Backward stability in polynomial and rational eigenvalue problems solved via linearization

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joint work with **Piers Lawrence, Javier Pérez, María C. Quintana** and **Paul Van Dooren**

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Different classes of matrix eigenvalue problems (I)

From a *simplified* point of view, we can consider the following matrix eigenvalue problems:

- **The basic eigenvalue problem (BEP).** Given $A \in \mathbb{C}^{n\times n}$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that

$$Av = \lambda v \iff \boxed{(\lambda I_n - A)\, v = 0}$$

- **The GENERALIZED eigenvalue problem (GEP).** Given $A, B \in \mathbb{C}^{m\times n}$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that

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- **The RATIONAL eigenvalue problem (REP).** Given a rational matrix $G(z) \in \mathbb{C}(z)^{m\times n}$, i.e., such that $G(z)_{ij}$ is a scalar rational function of $z \in \mathbb{C}$, for $1 \le i, j \le n$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that λ is not a pole of any $G(z)_{ij}$ and

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A key idea on matrix eigenvalue problems

1. **BEP**: $(\lambda I_n - A)\, v = 0$
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- **Key idea: PEPs and REPs can be solved by transforming the problem into a GEP** via a process known as **LINEARIZATION**.
- This transformation is **exact**, i.e., the obtained **GEP** contains (or allows us to easily extract) exactly all the eigen-information of the original **PEP** or **REP**.
- The use of **linearizations** is one of the **most reliable** approaches for solving numerically PEPs and REPs, because **there exist very reliable algorithms for solving GEPs**.
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- We will study the backward stability of solving a polynomial or a rational eigenvalue problem
- by applying a backward stable generalized eigenvalue algorithm to a wide family of its linearizations.
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- **GEPs, PEPs, REPs** may have also **infinite eigenvalues**.
- **GEPs, PEPs, REPs** may be **singular**, i.e., rectangular or square with identically zero determinant, (BEPs are always regular) and to have, in addition to eigenvalues, **minimal indices**.
- Moreover, **REPs** have **poles**.
- We define quickly these concepts.

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Finite and infinite eigenvalues of PEPs

Given $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0 \in \mathbb{C}[\lambda]^{m\times n}$,

- $\lambda_0 \in \mathbb{C}$ is a **finite eigenvalue** of $P(\lambda)$ if

$$\mathrm{rank} P(\lambda_0) < \max_{\lambda\in\mathbb{C}} \mathrm{rank} P(\lambda)$$

- The infinite eigenvalue of $P(\lambda)$ is defined through **the reversal polynomial**.
- The reversal of $P(\lambda)$ is

$$\mathrm{rev}P(\lambda) := \lambda^d P(\tfrac{1}{\lambda}) = P_0\lambda^d + \cdots + P_{d-1}\lambda + P_d \quad .$$

- Then the **infinite eigenvalue** (and its mutiplicities) of $P(\lambda)$ correspond to the **zero eigenvalue** (and its mutiplicities) of $\mathrm{rev}P(\lambda)$.

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Minimal indices of singular PEPs

- PEPs are **singular** when $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0$ is either **rectangular or square with** $\det P(\lambda) \equiv 0$.
- **Singular PEPs appear in applications**, in particular in Multivariable System Theory and Control Theory.
- In addition to eigenvalues, **singular matrix polynomials have** other "interesting numbers" called **minimal indices**,
- which are related to the fact that a singular $m \times n$ matrix polynomial $P(\lambda)$ has non-trivial left and/or right null-spaces over the field $\mathbb{C}(\lambda)$ of rational functions:

$$\begin{aligned} \mathcal{N}_\ell(P) &:= \left\{ y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times m} \,:\, y(\lambda)^T P(\lambda) \equiv 0^T \right\}, \\ \mathcal{N}_r(P) &:= \left\{ x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} \,:\, P(\lambda)x(\lambda) \equiv 0 \right\}. \end{aligned}$$

- They have bases consisting entirely of vector polynomials.
- The polynomial bases with "minimal sum of the degrees" of their vectors are the **minimal bases** of $P(\lambda)$. The **minimal indices** of $P(\lambda)$ are the degrees of the vectors of any minimal basis.

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The complete "eigenstructure" of a polynomial matrix

As a consequence of the previous discussion, we define:

Definition

The **complete "eigenstructure"** of a polynomial matrix $P(\lambda)$ is comprised of:

- its finite eigenvalues, together with their partial multiplicities,
- its infinite eigenvalue, together with its partial multiplicities,
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- its left minimal indices.

Remarks

- The partial multiplicities are defined through the Smith form of $P(\lambda)$ and for matrices and pencils they are just the sizes of the Jordan blocks associated to each eigenvalue.

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The complete "eigenstructure" of a rational matrix

Analogously, we define:

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Remarks

- The partial multiplicities are defined through the Smith-McMillan form of $G(\lambda)$.
- The infinite zeros and poles, together with its partial multiplicities, of $G(\lambda)$ are defined as the zeros and poles at $\lambda = 0$, together with its partial multiplicities, of $G(1/\lambda)$.

The complete "eigenstructure" of a rational matrix

Analogously, we define:

Definition

The **complete "eigenstructure"** of a rational matrix $G(\lambda)$ is comprised of:

- its finite zeros and **poles**, together with their partial multiplicities,
- its infinite zeros and **poles**, together with its partial multiplicities,
- its right minimal indices, and
- its left minimal indices.

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Outline

Definition: strong linearizations of polynomial matrices

Definition

- A **linear polynomial matrix (or matrix pencil)** $L(\lambda)$ is a **linearization** of $P(\lambda) = P_d\,\lambda^d + \cdots + P_1\lambda + P_0$ if there exist **unimodular** polynomial matrices $U(\lambda), V(\lambda)$ such that

$$U(\lambda)\,L(\lambda)\,V(\lambda) = \begin{bmatrix} I_s & 0 \\ 0 & P(\lambda) \end{bmatrix}.$$

- $L(\lambda)$ is a **strong linearization** of $P(\lambda)$ if, **in addition**, $\operatorname{rev} L(\lambda)$ is a linearization for $\operatorname{rev} P(\lambda)$, i.e.,

$$\widetilde{U}(\lambda)\,(\operatorname{rev} L(\lambda))\,\widetilde{V}(\lambda) = \begin{bmatrix} I_s & 0 \\ 0 & \operatorname{rev} P(\lambda) \end{bmatrix},$$

with $\widetilde{U}(\lambda)$ and $\widetilde{V}(\lambda)$ unimodular.

Theorem

A matrix pencil $L(\lambda)$ is a linearization *of a polynomial matrix* $P(\lambda)$ **if and only if**

(1) $L(\lambda)$ *and* $P(\lambda)$ **have the same number of right minimal indices**.

(2) $L(\lambda)$ *and* $P(\lambda)$ **have the same number of left minimal indices**.

(3) $L(\lambda)$ *and* $P(\lambda)$ **have the same finite eigenvalues** *with the same partial multiplicities.*

$L(\lambda)$ **is a strong linearization** *of* $P(\lambda)$ **if and only if** *(1), (2), (3) and*

(4) $L(\lambda)$ *and* $P(\lambda)$ **have the same infinite eigenvalues** *with the same partial multiplicities.*

Remark: The **minimal indices** of $L(\lambda)$ **may have arbitrarily different values** from those of $P(\lambda)$, though in the most important classes of linearizations they are easily related.

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The most famous strong linearization

The classical **Frobenius companion form** of the $m \times n$ matrix polynomial

$$P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0$$

is

$$C_1(\lambda) := \begin{bmatrix} \lambda P_d + P_{d-1} & P_{d-2} & \cdots & P_1 & P_0 \\ -I_n & \lambda I_n & & & \\ & \ddots & \ddots & & \\ & & \ddots & \lambda I_n & \\ & & & -I_n & \lambda I_n \end{bmatrix} \in \mathbb{C}[\lambda]^{(m+n(d-1))\times nd}$$

Theorem ($C_1(\lambda)$ is much more than a strong linearization!!)

(a) *If $0 \le \varepsilon_1 \le \cdots \le \varepsilon_p$ are the right minimal indices of $P(\lambda)$, then the right minimal indices of $C_1(\lambda)$ are $\varepsilon_1 + d - 1 \le \cdots \le \varepsilon_p + d - 1$.*

(b) *If $0 \le \eta_1 \le \cdots \le \eta_q$ are the left minimal indices of $P(\lambda)$, then the left minimal indices of $C_1(\lambda)$ are $\eta_1 \le \cdots \le \eta_q$.*

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Some comments on linearizations of rational matrices

- For brevity, I will not present the definition of linearizations and strong linearizations of rational matrices.
- In contrast with the polynomial case, there is no agreement in the community on the definition of **(strong) linearization** of a rational matrix.
- Pioneering works on linearizations of rational matrices where developed by Van Dooren and Verghese in late 70s & early 80s though they did not give a general definition.
- Amparan, D, Marcaida, and Zaballa, *Strong linearizations of rational matrices*, SIMAX 2018 introduced a definition of strong linearization of any rational matrix $R(\lambda)$ that reduces to the one for polynomials when $R(\lambda)$ is a polynomial matrix. They also constructed explicitly many of such linearizations.
- Another closely related approach for defining linearizations of rational matrices was initiated by Alam and Behera, *Linearizations for rational matrix functions and Rosenbrock system polynomials*, SIMAX 2016 and followed by other students of Alam.

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Some comments on linearizations of rational matrices (II)

- In simple words, a linearization $L(\lambda)$ of a rational matrix $R(\lambda)$ is a matrix pencil (i.e., a matrix polynomial of degree 1) whose eigenvalues are the finite zeros of $R(\lambda)$ and such that the eigenvalues of a certain square and nonsingular submatrix of $L(\lambda)$ are the finite poles of $R(\lambda)$,
- with equal partial multiplicities for zeros and poles.
- If $L(\lambda)$ contains also the pole-zero information of $R(\lambda)$ at infinity, then it is said to be a strong linearization of $R(\lambda)$.
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Outline

Two fundamental auxiliary polynomial matrices in the rest of the talk are

$$L_k(\lambda) := \begin{bmatrix} -1 & \lambda & & & \\ & -1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & -1 & \lambda \end{bmatrix} \in \mathbb{C}[\lambda]^{k\times(k+1)},$$

$$\Lambda_k(\lambda)^T := \begin{bmatrix} \lambda^k & \lambda^{k-1} & \cdots & \lambda & 1 \end{bmatrix} \in \mathbb{C}[\lambda]^{1\times(k+1)},$$

and their Kronecker products by identities

$$L_k(\lambda) \otimes I_n := \begin{bmatrix} -I_n & \lambda I_n & & & \\ & -I_n & \lambda I_n & & \\ & & \ddots & \ddots & \\ & & & -I_n & \lambda I_n \end{bmatrix} \in \mathbb{C}[\lambda]^{nk\times n(k+1)},$$

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We have seen one of these matrices before

in the Frobenius companion form of the $m \times n$ matrix polynomial $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0$

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which can be compactly written with the polynomials defined above as

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Observe also that

$$P(\lambda) = \begin{bmatrix} \lambda P_d + P_{d-1} & P_{d-2} & \cdots & P_1 & P_0 \end{bmatrix} (\Lambda_{d-1}(\lambda) \otimes I_n).$$

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Definition (D, Lawrence, Pérez, Van Dooren, Numer. Math., 2018)

Let $M(\lambda)$ be an arbitrary pencil. Then any pencil of the form

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M(\lambda) & L_\eta(\lambda)^T \otimes I_m \\ \hline L_\varepsilon(\lambda) \otimes I_n & 0 \end{array}\right] \begin{array}{l} \}\,(\eta+1)m \\ \}\,\varepsilon n \end{array},$$

$$\underbrace{\qquad\qquad}_{(\varepsilon+1)n} \quad \underbrace{\qquad\qquad}_{\eta m}$$

is called a block Kronecker pencil (one-block row and column cases included).

Theorem (D, Lawrence, Pérez, Van Dooren, Numer. Math., 2018)

Any block Kronecker pencil $\mathcal{L}(\lambda)$ is a strong linearization of the matrix polynomial

$$Q(\lambda) := (\Lambda_\eta(\lambda)^T \otimes I_m) M(\lambda) (\Lambda_\varepsilon(\lambda) \otimes I_n) \in \mathbb{C}[\lambda]^{m\times n},$$

the right minimal indices of $\mathcal{L}(\lambda)$ are those of $Q(\lambda)$ plus ε, and the left minimal indices of $\mathcal{L}(\lambda)$ are those of $Q(\lambda)$ plus η.

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Examples of block Kronecker linearizations of polynomial matrices (I)

(apart from the Frobenius companion form!!!)

$$P(\lambda) = \lambda^5 P_5 + \lambda^4 P_4 + \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0 \in \mathbb{C}[\lambda]^{m\times n}$$

$$\left[\begin{array}{ccc|cc} \lambda P_5 + P_4 & 0 & 0 & -I_m & 0 \\ 0 & \lambda P_3 + P_2 & 0 & \lambda I_m & -I_m \\ 0 & 0 & \lambda P_1 + P_0 & 0 & \lambda I_m \\ \hline -I_n & \lambda I_n & 0 & 0 & 0 \\ 0 & -I_n & \lambda I_n & 0 & 0 \end{array}\right]$$

Examples of block Kronecker linerizations of polynomial matrices (II)

$$P(\lambda) = \lambda^5 P_5 + \lambda^4 P_4 + \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0 \in \mathbb{C}[\lambda]^{m\times n}$$

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Outline

Rational matrices and their representations

- Rational matrices can be represented in different forms.
- In this talk, we consider that the rational matrix is represented as

$$R(\lambda) = R_p(\lambda) + D(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i \lambda^i \in \mathbb{C}(\lambda)^{m\times n},$$

  where the triple $\{A, B, C\}$ is a minimal state-space realization of the strictly proper part $R_p(\lambda)$, and d is the degree of the polynomial part.
- This minimality means that $\begin{bmatrix} \lambda_0 I_\ell - A \\ C \end{bmatrix}$ and $\begin{bmatrix} \lambda_0 I_\ell - A & B \end{bmatrix}$ have full column and row ranks, respectively, for any $\lambda_0 \in \mathbb{C}$.
- **Any rational matrix can be represented in this form, which is one of the most standard representations in linear systems theory**.

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This representation captures many rational matrices coming from NLEPs

- Loaded elastic string (Betcke et al., NLEVP, (2013); Solov'ëv (2006)):

$$R(\lambda) = A - \lambda B + \frac{\lambda}{\lambda - \sigma} E = (A + E) - \lambda B + \frac{\sigma}{\lambda - \sigma} E \in \mathbb{C}(\lambda)^{n \times n}.$$

- Damped vibration of a viscoelastic structure (Mehrmann & Voss, (2004)):

$$R(\lambda) = \lambda^2 M + K - \sum_{i=1}^{k} \frac{1}{1 + b_i \lambda} \Delta G_i \in \mathbb{C}(\lambda)^{n \times n}.$$

- El-Guide, Miedlar, Saad (2020) consider for approximating some NLEPs

$$R(\lambda) = -B_0 + \lambda A_0 + \frac{B_1}{\lambda - \sigma_1} + \cdots + \frac{B_s}{\lambda - \sigma_s} \in \mathbb{C}(\lambda)^{p \times p},$$

$$= -B_0 + \lambda A_0 + \begin{bmatrix} B_1 & \cdots & B_s \end{bmatrix} \begin{bmatrix} (\lambda - \sigma_1) I_p & & \\ & \ddots & \\ & & (\lambda - \sigma_s) I_p \end{bmatrix}^{-1} \begin{bmatrix} I_p \\ \vdots \\ I_p \end{bmatrix}$$

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$$R(\lambda) = -B_0 + \lambda A_0 + \frac{B_1}{\lambda - \sigma_1} + \cdots + \frac{B_s}{\lambda - \sigma_s} \in \mathbb{C}(\lambda)^{p \times p},$$

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This representation captures many rational matrices coming from NLEPs

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More auxiliary pencils

We have used so far

$$L_k(\lambda) := \begin{bmatrix} 1 & -\lambda & & & \\ & 1 & -\lambda & & \\ & & \ddots & \ddots & \\ & & & 1 & -\lambda \end{bmatrix} \in \mathbb{C}[\lambda]^{k\times(k+1)},$$

$$\Lambda_k(\lambda)^T := \begin{bmatrix} \lambda^k & \lambda^{k-1} & \cdots & \lambda & 1 \end{bmatrix} \in \mathbb{C}[\lambda]^{1\times(k+1)}.$$

and their Kronecker products by identities

$$K_1(\lambda) := L_\epsilon(\lambda) \otimes I_n,$$
$$K_2(\lambda) := L_\eta(\lambda) \otimes I_m.$$

Now, we introduce

$$\widehat{K}_1 := \mathbf{e}_{\epsilon+1}^T \otimes I_n = \underbrace{\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}}_{\epsilon+1} \otimes I_n,$$

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Theorem (Amparan, D, Marcaida, Zaballa, SIMAX, 2018)

Let

- $A \in \mathbb{C}^{\ell\times\ell}$, $B \in \mathbb{C}^{\ell\times n}$, $C \in \mathbb{C}^{m\times\ell}$ *be arbitrary constant matrices and* $M(\lambda)$ *be an arbitrary pencil of adequate size, and*
- $K_1(\lambda)$, $K_2(\lambda)$, $\widehat{K}_1$, $\widehat{K}_2$ *be the pencils and matrices in the previous slide.*

Let us consider the pencil

$$S(\lambda) = \left[\begin{array}{c|cc} A - \lambda I_\ell & B\widehat{K}_1 & 0 \\ \hline \widehat{K}_2^T C & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array}\right],$$

and the rational matrix

$$R(\lambda) = \underbrace{(\Lambda_\eta(\lambda)^T \otimes I_m) M(\lambda) (\Lambda_\epsilon(\lambda) \otimes I_n)}_{D(\lambda) := \text{poly. part}} + \underbrace{C(\lambda I_\ell - A)^{-1} B}_{\text{strict. proper. part}}.$$

If $\{A, B, C\}$ *is a minimal state-space realization,* **then** $S(\lambda)$ **is a strong linearization of** $R(\lambda)$**.**

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Example of block Kronecker lin: Van Dooren & De Wilde, LAA, (1983)

- Given the rational matrix:

$$R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i \lambda^i \in \mathbb{C}(\lambda)^{m\times n},$$

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Outline

The Setting

- We consider a **general $m \times n$ polynomial matrix** of degree d

$$P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0 \quad , \qquad P_i \in \mathbb{C}^{m\times n},$$

- and we assume that its **complete eigenstructure**
- has been computed by applying a **backward stable algorithm (QZ for regular, Staircase for singular)**
- to a **strong linearization $\mathcal{L}(\lambda)$ in the class of block Kronecker linearizations** of $P(\lambda)$.

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Backward stable algorithms on strong linearizations and question

- *The computed complete eigenstructure of $\mathcal{L}(\lambda)$ is the exact complete eigenstructure of a matrix pencil $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ such that*

$$\frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F} = O(\mathbf{u}),$$

where $\mathbf{u} \approx 10^{-16}$ is the unit roundoff and

- $\|\cdot\|_F$ is the Frobenius norm, i.e., for any matrix polynomial

$$\|Q_k\lambda^k + \cdots + Q_1\lambda + Q_0\|_F = \sqrt{\|Q_k\|_F^2 + \cdots + \|Q_1\|_F^2 + \|Q_0\|_F^2}\ .$$

- But, does this imply that the computed complete eigenstructure of $P(\lambda)$ is the exact complete eigenstructure of a polynomial matrix of the same degree $P(\lambda) + \Delta P(\lambda)$ such that

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Why is not obvious to answer this question?

because block Kronecker linearizations are highly structured pencils and perturbations destroy the structure!!

Example: The Frobenius Companion Form

$$C_1(\lambda) := \begin{bmatrix} \lambda P_d + P_{d-1} & P_{d-2} & \cdots & P_1 & P_0 \\ -I_n & \lambda I_n & & & \\ & \ddots & \ddots & & \\ & & \ddots & \lambda I_n & \\ & & & -I_n & \lambda I_n \end{bmatrix}$$

$C_1(\lambda) + \Delta\mathcal{L}(\lambda) =$

$$\begin{bmatrix} \lambda(P_d + E_{11}) + (P_{d-1} + F_{11}) & \lambda E_{12} + P_{d-2} + F_{12} & \cdots & \lambda E_{1,d-1} + P_1 + F_{1,d-1} & \cdots \\ \lambda E_{21} - I_n + F_{21} & \lambda(I_n + E_{22}) + F_{22} & \lambda E_{23} + F_{23} & & \\ \lambda E_{31} + F_{31} & \lambda E_{32} + F_{32} & \ddots & & \\ \vdots & \vdots & \ddots & \lambda(I_n + E_{d-1,d-1}) + F_{d-1,d-1} & \\ \lambda E_{d1} + F_{d1} & \lambda E_{d2} + F_{d2} & & \lambda E_{d,d-1} + F_{d,d-1} - I_n & \cdots \end{bmatrix}$$

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The matrix perturbation problems to be solved

- **Problem 1:** To establish conditions on $\|\Delta\mathcal{L}(\lambda)\|_F$ such that $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ is a strong linearization for some polynomial matrix $P(\lambda) + \Delta P(\lambda)$ of degree d.
- **Problem 2:** To prove a perturbation bound

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq C_{P,\mathcal{L}} \, \frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F},$$

  with $C_{P,\mathcal{L}}$ a number depending on $P(\lambda)$ and $\mathcal{L}(\lambda)$.
- For those $P(\lambda)$ and $\mathcal{L}(\lambda)$ s.t. $C_{P,\mathcal{L}}$ is moderate, to use global backward stable algorithms on $\mathcal{L}(\lambda)$ gives global backward stability for $P(\lambda)$.

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  with $C_{P,\mathcal{L}}$ a number depending on $P(\lambda)$ and $\mathcal{L}(\lambda)$.
- For those $P(\lambda)$ and $\mathcal{L}(\lambda)$ s.t. $C_{P,\mathcal{L}}$ is moderate, to use global backward stable algorithms on $\mathcal{L}(\lambda)$ gives global backward stability for $P(\lambda)$.

The matrix perturbation problems to be solved

- **Problem 1:** To establish conditions on $\|\Delta\mathcal{L}(\lambda)\|_F$ such that $\mathcal{L}(\lambda)+\Delta\mathcal{L}(\lambda)$ is a strong linearization for some polynomial matrix $P(\lambda)+\Delta P(\lambda)$ of degree d.
- **Problem 2:** To prove a perturbation bound

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq C_{P,\mathcal{L}}\,\frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F},$$

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The main perturbation theorem for polynomial matrices

Theorem (D, Lawrence, Pérez, Van Dooren, Numer. Math., 2018)

Let $\mathcal{L}(\lambda)$ be a block Kronecker pencil for $P(\lambda) = \sum_{i=0}^{d} P_i \lambda^i \in \mathbb{C}[\lambda]^{m\times n}$, i.e.,

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M(\lambda) & L_\eta(\lambda)^T \otimes I_m \\ \hline L_\varepsilon(\lambda) \otimes I_n & 0 \end{array}\right].$$

If $\Delta\mathcal{L}(\lambda)$ is any pencil with the same size as $\mathcal{L}(\lambda)$ and such that

$$\|\Delta\mathcal{L}(\lambda)\|_F < \frac{(\sqrt{2}-1)^2}{d^{5/2}} \frac{1}{1+\|M(\lambda)\|_F},$$

then $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ is a strong linearization of a polynomial matrix $P(\lambda) + \Delta P(\lambda)$ with grade d and such that

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq 14\, d^{5/2} \frac{\|\mathcal{L}(\lambda)\|_F}{\|P(\lambda)\|_F} \left(1 + \|M(\lambda)\|_F + \|M(\lambda)\|_F^2\right) \frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}.$$

Discussion of the perturbation bound for block Kronecker pencils

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M(\lambda) & L_\eta(\lambda)^T \otimes I_m \\ \hline L_\varepsilon(\lambda) \otimes I_n & 0 \end{array}\right].$$

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- It can be proved that if $\|P(\lambda)\|_F \ll 1$ or $\|P(\lambda)\|_F \gg 1$, then $C_{P,\mathcal{L}} \gg 1$,
- and that, if $\|M(\lambda)\|_F \gg 1$, then $C_{P,\mathcal{L}} \gg 1$.
- Therefore, **for getting "backward stability" from Block Kronecker linearizations**, one needs to normalize the matrix poly $\|P(\lambda)\|_F = 1$ and to use pencils such that $\|M(\lambda)\|_F \approx \|P(\lambda)\|_F$, then

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \lesssim d^3 \sqrt{m+n}\, \frac{\|\Delta \mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}.$$

For Fiedler, Frobenius, etc linearizations $\|M(\lambda)\|_F = \|P(\lambda)\|_F$.

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Outline

We need some additional norms....

- **We have already used the Frobenius norm of a matrix polynomial.** For $D(\lambda) = \sum_{i=0}^{d} D_i \lambda^i$, we define

$$\|D(\lambda)\|_F := \sqrt{\sum_{i=0}^{d} \|D_i\|_F^2}.$$

- **"Norm" of a rational matrix represented as**

$$R(\lambda) = C(\lambda I_\ell - A)^{-1} B + \sum_{i=0}^{d} D_i \lambda^i \,,$$

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- **Two-norm of a matrix pencil.**

$$\|S(\lambda)\|_2 = \|S_1 \lambda + S_0\|_2 := \| \begin{bmatrix} S_1 & S_0 \end{bmatrix} \|_2$$

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Statement of the problem (I)

- Given a rational matrix represented as

$$R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i \lambda^i \,,$$

- its zeros (and minimal indices) are computed by applying a backward stable algorithm (QZ for regular, Staircase for singular) for computing the eigenvalues (and minimal indices) of its block Kronecker linearization

$$S(\lambda) = \begin{bmatrix} M(\lambda) & \widehat{K}_2^T C & K_2(\lambda)^T \\ B\widehat{K}_1 & A - \lambda I_\ell & 0 \\ K_1(\lambda) & 0 & 0 \end{bmatrix},$$

where $\sum_{i=0}^{d} D_i \lambda^i = (\Lambda_\eta(\lambda)^T \otimes I_m) M(\lambda) (\Lambda_\epsilon(\lambda) \otimes I_n)\,.$

- This means that **we have computed the exact eigenvalues (and minimal indices) of a pencil**

$$\widehat{S}(\lambda) := S(\lambda) + \Delta_S(\lambda), \quad \text{with} \quad \frac{\|\Delta_S(\lambda)\|_F}{\|S(\lambda)\|_F} = O(\mathbf{u}),$$

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Statement of the problem (II)

- But, **does this imply that we have computed the exact zeros of a nearby rational matrix?**
- **Nearby in the following structural sense:**

$$\widetilde{R}(\lambda) = (C + \Delta C)(\lambda I_\ell - (A + \Delta A))^{-1}(B + \Delta B) + \sum_{i=0}^{d}(D_i + \Delta D_i)\lambda^i\,,$$

with

$$\frac{\|\Delta_R\|_F}{\|R(\lambda)\|_F} := \frac{\sqrt{\|\Delta A\|_F^2 + \|\Delta B\|_F^2 + \|\Delta C\|_F^2 + \sum_{i=0}^{d}\|\Delta D_i\|_F^2}}{\|R(\lambda)\|_F} = O(\mathbf{u})\ ??$$

- Again, this problem is not easy because **the perturbation pencil $\Delta_S(\lambda)$ destroys completely the delicate structure of $S(\lambda)$**.
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The main perturbation theorem

Theorem (D, Quintana, Van Dooren, Calcolo, 2023)

Given $R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i \lambda^i$, *let* $S(\lambda)$ *be any of its block Kronecker linearizations, and let us perturb it with a pencil to get*

$$\widehat{S}(\lambda) = S(\lambda) + \Delta_S(\lambda)$$

If $\|\Delta_S(\lambda)\|_F$ *is sufficiently small, then the eigenvalues and min. indices of* $\widehat{S}(\lambda)$ *are the zeros and shifted min. indices of*

$$\widetilde{R}(\lambda) = (C + \Delta C)(\lambda I_\ell - (A + \Delta A))^{-1}(B + \Delta B) + \sum_{i=0}^{d}(D_i + \Delta D_i)\lambda^i$$

such that, to first order in $\|\Delta_S(\lambda)\|_F$,

$$\frac{\sqrt{\|\Delta A\|_F^2 + \|\Delta B\|_F^2 + \|\Delta C\|_F^2 + \sum_{i=0}^{d}\|\Delta D_i\|_F^2}}{\|R(\lambda)\|_F} \leq C_{S,R}\,\frac{\|\Delta_S(\lambda)\|_F}{\|S(\lambda)\|_F},$$

where

$$C_{S,R} = \sqrt{2\min(\varepsilon+1, \eta+1)}\,(1 + f_1\|S(\lambda)\|_2)(1 + f_2\|S(\lambda)\|_2)(1 + f_3\|S(\lambda)\|_2)\frac{\|S(\lambda)\|_F}{\|R(\lambda)\|_F}$$

The main perturbation theorem (II)

Theorem (continuation)

where

$$\begin{aligned}
\alpha &:= 1 + 2\varepsilon \max(1, \|A\|_2^{\varepsilon}),\\
\beta &:= 1 + 2\eta \max(1, \|A\|_2^{\eta}),\\
\gamma &:= \frac{\varepsilon + \eta}{2\sqrt{2}},\\
s &:= \max(\alpha, \beta, \gamma) + \gamma(\beta\|B\|_2 + \alpha\|C\|_2)
\end{aligned}$$

and

$$\begin{aligned}
f_1 &:= \frac{4\sqrt{2}s}{2-\sqrt{3}},\\
f_2 &:= \frac{\sqrt{2}\,(4\max(\varepsilon,\eta) - 1)}{3},\\
f_3 &:= \sqrt{2}\,[1 + 2\max(\eta,\varepsilon)\max(1, \|A\|_2^{\max(\eta,\varepsilon)})]
\end{aligned}$$

only depend on the initial data.

The previous bound does not guarantee structural backward stability

- We should have

$$C_{S,R} = \sqrt{2\min(\varepsilon+1,\eta+1)}\,(1+f_1\|S(\lambda)\|_2)(1+f_2\|S(\lambda)\|_2)(1+f_3\|S(\lambda)\|_2)\frac{\|S(\lambda)\|_F}{\|R(\lambda)\|_F}$$

  moderate for guaranteeing structural backward stability.

- However this magnitude can be very large if $\|S(\lambda)\|_2$ is large or, under the weaker condition that $\|R(\lambda)\|_F$ is large,
- and we have performed numerical experiments that confirm that under such conditions the structured backward errors can be indeed very large.
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Theorem (D, Quintana, Van Dooren, Calcolo, 2023)

Let $R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i \lambda^i \in \mathbb{C}(\lambda)^{m\times n}$ *and* $S(\lambda)$ *be any of its block Kronecker linearizations. If*

$$\max(\|A\|_F, \|B\|_F, \|C\|_F, \|D(\lambda)\|_F) \leq 1 \quad \textit{and} \quad \|M(\lambda)\|_F \approx \|D(\lambda)\|_F$$

then

$$C_{S,R} \leq \; g\, d^5 \sqrt{m+n}\,,$$

where g *is a moderate number (a constant that does not depend on* d, m, n, ℓ*) and, to first order in* $\|\Delta_S(\lambda)\|_F$*,*

$$\frac{\sqrt{\|\Delta A\|_F^2 + \|\Delta B\|_F^2 + \|\Delta C\|_F^2 + \sum_{i=0}^{d} \|\Delta D_i\|_F^2}}{\|R(\lambda)\|_F} \lesssim d^5 \sqrt{m+n}\, \frac{\|\Delta_S(\lambda)\|_F}{\|S(\lambda)\|_F}\,.$$

What to do if these conditions do not hold? SCALING!!!

- **There are rational matrices for which**

$$\max(\|A\|_F, \|B\|_F, \|C\|_F, \|D(\lambda)\|_F) \leq 1$$

do not hold.

- **In such cases, we can scale the original rational matrix** $R(\lambda)$ as

$$\widehat{R}(\widehat{\lambda}) := \widehat{D}(\widehat{\lambda}) + \widehat{C}(\widehat{\lambda} I_\ell - \widehat{A})^{-1}\widehat{B} := d_R R(\widehat{\lambda}/d_\lambda)$$

where

$$\widehat{A} := d_\lambda T^{-1} A T, \quad \widehat{B} := \sqrt{d_\lambda d_R}\, T^{-1} B, \quad \widehat{C} := \sqrt{d_\lambda d_R}\, C T, \quad \widehat{D}_i := d_R d_\lambda^{-i} D_i,$$

$i = 0, 1, \ldots, d$, and $T = \text{diag}(d_1, \ldots, d_\ell)$,

- **to obtain a rational matrix** $\widehat{R}(\widehat{\lambda})$ **that satisfies the desired conditions.**
- Moreover, this can be done without errors with parameters that are integer powers of 2.
- However, **this requires to scale also the variable** λ in contrast with the polynomial case.

What to do if these conditions do not hold? SCALING!!!

- **There are rational matrices for which**

$$\max(\|A\|_F, \|B\|_F, \|C\|_F, \|D(\lambda)\|_F) \leq 1$$

do not hold.

- **In such cases, we can scale the original rational matrix** $R(\lambda)$ as

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Outline

Conclusions

- We have proved that the computation of the complete eigenstructure of a matrix polynomial $P(\lambda)$ (regular or singular) via block Kronecker linearizations is backward stable from the polynomial point of view
- if the polynomial is multiplied by a number in such a way that $\|P(\lambda)\|_F = 1$
- and the linearization is chosen to satisfy $\|M(\lambda)\|_F \approx \|P(\lambda)\|_F$, which happens for many of the linearizations used in the literature.
- We have obtained similar results for rational matrices expressed as $R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i \lambda^i \in \mathbb{C}(\lambda)^{m\times n}$
- though in this case a delicate scaling of the rational matrix, which requires to scale the variable, is needed.
- We have also analyzed (D, Pérez, Van Dooren, Math. Comp., 2019) structured polynomial matrices (Hermitian, alternating, palindromic, ...) with structure preserving backward errors and we have obtained similar positive results.

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