_{\beta-1}^{T}, \ldots, \underbrace{\mathcal{L}_{\beta-1}^{T}}_{m-r-t}, \underbrace{\mathcal{L}_{\beta-1}^{T}, \ldots, \underbrace{\mathcal{L}_{\beta-1}^{T}}_{m-r-t}, \underbrace{\mathcal{L}_{\beta-1}^{T}, \ldots, \underbrace{\mathcal{L}_{\beta-1}^{T}, \ldots, \underbrace{\mathcal{L}_{\beta-1}^{T}}_{m-r-t}, \underbrace{\mathcal{L}_{\beta-1}^{T}, \ldots, \underbrace{\mathcal{L}_{\beta-1}^{T},$$

with  $\alpha = \lfloor a/(n-r) \rfloor$  and  $s = a \mod (n-r)$ ,  $\beta = \lfloor (r-a)/(m-r) \rfloor$  and  $t = (r-a) \mod (m-r)$ .

Moreover,  $\overline{O}(\mathcal{K}_a) \cap O(\mathcal{K}_{a'}) = \emptyset$  whenever  $a \neq a'$  (but  $\overline{O}(\mathcal{K}_a) \cap \overline{O}(\mathcal{K}_{a'}) \neq \emptyset$ ).

 $\bigcup_{0 \le a \le r} O(\mathcal{K}_a) \text{ is an open dense subset of } PENCIL_r^{m \times n}. \text{ So, generically, the } m \times n \text{ pencils}$ with rank at most r have only r + 1 possible KCFs given by  $\mathcal{K}_a$  for  $a = 0, 1, \ldots, r$ .

- do not have eigenvalues (finite or infinite),
- have n r right minimal indices differing at most by 1 (almost homogeneous) and summing up a, and
- have *m* − *r* left minimal indices differing at most by 1 (almost homogeneous) and summing up *r* − *a*.
- The parameter a = 0, 1, ..., r determines how much rank is attached to the right minimal indices and, then, the rank attached to the left minimal indices is determined by the index sum theorem.
- These are the generic eigenstructures in  $\text{PENCIL}_r^{m \times n}$ .

- do not have eigenvalues (finite or infinite),
- have n r right minimal indices differing at most by 1 (almost homogeneous) and summing up a, and
- have *m* − *r* left minimal indices differing at most by 1 (almost homogeneous) and summing up *r* − *a*.
- The parameter a = 0, 1, ..., r determines how much rank is attached to the right minimal indices and, then, the rank attached to the left minimal indices is determined by the index sum theorem.
- These are the generic eigenstructures in  $\text{PENCIL}_r^{m \times n}$ .

A B A B A B A
 A B A
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 B
 A
 A
 A

- do not have eigenvalues (finite or infinite),
- have n r right minimal indices differing at most by 1 (almost homogeneous) and summing up a, and
- have *m* − *r* left minimal indices differing at most by 1 (almost homogeneous) and summing up *r* − *a*.
- The parameter a = 0, 1, ..., r determines how much rank is attached to the right minimal indices and, then, the rank attached to the left minimal indices is determined by the index sum theorem.
- These are the generic eigenstructures in  $\text{PENCIL}_r^{m \times n}$ .

(日)

- do not have eigenvalues (finite or infinite),
- have n r right minimal indices differing at most by 1 (almost homogeneous) and summing up a, and
- have *m* − *r* left minimal indices differing at most by 1 (almost homogeneous) and summing up *r* − *a*.
- The parameter *a* = 0, 1, ..., *r* determines how much rank is attached to the right minimal indices and, then, the rank attached to the left minimal indices is determined by the index sum theorem.
- These are the generic eigenstructures in  $\operatorname{PENCIL}_{r}^{m \times n}$ .

< 日 > < 同 > < 回 > < 回 > < □ > <

- do not have eigenvalues (finite or infinite),
- have n r right minimal indices differing at most by 1 (almost homogeneous) and summing up a, and
- have *m* − *r* left minimal indices differing at most by 1 (almost homogeneous) and summing up *r* − *a*.
- The parameter *a* = 0, 1, ..., *r* determines how much rank is attached to the right minimal indices and, then, the rank attached to the left minimal indices is determined by the index sum theorem.
- These are the generic eigenstructures in  $\text{PENCIL}_r^{m \times n}$ .

### Theorem (De Terán, D., Landsberg, LAA, 2017)

$$\operatorname{PENCIL}_{r}^{m \times n} = \left\{ \begin{array}{l} m \times n \text{ complex matrix pencils} \\ \text{with rank at most } r < \min\{m, n\} \end{array} \right\} = \bigcup_{0 \le a \le r} \mathcal{C}_{a}$$

where, for a = 0, 1, ..., r,

$$\mathcal{C}_{a} := \left\{ \begin{array}{c} L(\lambda) \in \operatorname{PENCIL}_{r}^{m \times r}, \ R(\lambda) \in \operatorname{PENCIL}_{r}^{r \times n}, \\ L(\lambda)R(\lambda) : \ \deg \operatorname{row}_{i}(R) = 0, \quad \text{for } i = a + 1, \dots, r, \\ \deg \operatorname{row}_{i}(L) = 0, \quad \text{for } i = 1, \dots, a \end{array} \right\}$$

Moreover,

 $\mathcal{C}_a = \overline{\mathrm{O}}(\mathcal{K}_a),$ 

where  $\mathcal{K}_a$  are the  $m \times n$  pencils with rank exactly r and with the generic eigenstructures defined in the previous slides.

F. De Terán, F.M. Dopico, J. Landsberg, An explicit description of the irreducible components of the set of matrix pencils with bounded normal rank, Linear Algebra Appl., 520 (2017) 80-103.

June 12, 2023

13/23

### Theorem (De Terán, D., Landsberg, LAA, 2017)

$$\operatorname{PENCIL}_{r}^{m \times n} = \left\{ \begin{array}{l} m \times n \text{ complex matrix pencils} \\ \text{with rank at most } r < \min\{m, n\} \end{array} \right\} = \bigcup_{0 \le a \le r} \mathcal{C}_{a}$$

where, for a = 0, 1, ..., r,

$$\mathcal{C}_{a} \coloneqq \begin{cases} L(\lambda) \in \operatorname{PENCIL}_{r}^{m \times r}, \ R(\lambda) \in \operatorname{PENCIL}_{r}^{r \times n}, \\ \operatorname{deg} \operatorname{row}_{i}(R) = 0, \quad \text{for } i = a + 1, \dots, r, \\ \operatorname{deg} \operatorname{rol}_{i}(L) = 0, \quad \text{for } i = 1, \dots, a \end{cases} \end{cases}$$

Moreover,

$$\mathcal{C}_a = \overline{\mathrm{O}}(\mathcal{K}_a),$$

where  $\mathcal{K}_a$  are the  $m \times n$  pencils with rank exactly r and with the generic eigenstructures defined in the previous slides.

F. De Terán, F.M. Dopico, J. Landsberg, An explicit description of the irreducible components of the set of matrix pencils with bounded normal rank, Linear Algebra Appl., 520 (2017) 80-103.

ヘロア 人間 アイヨア・

13/23

### Theorem (De Terán, D., Landsberg, LAA, 2017)

$$\operatorname{PENCIL}_{r}^{m \times n} = \left\{ \begin{array}{l} m \times n \text{ complex matrix pencils} \\ \text{with rank at most } r < \min\{m, n\} \end{array} \right\} = \bigcup_{0 \le a \le r} \mathcal{C}_{a}$$

where, for a = 0, 1, ..., r,

$$\mathcal{C}_{a} := \left\{ \begin{array}{c} L(\lambda) \in \operatorname{PENCIL}_{r}^{m \times r}, \ R(\lambda) \in \operatorname{PENCIL}_{r}^{r \times n}, \\ L(\lambda)R(\lambda) : \ \deg \operatorname{row}_{i}(R) = 0, \quad \text{for } i = a + 1, \dots, r, \\ \deg \operatorname{rov}_{i}(L) = 0, \quad \text{for } i = 1, \dots, a \end{array} \right\}$$

Moreover,

$$\mathcal{C}_a=\overline{\mathrm{O}}(\mathcal{K}_a),$$

where  $\mathcal{K}_a$  are the  $m \times n$  pencils with rank exactly r and with the generic eigenstructures defined in the previous slides.

F. De Terán, F.M. Dopico, J. Landsberg, An explicit description of the irreducible components of the set of matrix pencils with bounded normal rank, Linear Algebra Appl., 520 (2017) 80-103.

F. M. Dopico (U. Carlos III, Madrid)

Minimal rank factorizations matrix polys

June 12, 2023

(日)

13/23

# • There are no closures involved in $\text{PENCIL}_r^{m \times n} = \bigcup_{0 \le a \le r} C_a$ .

- The conditions of the theorem guarantee the pairing  $\deg \operatorname{col}_i(L) + \deg \operatorname{col}_i(R) \le 1$  for  $i = 1, \ldots, r$  (generically we will have equality) of the column degrees of  $L(\lambda)$  and of the row degrees of  $R(\lambda)$ .
- Since L(λ) and R(λ) are pencils the degrees of its columns and rows, respectively, differ automatically at most by 1.
- The factors  $L(\lambda)$  and  $R(\lambda)$  can be easily parameterized.
- The parameter a = 0, 1, ..., r determines the (maximal) sum of the degrees of the rows of R(λ).
- In addition, it was proved that C<sub>0</sub>, C<sub>1</sub>,..., C<sub>r</sub> are the irreducible components of the closed set PENCIL<sup>m×n</sup> (in the Zariski topology).

<ロ> <問> <問> < 回> < 回> 、

- There are no closures involved in  $\text{PENCIL}_r^{m \times n} = \bigcup_{0 \le a \le r} C_a$ .
- The conditions of the theorem guarantee the pairing  $\deg \operatorname{col}_i(L) + \deg \operatorname{col}_i(R) \le 1$  for  $i = 1, \ldots, r$  (generically we will have equality) of the column degrees of  $L(\lambda)$  and of the row degrees of  $R(\lambda)$ .
- Since L(λ) and R(λ) are pencils the degrees of its columns and rows, respectively, differ automatically at most by 1.
- The factors  $L(\lambda)$  and  $R(\lambda)$  can be easily parameterized.
- The parameter a = 0, 1, ..., r determines the (maximal) sum of the degrees of the rows of R(λ).
- In addition, it was proved that C<sub>0</sub>, C<sub>1</sub>,..., C<sub>r</sub> are the irreducible components of the closed set PENCIL<sup>m×n</sup> (in the Zariski topology).

<ロ> <問> <問> < 回> < 回> 、

- There are no closures involved in  $\text{PENCIL}_r^{m \times n} = \bigcup_{0 \le a \le r} C_a$ .
- The conditions of the theorem guarantee the pairing  $\deg \operatorname{col}_i(L) + \deg \operatorname{col}_i(R) \le 1$  for  $i = 1, \ldots, r$  (generically we will have equality) of the column degrees of  $L(\lambda)$  and of the row degrees of  $R(\lambda)$ .
- Since L(λ) and R(λ) are pencils the degrees of its columns and rows, respectively, differ automatically at most by 1.
- The factors  $L(\lambda)$  and  $R(\lambda)$  can be easily parameterized.
- The parameter a = 0, 1, ..., r determines the (maximal) sum of the degrees of the rows of R(λ).
- In addition, it was proved that C<sub>0</sub>, C<sub>1</sub>,..., C<sub>r</sub> are the irreducible components of the closed set PENCIL<sup>m×n</sup> (in the Zariski topology).

<ロ> <問> <問> < 回> < 回> 、

- There are no closures involved in  $\text{PENCIL}_r^{m \times n} = \bigcup_{0 \le a \le r} C_a$ .
- The conditions of the theorem guarantee the pairing  $\deg \operatorname{col}_i(L) + \deg \operatorname{col}_i(R) \le 1$  for  $i = 1, \ldots, r$  (generically we will have equality) of the column degrees of  $L(\lambda)$  and of the row degrees of  $R(\lambda)$ .
- Since L(λ) and R(λ) are pencils the degrees of its columns and rows, respectively, differ automatically at most by 1.
- The factors  $L(\lambda)$  and  $R(\lambda)$  can be easily parameterized.
- The parameter a = 0, 1, ..., r determines the (maximal) sum of the degrees of the rows of R(λ).
- In addition, it was proved that C<sub>0</sub>, C<sub>1</sub>,..., C<sub>r</sub> are the irreducible components of the closed set PENCIL<sup>m×n</sup> (in the Zariski topology).

イロン イ理 とく ヨン イヨン

- There are no closures involved in  $\text{PENCIL}_r^{m \times n} = \bigcup_{0 \le a \le r} C_a$ .
- The conditions of the theorem guarantee the pairing  $\deg \operatorname{col}_i(L) + \deg \operatorname{col}_i(R) \le 1$  for  $i = 1, \ldots, r$  (generically we will have equality) of the column degrees of  $L(\lambda)$  and of the row degrees of  $R(\lambda)$ .
- Since L(λ) and R(λ) are pencils the degrees of its columns and rows, respectively, differ automatically at most by 1.
- The factors  $L(\lambda)$  and  $R(\lambda)$  can be easily parameterized.
- The parameter a = 0, 1, ..., r determines the (maximal) sum of the degrees of the rows of R(λ).
- In addition, it was proved that C<sub>0</sub>, C<sub>1</sub>,..., C<sub>r</sub> are the irreducible components of the closed set PENCIL<sup>m×n</sup> (in the Zariski topology).

< 日 > < 同 > < 回 > < 回 > < □ > <

- There are no closures involved in  $\text{PENCIL}_r^{m \times n} = \bigcup_{0 \le a \le r} C_a$ .
- The conditions of the theorem guarantee the pairing  $\deg \operatorname{col}_i(L) + \deg \operatorname{col}_i(R) \le 1$  for  $i = 1, \ldots, r$  (generically we will have equality) of the column degrees of  $L(\lambda)$  and of the row degrees of  $R(\lambda)$ .
- Since L(λ) and R(λ) are pencils the degrees of its columns and rows, respectively, differ automatically at most by 1.
- The factors  $L(\lambda)$  and  $R(\lambda)$  can be easily parameterized.
- The parameter a = 0, 1, ..., r determines the (maximal) sum of the degrees of the rows of R(λ).
- In addition, it was proved that C<sub>0</sub>, C<sub>1</sub>,..., C<sub>r</sub> are the irreducible components of the closed set PENCIL<sup>m×n</sup> (in the Zariski topology).

A review of the results for pencils

2 The set of matrix polynomials with bounded rank and degree in terms of eigenstructures

3 The set of matrix polynomials with bounded rank and degree in terms of factors

A (10) A (10) A (10)

$$P(\lambda) = \lambda^d P_d + \dots + \lambda P_1 + P_0, \quad P_i \in \mathbb{C}^{m \times n}$$

- Essentially the same as in pencils but definitions more complicated since there is NO KCF.
- Finite and infinite eigenvalues and their elementary divisors defined with Smith Form under unimodular equivalence of P(λ) and revP(λ):

 $U(\lambda)P(\lambda)V(\lambda) = \operatorname{diag}\left(g_1(\lambda), \dots, g_r(\lambda)\right) \oplus 0_{(m-r)\times(n-r)}, \quad g_j(\lambda) \mid g_{j+1}(\lambda).$ 

Invariant polynomials:  $g_j(\lambda) = (\lambda - \alpha_1)^{\delta_{j1}} \cdot (\lambda - \alpha_2)^{\delta_{j2}} \cdot \ldots \cdot (\lambda - \alpha_{l_j})^{\delta_{jl_j}}$ . Elementary divisors:  $(\lambda - \alpha_k)^{\delta_{jk}}$ .

• Left and right minimal indices defined through the minimal bases of left and right rational null spaces of  $P(\lambda)$ :

$$\mathcal{N}_{\text{left}}(P) \coloneqq \{ y(\lambda)^T \in \mathbb{C}(\lambda)^{1 \times m} : y(\lambda)^T P(\lambda) = 0_{1 \times n} \}, \\ \mathcal{N}_{\text{right}}(P) \coloneqq \{ x(\lambda) \in \mathbb{C}(\lambda)^{n \times 1} : P(\lambda) x(\lambda) = 0_{m \times 1} \}.$$

# • The definition of orbit does not involve a group action

 $(P) = \begin{cases} \text{matrix polynomials of the same size, degree,} \\ \text{and with the same complete eigenstructure as } P(\lambda) \end{cases}$ 

F. M. Dopico (U. Carlos III, Madrid)

$$P(\lambda) = \lambda^d P_d + \dots + \lambda P_1 + P_0, \quad P_i \in \mathbb{C}^{m \times n}$$

- Essentially the same as in pencils but definitions more complicated since there is NO KCF.
- Finite and infinite eigenvalues and their elementary divisors defined with Smith Form under unimodular equivalence of P(λ) and revP(λ):

 $U(\lambda)P(\lambda)V(\lambda) = \operatorname{diag}\left(g_1(\lambda), \dots, g_r(\lambda)\right) \oplus 0_{(m-r)\times(n-r)}, \quad g_j(\lambda) \mid g_{j+1}(\lambda).$ 

Invariant polynomials:  $g_j(\lambda) = (\lambda - \alpha_1)^{\delta_{j1}} \cdot (\lambda - \alpha_2)^{\delta_{j2}} \cdot \ldots \cdot (\lambda - \alpha_{l_j})^{\delta_{jl_j}}$ . Elementary divisors:  $(\lambda - \alpha_k)^{\delta_{jk}}$ .

 Left and right minimal indices defined through the minimal bases of left and right rational null spaces of P(λ):

$$\mathcal{N}_{\text{left}}(P) \coloneqq \{ y(\lambda)^T \in \mathbb{C}(\lambda)^{1 \times m} : y(\lambda)^T P(\lambda) = 0_{1 \times n} \},$$
  
$$\mathcal{N}_{\text{right}}(P) \coloneqq \{ x(\lambda) \in \mathbb{C}(\lambda)^{n \times 1} : P(\lambda) x(\lambda) = 0_{m \times 1} \}.$$

#### The definition of orbit does not involve a group action

 $(P) = \begin{cases} \text{matrix polynomials of the same size, degree,} \\ \text{and with the same complete eigenstructure as } P(\lambda) \end{cases}$ 

F. M. Dopico (U. Carlos III, Madrid)

$$P(\lambda) = \lambda^d P_d + \dots + \lambda P_1 + P_0, \qquad P_i \in \mathbb{C}^{m \times n}$$

- Essentially the same as in pencils but definitions more complicated since there is NO KCF.
- Finite and infinite eigenvalues and their elementary divisors defined with Smith Form under unimodular equivalence of P(λ) and revP(λ):

$$U(\lambda)P(\lambda)V(\lambda) = \operatorname{diag}\left(g_1(\lambda), \dots, g_r(\lambda)\right) \oplus 0_{(m-r)\times(n-r)}, \quad g_j(\lambda) \mid g_{j+1}(\lambda).$$

Invariant polynomials:  $g_j(\lambda) = (\lambda - \alpha_1)^{\delta_{j1}} \cdot (\lambda - \alpha_2)^{\delta_{j2}} \cdot \ldots \cdot (\lambda - \alpha_{l_j})^{\delta_{jl_j}}$ . Elementary divisors:  $(\lambda - \alpha_k)^{\delta_{jk}}$ .

 Left and right minimal indices defined through the minimal bases of left and right rational null spaces of P(λ):

$$\mathcal{N}_{\text{left}}(P) \coloneqq \{ y(\lambda)^T \in \mathbb{C}(\lambda)^{1 \times m} : y(\lambda)^T P(\lambda) = 0_{1 \times n} \}, \\ \mathcal{N}_{\text{right}}(P) \coloneqq \{ x(\lambda) \in \mathbb{C}(\lambda)^{n \times 1} : P(\lambda) x(\lambda) = 0_{m \times 1} \}.$$

#### • The definition of orbit does not involve a group action

 $(P) = \begin{cases} \text{matrix polynomials of the same size, degree,} \\ \text{and with the same complete eigenstructure as } P(\lambda) \end{cases}$ 

F. M. Dopico (U. Carlos III, Madrid)

$$P(\lambda) = \lambda^d P_d + \dots + \lambda P_1 + P_0, \qquad P_i \in \mathbb{C}^{m \times n}$$

- Essentially the same as in pencils but definitions more complicated since there is NO KCF.
- Finite and infinite eigenvalues and their elementary divisors defined with Smith Form under unimodular equivalence of P(λ) and revP(λ):

$$U(\lambda)P(\lambda)V(\lambda) = \operatorname{diag}\left(g_1(\lambda), \dots, g_r(\lambda)\right) \oplus 0_{(m-r)\times(n-r)}, \quad g_j(\lambda) \mid g_{j+1}(\lambda).$$

Invariant polynomials:  $g_j(\lambda) = (\lambda - \alpha_1)^{\delta_{j1}} \cdot (\lambda - \alpha_2)^{\delta_{j2}} \cdot \ldots \cdot (\lambda - \alpha_{l_j})^{\delta_{jl_j}}$ . Elementary divisors:  $(\lambda - \alpha_k)^{\delta_{jk}}$ .

 Left and right minimal indices defined through the minimal bases of left and right rational null spaces of P(λ):

$$\mathcal{N}_{\text{left}}(P) \coloneqq \{ y(\lambda)^T \in \mathbb{C}(\lambda)^{1 \times m} : y(\lambda)^T P(\lambda) = 0_{1 \times n} \},$$
  
$$\mathcal{N}_{\text{right}}(P) \coloneqq \{ x(\lambda) \in \mathbb{C}(\lambda)^{n \times 1} : P(\lambda) x(\lambda) = 0_{m \times 1} \}.$$

#### The definition of orbit does not involve a group action

 $O(P) = \begin{cases} matrix polynomials of the same size, degree, \\ and with the same complete eigenstructure as <math>P(\lambda) \end{cases}$ 

F. M. Dopico (U. Carlos III, Madrid)

$$P(\lambda) = \lambda^d P_d + \dots + \lambda P_1 + P_0, \quad P_i \in \mathbb{C}^{m \times n}$$

- Essentially the same as in pencils but definitions more complicated since there is NO KCF.
- Finite and infinite eigenvalues and their elementary divisors defined with Smith Form under unimodular equivalence of P(λ) and revP(λ):

$$U(\lambda)P(\lambda)V(\lambda) = \operatorname{diag}\left(g_1(\lambda),\ldots,g_r(\lambda)\right) \oplus 0_{(m-r)\times(n-r)}, \quad g_j(\lambda) \mid g_{j+1}(\lambda).$$

Invariant polynomials:  $g_j(\lambda) = (\lambda - \alpha_1)^{\delta_{j1}} \cdot (\lambda - \alpha_2)^{\delta_{j2}} \cdot \ldots \cdot (\lambda - \alpha_{l_j})^{\delta_{jl_j}}$ . Elementary divisors:  $(\lambda - \alpha_k)^{\delta_{jk}}$ .

• Left and right minimal indices defined through the minimal bases of left and right rational null spaces of  $P(\lambda)$ :

$$\mathcal{N}_{\text{left}}(P) \coloneqq \{ y(\lambda)^T \in \mathbb{C}(\lambda)^{1 \times m} : y(\lambda)^T P(\lambda) = 0_{1 \times n} \},$$
  
$$\mathcal{N}_{\text{right}}(P) \coloneqq \{ x(\lambda) \in \mathbb{C}(\lambda)^{n \times 1} : P(\lambda) x(\lambda) = 0_{m \times 1} \}.$$

# The definition of orbit does not involve a group action

 $O(P) = \begin{cases} matrix polynomials of the same size, degree, \\ and with the same complete eigenstructure as <math>P(\lambda) \end{cases}$ 

F. M. Dopico (U. Carlos III, Madrid)

Minimal rank factorizations matrix polys

 $\operatorname{POL}_{d,r}^{m \times n} = \begin{cases} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d, \text{ with rank at most } r < \min\{m, n\} \end{cases}$ 

$$=\bigcup_{0\leq a\leq rd}\overline{\mathcal{O}}(K_a)$$

where the  $m \times n$  complex matrix polynomial  $K_a, a = 0, 1, \dots, rd$ , has

- degree exactly *d*, rank exactly *r*, and
- the complete eigenstructure

$$\mathbf{K}_{a}: \{\underbrace{\alpha+1,\ldots,\alpha+1}_{s},\underbrace{\alpha,\ldots,\alpha}_{n-r-s},\underbrace{\beta+1,\ldots,\beta+1}_{t},\underbrace{\beta,\ldots,\beta}_{m-r-t}\},$$
where  $\alpha = \lfloor a/(n-r) \rfloor$  and  $s = a \mod (n-r),$   
 $\beta = \lfloor (rd-a)/(m-r) \rfloor$  and  $t = (rd-a) \mod (m-r).$ 

Moreover,  $\overline{O}(K_a) \cap O(K_{a'}) = \emptyset$  whenever  $a \neq a'$  (but  $\overline{O}(K_a) \cap \overline{O}(K_{a'}) \neq \emptyset$ ).

A. Dmytryshyn and F.M. Dopico, Generic complete eigenstructures for sets of matrix polynomials with bounded rank and degree, Linear Algebra Appl., 535 (2017) 213–230

 $\operatorname{POL}_{d,r}^{m \times n} = \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d, \text{ with rank at most } r < \min\{m, n\} \end{array} \right\} = \bigcup_{0 \le a \le rd} \overline{O}(K_a),$ 

where the  $m \times n$  complex matrix polynomial  $K_a, a = 0, 1, ..., rd$ , has

- degree exactly d, rank exactly r, and
- the complete eigenstructure

$$\mathbf{K}_{a}: \{\overbrace{\alpha+1,\ldots,\alpha+1}^{right minimal indices}, \overbrace{\beta+1,\ldots,\beta+1}^{left minimal indices}, \overbrace{\beta+1,\ldots,\beta+1}^{left minimal indices}, \overbrace{\beta-r-r-t}^{left minimal indices}, [f_{a}], where \alpha = \lfloor a/(n-r) \rfloor and s = a \mod (n-r), \\ \beta = \lfloor (rd-a)/(m-r) \rfloor and t = (rd-a) \mod (m-r).$$

A. Dmytryshyn and F.M. Dopico, Generic complete eigenstructures for sets of matrix polynomials with bounded rank and degree, Linear Algebra Appl., 535 (2017) 213–230

F. M. Dopico (U. Carlos III, Madrid)

Minimal rank factorizations matrix polys

June 12, 2023

 $\operatorname{POL}_{d,r}^{m \times n} = \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d, \text{ with rank at most } r < \min\{m, n\} \end{array} \right\} = \bigcup_{0 \le a \le rd} \overline{O}(K_a),$ 

where the  $m \times n$  complex matrix polynomial  $K_a, a = 0, 1, ..., rd$ , has

- degree exactly d, rank exactly r, and
- the complete eigenstructure

$$\mathbf{K}_{a}:\{\overbrace{\alpha+1,\ldots,\alpha+1}^{right minimal indices}, \overbrace{\beta+1,\ldots,\beta+1}^{left minimal indices}, \overbrace{\beta+1,\ldots,\beta+1}^{left minimal indices}, \overbrace{\beta+1,\ldots,\beta+1}^{left minimal indices}, \overbrace{\beta+1,\ldots,\beta+1}^{m-r-t}, \overbrace{m-r-t}^{m-r-t}\}$$
  
where  $\alpha = \lfloor a/(n-r) \rfloor$  and  $s = a \mod (n-r)$ ,  
 $\beta = \lfloor (rd-a)/(m-r) \rfloor$  and  $t = (rd-a) \mod (m-r)$ .

Moreover,  $\overline{O}(K_a) \cap O(K_{a'}) = \emptyset$  whenever  $a \neq a'$  (but  $\overline{O}(K_a) \cap \overline{O}(K_{a'}) \neq \emptyset$ ).

A. Dmytryshyn and F.M. Dopico, Generic complete eigenstructures for sets of matrix polynomials with bounded rank and degree, Linear Algebra Appl., 535 (2017) 213–230

F. M. Dopico (U. Carlos III, Madrid)

 $\operatorname{POL}_{d,r}^{m \times n} = \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d, \text{ with rank at most } r < \min\{m, n\} \end{array} \right\} = \bigcup_{0 \le a \le rd} \overline{O}(K_a),$ 

where the  $m \times n$  complex matrix polynomial  $K_a, a = 0, 1, ..., rd$ , has

- degree exactly d, rank exactly r, and
- the complete eigenstructure

$$\mathbf{K}_{a}: \{\underbrace{\alpha+1,\ldots,\alpha+1}_{s},\underbrace{\alpha,\ldots,\alpha}_{n-r-s},\underbrace{\beta+1,\ldots,\beta+1}_{t},\underbrace{\beta,\ldots,\beta}_{m-r-t}\},$$
where  $\alpha = \lfloor a/(n-r) \rfloor$  and  $s = a \mod (n-r),$   
 $\beta = \lfloor (rd-a)/(m-r) \rfloor$  and  $t = (rd-a) \mod (m-r).$ 

Moreover,  $\overline{O}(K_a) \cap O(K_{a'}) = \emptyset$  whenever  $a \neq a'$  (but  $\overline{O}(K_a) \cap \overline{O}(K_{a'}) \neq \emptyset$ ).

A. Dmytryshyn and F.M. Dopico, Generic complete eigenstructures for sets of matrix polynomials with bounded rank and degree, Linear Algebra Appl., 535 (2017) 213–230

F. M. Dopico (U. Carlos III, Madrid)

 $\operatorname{POL}_{d,r}^{m \times n} = \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d, \text{ with rank at most } r < \min\{m, n\} \end{array} \right\} = \overline{\bigcup_{0 \le a \le rd} \operatorname{O}(K_a)},$ 

where the  $m \times n$  complex matrix polynomial  $K_a, a = 0, 1, ..., rd$ , has degree exactly d, rank exactly r, and the complete eigenstructure

$$\mathbf{K}_{a}:\{\underbrace{\alpha+1,\ldots,\alpha+1}_{s},\underbrace{\alpha,\ldots,\alpha}_{n-r-s},\underbrace{\beta+1,\ldots,\beta+1}_{t},\underbrace{\beta,\ldots,\beta}_{m-r-t}\},$$

where  $\alpha = \lfloor a/(n-r) \rfloor$  and  $s = a \mod (n-r)$ ,  $\beta = \lfloor (rd-a)/(m-r) \rfloor$  and  $t = (rd-a) \mod (m-r)$ .

Moreover,  $\overline{O}(K_a) \cap O(K_{a'}) = \emptyset$  whenever  $a \neq a'$  (but  $\overline{O}(K_a) \cap \overline{O}(K_{a'}) \neq \emptyset$ ).

 $\bigcup_{0 \le a \le rd} O(K_a) \text{ is an open dense subset of } POL_{d,r}^{m \times n}. \text{ So, generically, the } m \times n \text{ matrix}$ polys with degree at most *d* and with rank at most *r* have only rd + 1 possible complete eigenstructures given by  $K_a$  for  $a = 0, 1, \ldots, rd$ .

- do not have eigenvalues (finite or infinite),
- have n r right minimal indices differing at most by 1 (almost homogeneous) and summing up a, and
- have *m* − *r* left minimal indices differing at most by 1 (almost homogeneous) and summing up *rd* − *a*.
- The parameter a = 0, 1, ..., rd determines in the index sum theorem

 $(\sum right minimal indices) + (\sum left minimal indices) = rd$ 

how much of the total sum corresponds to the right minimal indices.

• These are the generic eigenstructures in  $POL_{d,r}^{m \times n}$ .

This description in terms of eigenstructures is very similar to the result for pencils, but the description of  $POL_{d,r}^{m \times n}$  in terms of factors is missing.

э

- do not have eigenvalues (finite or infinite),
- have n r right minimal indices differing at most by 1 (almost homogeneous) and summing up a, and
- have *m* − *r* left minimal indices differing at most by 1 (almost homogeneous) and summing up *rd* − *a*.
- The parameter a = 0, 1, ..., rd determines in the index sum theorem

 $(\sum right minimal indices) + (\sum left minimal indices) = rd$ 

how much of the total sum corresponds to the right minimal indices.

• These are the generic eigenstructures in  $POL_{d,r}^{m \times n}$ .

This description in terms of eigenstructures is very similar to the result for pencils, but the description of  $POL_{d,r}^{m \times n}$  in terms of factors is missing.

э

- do not have eigenvalues (finite or infinite),
- have n r right minimal indices differing at most by 1 (almost homogeneous) and summing up a, and
- have *m* − *r* left minimal indices differing at most by 1 (almost homogeneous) and summing up *rd* − *a*.
- The parameter a = 0, 1, ..., rd determines in the index sum theorem

 $(\sum right minimal indices) + (\sum left minimal indices) = rd$ 

how much of the total sum corresponds to the right minimal indices.

• These are the generic eigenstructures in  $POL_{d,r}^{m \times n}$ .

This description in terms of eigenstructures is very similar to the result for pencils, but the description of  $POL_{d,r}^{m \times n}$  in terms of factors is missing.

э

- do not have eigenvalues (finite or infinite),
- have n r right minimal indices differing at most by 1 (almost homogeneous) and summing up a, and
- have *m* − *r* left minimal indices differing at most by 1 (almost homogeneous) and summing up *rd* − *a*.
- The parameter a = 0, 1, ..., rd determines in the index sum theorem

 $(\sum right minimal indices) + (\sum left minimal indices) = rd$ 

how much of the total sum corresponds to the right minimal indices.

• These are the generic eigenstructures in  $POL_{d,r}^{m \times n}$ .

This description in terms of eigenstructures is very similar to the result for pencils, but the description of  $POL_{d,r}^{m \times n}$  in terms of factors is missing.

э

- do not have eigenvalues (finite or infinite),
- have n r right minimal indices differing at most by 1 (almost homogeneous) and summing up a, and
- have *m* − *r* left minimal indices differing at most by 1 (almost homogeneous) and summing up *rd* − *a*.
- The parameter a = 0, 1, ..., rd determines in the index sum theorem

 $(\sum right minimal indices) + (\sum left minimal indices) = rd$ 

how much of the total sum corresponds to the right minimal indices.

• These are the generic eigenstructures in  $POL_{d,r}^{m \times n}$ .

This description in terms of eigenstructures is very similar to the result for pencils, but the description of  $POL_{d,r}^{m \times n}$  in terms of factors is missing.
The matrix polynomials in  $O(K_a) \subset POL_{d,r}^{m \times n}$  of degree exactly *d* and rank exactly *r* 

- do not have eigenvalues (finite or infinite),
- have n r right minimal indices differing at most by 1 (almost homogeneous) and summing up a, and
- have *m* − *r* left minimal indices differing at most by 1 (almost homogeneous) and summing up *rd* − *a*.
- The parameter a = 0, 1, ..., rd determines in the index sum theorem

 $(\sum right minimal indices) + (\sum left minimal indices) = rd$ 

how much of the total sum corresponds to the right minimal indices.

• These are the generic eigenstructures in  $POL_{d,r}^{m \times n}$ .

This description in terms of eigenstructures is very similar to the result for pencils, but the description of  $POL_{d,r}^{m \times n}$  in terms of factors is missing.

ъ

イロト 不得 トイヨト イヨト

A review of the results for pencils

2 The set of matrix polynomials with bounded rank and degree in terms of eigenstructures

3 The set of matrix polynomials with bounded rank and degree in terms of factors

< 同 ト < 三 ト < 三 ト

## Ideas, difficulties, and simple statement of results

• Any  $m \times n$  constant matrix A of rank r can be written as

 $A = LR, \text{ where } \begin{cases} L \text{ is } m \times r \text{ and } \operatorname{rank} L = r, \\ R \text{ is } r \times n \text{ and } \operatorname{rank} R = r. \end{cases}$ 

- The idea is to get a similar description of POL<sup>*m×n*</sup><sub>*d,r*</sub> but the degree of the factors makes the problem not trivial: it might be cancellations of "high degrees", how to distribute degrees between the factors, etc.
- Nevertheless, generically, i.e., using closures of open dense sets, we can prove that if  $P(\lambda) \in \text{POL}_{d,r}^{m \times n}$

 $P(\lambda) = L(\lambda)R(\lambda),$ 

### where

- **1**  $L(\lambda)$  is an  $m \times r$  matrix polynomial, rank  $L(\lambda) = r$ , and degrees of its columns differ at most by one,
- R(λ) is an r×n matrix polynomial, rank R(λ) = r, and degrees of its rows differ at most by one, and
- 3  $\deg \operatorname{\mathsf{col}}_i(L(\lambda)) + \deg \operatorname{\mathsf{row}}_i(R(\lambda)) = d$ , for  $i = 1, \ldots$

## Ideas, difficulties, and simple statement of results

• Any  $m \times n$  constant matrix A of rank r can be written as

$$A = LR, \text{ where } \begin{cases} L \text{ is } m \times r \text{ and } \operatorname{rank} L = r, \\ R \text{ is } r \times n \text{ and } \operatorname{rank} R = r. \end{cases}$$

- The idea is to get a similar description of POL<sup>m×n</sup><sub>d,r</sub> but the degree of the factors makes the problem not trivial: it might be cancellations of "high degrees", how to distribute degrees between the factors, etc.
- Nevertheless, generically, i.e., using closures of open dense sets, we can prove that if P(λ) ∈ POL<sup>m×n</sup><sub>d,r</sub>

 $P(\lambda) = L(\lambda)R(\lambda),$ 

### where

- **1**  $L(\lambda)$  is an  $m \times r$  matrix polynomial, rank  $L(\lambda) = r$ , and degrees of its columns differ at most by one,
- R(λ) is an r×n matrix polynomial, rank R(λ) = r, and degrees of its rows differ at most by one, and
- 3  $\deg \operatorname{\mathsf{col}}_i(L(\lambda)) + \deg \operatorname{\mathsf{row}}_i(R(\lambda)) = d$ , for  $i = 1, \ldots$

## Ideas, difficulties, and simple statement of results

• Any  $m \times n$  constant matrix A of rank r can be written as

$$A = LR, \text{ where } \begin{cases} L \text{ is } m \times r \text{ and } \operatorname{rank} L = r, \\ R \text{ is } r \times n \text{ and } \operatorname{rank} R = r. \end{cases}$$

- The idea is to get a similar description of POL<sup>m×n</sup><sub>d,r</sub> but the degree of the factors makes the problem not trivial: it might be cancellations of "high degrees", how to distribute degrees between the factors, etc.
- Nevertheless, generically, i.e., using closures of open dense sets, we can prove that if P(λ) ∈ POL<sup>m×n</sup><sub>d,r</sub>

 $P(\lambda) = L(\lambda)R(\lambda),$ 

### where

- $L(\lambda)$  is an  $m \times r$  matrix polynomial, rank  $L(\lambda) = r$ , and degrees of its columns differ at most by one,
- 2  $R(\lambda)$  is an  $r \times n$  matrix polynomial, rank  $R(\lambda) = r$ , and degrees of its rows differ at most by one, and
- deg  $\operatorname{col}_i(L(\lambda))$  + deg  $\operatorname{row}_i(R(\lambda)) = d$ , for  $i = 1, \dots, r$ .

## Theorem (Dmytryshyn, D., Van Dooren)

$$\begin{split} & \text{POL}_{d,r}^{m \times n} = \begin{cases} m \times n \text{ complex matrix polynomials} \\ & \text{with degree at most } d, \text{ with rank at most } r < \min\{m, n\} \end{cases} = \bigcup_{0 \le a \le rd} \overline{\mathcal{B}_a} \,, \\ & \text{where, for } a = 0, 1, \dots, rd, \\ & \mathcal{B}_a := \begin{cases} L(\lambda)R(\lambda) : & L(\lambda) \in \mathbb{C}[\lambda]^{m \times r}, R(\lambda) \in \mathbb{C}[\lambda]^{r \times n}, \\ & \deg row_i(R) = d_R + 1, & \text{for } i = 1, \dots, r_R, \\ & \deg row_i(R) = d_R, & \text{for } i = t_R + 1, \dots, r, \\ & \deg row_i(R) = d - \deg row_i(R), & \text{for } i = 1, \dots, r \end{cases} \end{cases}, \end{split}$$

with  $d_R = \lfloor a/r \rfloor$  and  $t_R = a \mod r$ . Moreover,

# $\overline{\mathcal{B}_a}=\overline{\mathrm{O}}(K_a),$

where  $K_a$  are the  $m \times n$  matrix polynomials of degree exactly d and rank exactly r with the generic eigenstructures defined in the previous section.

F. M. Dopico (U. Carlos III, Madrid)

イロン イ理 とく ヨン イヨン

3

## Theorem (Dmytryshyn, D., Van Dooren)

 $\operatorname{POL}_{d,r}^{m \times n} = \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d, \text{ with rank at most } r < \min\{m, n\} \end{array} \right\} = \bigcup_{0 \le a \le rd} \overline{\mathcal{B}_a},$ 

where, for a = 0, 1, ..., rd,

$$\mathcal{B}_{a} := \begin{cases} L(\lambda) \in \mathbb{C}[\lambda]^{m \times r}, R(\lambda) \in \mathbb{C}[\lambda]^{r \times n}, \\ \deg \operatorname{row}_{i}(R) = d_{R} + 1, \quad \text{for } i = 1, \dots, t_{R}, \\ \deg \operatorname{row}_{i}(R) = d_{R}, \quad \text{for } i = t_{R} + 1, \dots, r, \\ \deg \operatorname{row}_{i}(L) = d - \deg \operatorname{row}_{i}(R), \quad \text{for } i = 1, \dots, r \end{cases} \end{cases}$$

with  $d_R = \lfloor a/r \rfloor$  and  $t_R = a \mod r$ . Moreover,

 $\overline{\mathcal{B}_a}=\overline{\mathrm{O}}(K_a),$ 

where  $K_a$  are the  $m \times n$  matrix polynomials of degree exactly d and rank exactly r with the generic eigenstructures defined in the previous section.

< 口 > < 同 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ >

• The factors  $L(\lambda)$  and  $R(\lambda)$  can be easily parameterized.

- The parameter *a* = 0, 1, ..., *rd* determines the sum of the degrees of the rows of *R*(λ).
- Though  $\overline{\mathcal{B}}_a = \overline{O}(K_a)$ , it is easy to see that, in general,  $\mathcal{B}_a \neq O(K_a)$ , even more  $\mathcal{B}_a \notin O(K_a)$  and  $O(K_a) \notin \mathcal{B}_a$ .
- The proof of the main theorem is rather technical and needs several preliminary results but a key idea is the fact that  $r \times n$  (r < n) rectangular matrix polynomials with given row degrees are generically minimal bases with almost homogeneous right minimal indices.

E.M. Dopice and P. Van Dooren, Robustness and perturbations of minimal bases II: The case with given row degrees, Linear Algebra and its Applications, 576 (2019) 268-300

- The factors  $L(\lambda)$  and  $R(\lambda)$  can be easily parameterized.
- The parameter a = 0, 1, ..., rd determines the sum of the degrees of the rows of R(λ).
- Though  $\overline{\mathcal{B}}_a = \overline{O}(K_a)$ , it is easy to see that, in general,  $\mathcal{B}_a \neq O(K_a)$ , even more  $\mathcal{B}_a \notin O(K_a)$  and  $O(K_a) \notin \mathcal{B}_a$ .
- The proof of the main theorem is rather technical and needs several preliminary results but a key idea is the fact that  $r \times n$  (r < n) rectangular matrix polynomials with given row degrees are generically minimal bases with almost homogeneous right minimal indices.

E.M. Dopice and P. Van Dooren, Robustness and perturbations of minimal bases II: The case with given row degrees, Linear Algebra and its Applications, 576 (2019) 268-300

- The factors  $L(\lambda)$  and  $R(\lambda)$  can be easily parameterized.
- The parameter a = 0, 1, ..., rd determines the sum of the degrees of the rows of R(λ).
- Though  $\overline{\mathcal{B}_a} = \overline{O}(K_a)$ , it is easy to see that, in general,  $\mathcal{B}_a \neq O(K_a)$ , even more  $\mathcal{B}_a \notin O(K_a)$  and  $O(K_a) \notin \mathcal{B}_a$ .
- The proof of the main theorem is rather technical and needs several preliminary results but a key idea is the fact that  $r \times n$  (r < n) rectangular matrix polynomials with given row degrees are generically minimal bases with almost homogeneous right minimal indices.

E.M. Dopice and P. Van Dooren, Robustness and perturbations of minimal bases II: The case with given row degrees, Linear Algebra and its Applications, 576 (2019) 268-300

- The factors  $L(\lambda)$  and  $R(\lambda)$  can be easily parameterized.
- The parameter a = 0, 1, ..., rd determines the sum of the degrees of the rows of R(λ).
- Though  $\overline{\mathcal{B}}_a = \overline{O}(K_a)$ , it is easy to see that, in general,  $\mathcal{B}_a \neq O(K_a)$ , even more  $\mathcal{B}_a \notin O(K_a)$  and  $O(K_a) \notin \mathcal{B}_a$ .
- The proof of the main theorem is rather technical and needs several preliminary results but a key idea is the fact that r × n (r < n) rectangular matrix polynomials with given row degrees are generically minimal bases with almost homogeneous right minimal indices.

F.M. Dopico and P. Van Dooren, Robustness and perturbations of minimal bases II: The case with given row degrees, Linear Algebra and its Applications, 576 (2019) 268-300

- The factors  $L(\lambda)$  and  $R(\lambda)$  can be easily parameterized.
- The parameter a = 0, 1, ..., rd determines the sum of the degrees of the rows of R(λ).
- Though  $\overline{\mathcal{B}}_a = \overline{O}(K_a)$ , it is easy to see that, in general,  $\mathcal{B}_a \neq O(K_a)$ , even more  $\mathcal{B}_a \notin O(K_a)$  and  $O(K_a) \notin \mathcal{B}_a$ .
- The proof of the main theorem is rather technical and needs several preliminary results but a key idea is the fact that r × n (r < n) rectangular matrix polynomials with given row degrees are generically minimal bases with almost homogeneous right minimal indices.

F.M. Dopico and P. Van Dooren, Robustness and perturbations of minimal bases II: The case with given row degrees, Linear Algebra and its Applications, 576 (2019) 268-300

# THANK YOU VERY MUCH FOR YOUR ATTENTION!!