Backward stability in polynomial and rational eigenvalue problems solved via linearization

Froilán M. Dopico

joint work with **Piers Lawrence, Javier Pérez, María C. Quintana** and **Paul Van Dooren**

Depto de Matemáticas, Universidad Carlos III de Madrid, Spain

Part of "Proyecto de I+D+i PID2019-106362GB-I00 financiado por MCIN/AEI/10.13039/501100011033"

Numerical Analysis and Scientific Computation with Applications (NASCA23)

National and Kapodistrian University of Athens. 3-6 July, 2023

GOBIERNO DE ESPAÑA MINISTERIO DE CIENCIA E INNOVACIÓN

AGENCIA ESTATAL DE INVESTIGACIÓN

uc3m | Universidad Carlos III de Madrid

Different classes of matrix eigenvalue problems (I)

From a *simplified* point of view, we can consider the following matrix eigenvalue problems:

- **The basic eigenvalue problem (BEP).** Given $A \in \mathbb{C}^{n\times n}$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that
$$Av = \lambda v \iff \boxed{(\lambda I_n - A)\, v = 0}$$

- **The GENERALIZED eigenvalue problem (GEP).** Given $A, B \in \mathbb{C}^{m\times n}$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that
$$Av = \lambda B v \iff \boxed{(\lambda B - A)\, v = 0},$$
often (but not always) under the regularity assumption that A and B are square and $\det(zB - A)$ is not zero for all $z \in \mathbb{C}$.

From a *simplified* point of view, we can consider the following matrix eigenvalue problems:

- **The basic eigenvalue problem (BEP).** Given $A \in \mathbb{C}^{n\times n}$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that

$$Av = \lambda v \iff \boxed{(\lambda I_n - A)\, v = 0}$$

- **The GENERALIZED eigenvalue problem (GEP).** Given $A, B \in \mathbb{C}^{m\times n}$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that

$$Av = \lambda Bv \iff \boxed{(\lambda B - A)\, v = 0},$$

often (but not always) under the regularity assumption that A and B are square and $\det(zB - A)$ is not zero for all $z \in \mathbb{C}$.

From a *simplified* point of view, we can consider the following matrix eigenvalue problems:

- **The basic eigenvalue problem (BEP).** Given $A \in \mathbb{C}^{n\times n}$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that

$$Av = \lambda v \iff \boxed{(\lambda I_n - A)\, v = 0}$$

- **The GENERALIZED eigenvalue problem (GEP).** Given $A, B \in \mathbb{C}^{m\times n}$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that

$$Av = \lambda Bv \iff \boxed{(\lambda B - A)\, v = 0},$$

often (but not always) under the regularity assumption that A and B are square and $\det(zB - A)$ is not zero for all $z \in \mathbb{C}$.

- **The POLYNOMIAL eigenvalue problem (PEP).** Given $P_0, P_1, \ldots, P_d \in \mathbb{C}^{m\times n}$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that

$$\boxed{(P_d\lambda^d + \cdots + P_1\lambda + P_0)v = 0},$$

often (but not always) under the regularity assumption that P_i are square and $\det(P_d z^d + \cdots + P_1 z + P_0) \not\equiv 0$.

- **The RATIONAL eigenvalue problem (REP).** Given a rational matrix $G(z) \in \mathbb{C}(z)^{m\times n}$, i.e., such that $G(z)_{ij}$ is a scalar rational function of $z \in \mathbb{C}$, for $1 \leq i, j \leq n$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that λ is not a pole of any $G(z)_{ij}$ and

$$\boxed{G(\lambda)v = 0},$$

often (but not always) under the regularity assumption $\det(G(z)) \not\equiv 0$.

We focus in this talk on PEPs and REPs, which are important by themselves but also as approximations of more general nonlinear eigenvalue problems.

- **The POLYNOMIAL eigenvalue problem (PEP).** Given $P_0, P_1, \ldots, P_d \in \mathbb{C}^{m\times n}$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that

$$\boxed{(P_d\lambda^d + \cdots + P_1\lambda + P_0)v = 0},$$

often (but not always) under the regularity assumption that P_i are square and $\det(P_d z^d + \cdots + P_1 z + P_0) \not\equiv 0$.

- **The RATIONAL eigenvalue problem (REP).** Given a rational matrix $G(z) \in \mathbb{C}(z)^{m\times n}$, i.e., such that $G(z)_{ij}$ is a scalar rational function of $z \in \mathbb{C}$, for $1 \le i, j \le n$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that λ is not a pole of any $G(z)_{ij}$ and

$$\boxed{G(\lambda)v = 0},$$

often (but not always) under the regularity assumption $\det(G(z)) \not\equiv 0$.

We focus in this talk on PEPs and REPs, which are important by themselves but also as approximations of more general nonlinear eigenvalue problems.

- **The POLYNOMIAL eigenvalue problem (PEP).** Given $P_0, P_1, \ldots, P_d \in \mathbb{C}^{m\times n}$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that

$$\boxed{(P_d\lambda^d + \cdots + P_1\lambda + P_0)v = 0},$$

  often (but not always) under the regularity assumption that P_i are square and $\det(P_d z^d + \cdots + P_1 z + P_0) \not\equiv 0$.

- **The RATIONAL eigenvalue problem (REP).** Given a rational matrix $G(z) \in \mathbb{C}(z)^{m\times n}$, i.e., such that $G(z)_{ij}$ is a scalar rational function of $z \in \mathbb{C}$, for $1 \leq i, j \leq n$, compute scalars λ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that λ is not a pole of any $G(z)_{ij}$ and

$$\boxed{G(\lambda)v = 0},$$

  often (but not always) under the regularity assumption $\det(G(z)) \not\equiv 0$.

We focus in this talk on PEPs and REPs, which are important by themselves but also as approximations of more general nonlinear eigenvalue problems.

A key idea on matrix eigenvalue problems

1. **BEP**: $(\lambda I_n - A)\, v = 0$
2. **GEP**: $(\lambda B - A)\, v = 0$!!!!
3. **PEP**: $(P_d \lambda^d + \cdots + P_1 \lambda + P_0) v = 0$
4. **REP**: $G(\lambda) v = 0$

- **Key idea: PEPs and REPs can be solved by transforming the problem into a GEP** via a process known as **LINEARIZATION**.
- This transformation is **exact**, i.e., the obtained **GEP** contains (or allows us to easily extract) exactly all the eigen-information of the original **PEP** or **REP**.
- The use of **linearizations** is one of the **most reliable** approaches for solving numerically PEPs and REPs, because **there exist very reliable algorithms for solving GEPs**.
- This approach has been studied by many researchers in the last two decades.

A key idea on matrix eigenvalue problems

1. **BEP**: $(\lambda I_n - A)\, v = 0$
2. **GEP**: $(\lambda B - A)\, v = 0$!!!!
3. **PEP**: $(P_d\lambda^d + \cdots + P_1\lambda + P_0)v = 0$
4. **REP**: $G(\lambda)v = 0$

- **Key idea: PEPs and REPs can be solved by transforming the problem into a GEP** via a process known as **LINEARIZATION**.
- This transformation is **exact**, i.e., the obtained **GEP** contains (or allows us to easily extract) exactly all the eigen-information of the original **PEP** or **REP**.
- The use of **linearizations** is one of the **most reliable** approaches for solving numerically PEPs and REPs, because **there exist very reliable algorithms for solving GEPs**.
- This approach has been studied by many researchers in the last two decades.

A key idea on matrix eigenvalue problems

1. **BEP**: $(\lambda I_n - A)\, v = 0$
2. **GEP**: $(\lambda B - A)\, v = 0$!!!!
3. **PEP**: $(P_d \lambda^d + \cdots + P_1 \lambda + P_0) v = 0$
4. **REP**: $G(\lambda) v = 0$

- **Key idea: PEPs and REPs can be solved by transforming the problem into a GEP** via a process known as **LINEARIZATION**.
- This transformation is **exact**, i.e., the obtained **GEP** contains (or allows us to easily extract) exactly all the eigen-information of the original **PEP** or **REP**.
- The use of **linearizations** is one of the **most reliable** approaches for solving numerically PEPs and REPs, because **there exist very reliable algorithms for solving GEPs**.
- This approach has been studied by many researchers in the last two decades.

A key idea on matrix eigenvalue problems

1. **BEP**: $(\lambda I_n - A)\, v = 0$
2. **GEP**: $(\lambda B - A)\, v = 0$ **!!!!**
3. **PEP**: $(P_d \lambda^d + \cdots + P_1 \lambda + P_0) v = 0$
4. **REP**: $G(\lambda) v = 0$

- **Key idea: PEPs and REPs can be solved by transforming the problem into a GEP** via a process known as **LINEARIZATION**.
- This transformation is **exact**, i.e., the obtained **GEP** contains (or allows us to easily extract) exactly all the eigen-information of the original **PEP** or **REP**.
- The use of **linearizations** is one of the **most reliable** approaches for solving numerically PEPs and REPs, because **there exist very reliable algorithms for solving GEPs**.
- This approach has been studied by many researchers in the last two decades.

A key idea on matrix eigenvalue problems

1. **BEP**: $(\lambda I_n - A)\, v = 0$
2. **GEP**: $(\lambda B - A)\, v = 0$ **!!!!**
3. **PEP**: $(P_d \lambda^d + \cdots + P_1 \lambda + P_0) v = 0$
4. **REP**: $G(\lambda) v = 0$

- **Key idea: PEPs and REPs can be solved by transforming the problem into a GEP** via a process known as **LINEARIZATION**.
- This transformation is **exact**, i.e., the obtained **GEP** contains (or allows us to easily extract) exactly all the eigen-information of the original **PEP** or **REP**.
- The use of **linearizations** is one of the **most reliable** approaches for solving numerically PEPs and REPs, because **there exist very reliable algorithms for solving GEPs**.
- This approach has been studied by many researchers in the last two decades.

The goal of the talk

- We will study the backward stability of solving a polynomial or a rational eigenvalue problem
- by applying a backward stable generalized eigenvalue algorithm to a wide family of its linearizations.
- As we will see, this backward stability problem is nontrivial because the linearizations are highly structured pencils and the backward errors of the generalized eigenvalue algorithm destroy such structures.

The goal of the talk

- We will study the backward stability of solving a polynomial or a rational eigenvalue problem
- by applying a backward stable generalized eigenvalue algorithm to a wide family of its linearizations.
- As we will see, this backward stability problem is nontrivial because the linearizations are highly structured pencils and the backward errors of the generalized eigenvalue algorithm destroy such structures.

The goal of the talk

- We will study the backward stability of solving a polynomial or a rational eigenvalue problem
- by applying a backward stable generalized eigenvalue algorithm to a wide family of its linearizations.
- As we will see, this backward stability problem is nontrivial because the linearizations are highly structured pencils and the backward errors of the generalized eigenvalue algorithm destroy such structures.

Outline

Outline

GEPs-PEPs-REPs have more spectral “structural” data than BEPs

1. **BEP**: $(\lambda I_n - A)\, v = 0$
2. **GEP**: $(\lambda B - A)\, v = 0$
3. **PEP**: $(P_d \lambda^d + \cdots + P_1 \lambda + P_0) v = 0$
4. **REP**: $G(\lambda) v = 0$

- So far, we have only considered informally **finite eigenvalues**, but
- **GEPs, PEPs, REPs** may have also **infinite eigenvalues**.
- **GEPs, PEPs, REPs** may be **singular**, i.e., rectangular or square with identically zero determinant, (BEPs are always regular) and to have, in addition to eigenvalues, **minimal indices**.
- Moreover, **REPs** have **poles**.
- We define quickly these concepts.

GEPs-PEPs-REPs have more spectral "structural" data than BEPs

1. **BEP**: $(\lambda I_n - A)\, v = 0$
2. **GEP**: $(\lambda B - A)\, v = 0$
3. **PEP**: $(P_d \lambda^d + \cdots + P_1 \lambda + P_0) v = 0$
4. **REP**: $G(\lambda) v = 0$

- So far, we have only considered informally **finite eigenvalues**, but
- **GEPs, PEPs, REPs** may have also **infinite eigenvalues**.
- **GEPs, PEPs, REPs** may be **singular**, i.e., rectangular or square with identically zero determinant, (BEPs are always regular) and to have, in addition to eigenvalues, **minimal indices**.
- Moreover, **REPs** have **poles**.
- We define quickly these concepts.

1. **BEP**: $(\lambda I_n - A)\, v = 0$
2. **GEP**: $(\lambda B - A)\, v = 0$
3. **PEP**: $(P_d \lambda^d + \cdots + P_1 \lambda + P_0) v = 0$
4. **REP**: $G(\lambda) v = 0$

- So far, we have only considered informally **finite eigenvalues**, but
- **GEPs, PEPs, REPs** may have also **infinite eigenvalues**.
- **GEPs, PEPs, REPs** may be **singular**, i.e., rectangular or square with identically zero determinant, (BEPs are always regular) and to have, in addition to eigenvalues, **minimal indices**.
- Moreover, **REPs** have **poles**.
- We define quickly these concepts.

GEPs-PEPs-REPs have more spectral "structural" data than BEPs

1. **BEP**: $(\lambda I_n - A)\, v = 0$
2. **GEP**: $(\lambda B - A)\, v = 0$
3. **PEP**: $(P_d \lambda^d + \cdots + P_1 \lambda + P_0) v = 0$
4. **REP**: $G(\lambda) v = 0$

- So far, we have only considered informally **finite eigenvalues**, but
- **GEPs, PEPs, REPs** may have also **infinite eigenvalues**.
- **GEPs, PEPs, REPs** may be **singular**, i.e., rectangular or square with identically zero determinant, (BEPs are always regular) and to have, in addition to eigenvalues, **minimal indices**.
- Moreover, **REPs** have **poles**.
- We define quickly these concepts.

GEPs-PEPs-REPs have more spectral “structural” data than BEPs

1. **BEP**: $(\lambda I_n - A)\, v = 0$
2. **GEP**: $(\lambda B - A)\, v = 0$
3. **PEP**: $(P_d \lambda^d + \cdots + P_1 \lambda + P_0) v = 0$
4. **REP**: $G(\lambda) v = 0$

- So far, we have only considered informally **finite eigenvalues**, but
- **GEPs, PEPs, REPs** may have also **infinite eigenvalues**.
- **GEPs, PEPs, REPs** may be **singular**, i.e., rectangular or square with identically zero determinant, (BEPs are always regular) and to have, in addition to eigenvalues, **minimal indices**.
- Moreover, **REPs** have **poles**.
- We define quickly these concepts.

GEPs-PEPs-REPs have more spectral “structural” data than BEPs

1. **BEP**: $(\lambda I_n - A)\, v = 0$
2. **GEP**: $(\lambda B - A)\, v = 0$
3. **PEP**: $(P_d \lambda^d + \cdots + P_1 \lambda + P_0) v = 0$
4. **REP**: $G(\lambda) v = 0$

- So far, we have only considered informally **finite eigenvalues**, but
- **GEPs, PEPs, REPs** may have also **infinite eigenvalues**.
- **GEPs, PEPs, REPs** may be **singular**, i.e., rectangular or square with identically zero determinant, (BEPs are always regular) and to have, in addition to eigenvalues, **minimal indices**.
- Moreover, **REPs** have **poles**.
- We define quickly these concepts.

Finite and infinite eigenvalues of PEPs

Given $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0 \in \mathbb{C}[\lambda]^{m\times n}$,

- $\lambda_0 \in \mathbb{C}$ is a **finite eigenvalue** of $P(\lambda)$ if

$$\operatorname{rank} P(\lambda_0) < \max_{\lambda\in\mathbb{C}} \operatorname{rank} P(\lambda)$$

- The infinite eigenvalue of $P(\lambda)$ is defined through **the reversal polynomial**.
- The reversal of $P(\lambda)$ is

$$\operatorname{rev} P(\lambda) := \lambda^d P\left(\tfrac{1}{\lambda}\right) = P_0\lambda^d + \cdots + P_{d-1}\lambda + P_d .$$

- Then the **infinite eigenvalue** (and its mutiplicities) of $P(\lambda)$ correspond to the **zero eigenvalue** (and its mutiplicities) of $\operatorname{rev}P(\lambda)$.

Finite and infinite eigenvalues of PEPs

Given $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0 \in \mathbb{C}[\lambda]^{m\times n}$,

- $\lambda_0 \in \mathbb{C}$ is a **finite eigenvalue** of $P(\lambda)$ if

$$\operatorname{rank} P(\lambda_0) < \max_{\lambda\in\mathbb{C}} \operatorname{rank} P(\lambda)$$

- The infinite eigenvalue of $P(\lambda)$ is defined through **the reversal polynomial**.
- The reversal of $P(\lambda)$ is

$$\operatorname{rev} P(\lambda) := \lambda^d P\left(\tfrac{1}{\lambda}\right) = P_0\lambda^d + \cdots + P_{d-1}\lambda + P_d \, .$$

- Then the **infinite eigenvalue** (and its mutiplicities) of $P(\lambda)$ correspond to the **zero eigenvalue** (and its mutiplicities) of $\operatorname{rev} P(\lambda)$.

Given $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0 \in \mathbb{C}[\lambda]^{m\times n}$,

- $\lambda_0 \in \mathbb{C}$ is a **finite eigenvalue** of $P(\lambda)$ if

$$\operatorname{rank} P(\lambda_0) < \max_{\lambda\in\mathbb{C}} \operatorname{rank} P(\lambda)$$

- The infinite eigenvalue of $P(\lambda)$ is defined through **the reversal polynomial**.
- The reversal of $P(\lambda)$ is

$$\text{rev}P(\lambda) := \lambda^d P\left(\tfrac{1}{\lambda}\right) = P_0\lambda^d + \cdots + P_{d-1}\lambda + P_d \quad .$$

- Then the **infinite eigenvalue** (and its mutiplicities) of $P(\lambda)$ correspond to the **zero eigenvalue** (and its mutiplicities) of rev$P(\lambda)$.

Finite and infinite eigenvalues of PEPs

Given $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0 \in \mathbb{C}[\lambda]^{m\times n}$,

- $\lambda_0 \in \mathbb{C}$ is a **finite eigenvalue** of $P(\lambda)$ if

$$\mathrm{rank} P(\lambda_0) < \max_{\lambda\in\mathbb{C}} \mathrm{rank} P(\lambda)$$

- The infinite eigenvalue of $P(\lambda)$ is defined through **the reversal polynomial**.
- The reversal of $P(\lambda)$ is

$$\text{rev}P(\lambda) := \lambda^d P(\tfrac{1}{\lambda}) = P_0\lambda^d + \cdots + P_{d-1}\lambda + P_d \quad .$$

- Then the **infinite eigenvalue** (and its mutiplicities) of $P(\lambda)$ correspond to the **zero eigenvalue** (and its mutiplicities) of rev$P(\lambda)$.

Minimal indices of singular PEPs

- PEPs are **singular** when $\boxed{P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0}$ is either **rectangular or square with** $\det P(\lambda) \equiv 0$.
- **Singular PEPs appear in applications**, in particular in Multivariable System Theory and Control Theory.
- In addition to eigenvalues, **singular matrix polynomials have** other "interesting numbers" called **minimal indices**,
- which are related to the fact that a singular $m \times n$ matrix polynomial $P(\lambda)$ has non-trivial left and/or right null-spaces over the field $\mathbb{C}(\lambda)$ of rational functions:

$$\begin{aligned}\mathcal{N}_\ell(P) &:= \left\{ y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times m} \,:\, y(\lambda)^T P(\lambda) \equiv 0^T \right\},\\ \mathcal{N}_r(P) &:= \left\{ x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} \,:\, P(\lambda)x(\lambda) \equiv 0 \right\}.\end{aligned}$$

- They have bases consisting entirely of vector polynomials.
- The polynomial bases with "minimal sum of the degrees" of their vectors are the **minimal bases** of $P(\lambda)$. The **minimal indices** of $P(\lambda)$ are the degrees of the vectors of any minimal basis.

- PEPs are **singular** when $\boxed{P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0}$ is either **rectangular or square with** $\det P(\lambda) \equiv 0$.
- **Singular PEPs appear in applications**, in particular in Multivariable System Theory and Control Theory.
- In addition to eigenvalues, **singular matrix polynomials have** other "interesting numbers" called **minimal indices**,
- which are related to the fact that a singular $m \times n$ matrix polynomial $P(\lambda)$ has non-trivial left and/or right null-spaces over the field $\mathbb{C}(\lambda)$ of rational functions:

$$\begin{aligned} \mathcal{N}_\ell(P) &:= \left\{ y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times m} \, : \, y(\lambda)^T P(\lambda) \equiv 0^T \right\}, \\ \mathcal{N}_r(P) &:= \left\{ x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} \, : \, P(\lambda) x(\lambda) \equiv 0 \right\}. \end{aligned}$$

- They have bases consisting entirely of vector polynomials.
- The polynomial bases with "minimal sum of the degrees" of their vectors are the **minimal bases** of $P(\lambda)$. The **minimal indices** of $P(\lambda)$ are the degrees of the vectors of any minimal basis.

Minimal indices of singular PEPs

- PEPs are **singular** when $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0$ is either **rectangular or square with** $\det P(\lambda) \equiv 0$.
- **Singular PEPs appear in applications**, in particular in Multivariable System Theory and Control Theory.
- In addition to eigenvalues, **singular matrix polynomials have** other "interesting numbers" called **minimal indices**,
- which are related to the fact that a singular $m \times n$ matrix polynomial $P(\lambda)$ has non-trivial left and/or right null-spaces over the field $\mathbb{C}(\lambda)$ of rational functions:

$$\begin{aligned} \mathcal{N}_\ell(P) &:= \left\{ y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times m} : y(\lambda)^T P(\lambda) \equiv 0^T \right\}, \\ \mathcal{N}_r(P) &:= \left\{ x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} : P(\lambda)x(\lambda) \equiv 0 \right\}. \end{aligned}$$

- They have bases consisting entirely of vector polynomials.
- The polynomial bases with "minimal sum of the degrees" of their vectors are the **minimal bases** of $P(\lambda)$. The **minimal indices** of $P(\lambda)$ are the degrees of the vectors of any minimal basis.

Minimal indices of singular PEPs

- PEPs are **singular** when $P(\lambda) = P_d \lambda^d + \cdots + P_1 \lambda + P_0$ is either **rectangular or square with** $\det P(\lambda) \equiv 0$.
- **Singular PEPs appear in applications**, in particular in Multivariable System Theory and Control Theory.
- In addition to eigenvalues, **singular matrix polynomials have** other "interesting numbers" called **minimal indices**,
- which are related to the fact that a singular $m \times n$ matrix polynomial $P(\lambda)$ has non-trivial left and/or right null-spaces over the field $\mathbb{C}(\lambda)$ of rational functions:

$$\begin{aligned} \mathcal{N}_\ell(P) &:= \left\{ y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times m} \, : \, y(\lambda)^T P(\lambda) \equiv 0^T \right\}, \\ \mathcal{N}_r(P) &:= \left\{ x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} \, : \, P(\lambda) x(\lambda) \equiv 0 \right\}. \end{aligned}$$

- They have bases consisting entirely of vector polynomials.
- The polynomial bases with "minimal sum of the degrees" of their vectors are the **minimal bases** of $P(\lambda)$. The **minimal indices** of $P(\lambda)$ are the degrees of the vectors of any minimal basis.

Minimal indices of singular PEPs

- PEPs are **singular** when $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0$ is either **rectangular or square with** $\det P(\lambda) \equiv 0$.
- **Singular PEPs appear in applications**, in particular in Multivariable System Theory and Control Theory.
- In addition to eigenvalues, **singular matrix polynomials have** other "interesting numbers" called **minimal indices**,
- which are related to the fact that a singular $m \times n$ matrix polynomial $P(\lambda)$ has non-trivial left and/or right null-spaces over the field $\mathbb{C}(\lambda)$ of rational functions:

$$\begin{aligned} \mathcal{N}_\ell(P) &:= \left\{ y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times m} \; : \; y(\lambda)^T P(\lambda) \equiv 0^T \right\}, \\ \mathcal{N}_r(P) &:= \left\{ x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} \; : \; P(\lambda) x(\lambda) \equiv 0 \right\}. \end{aligned}$$

- They have bases consisting entirely of vector polynomials.
- The polynomial bases with "minimal sum of the degrees" of their vectors are the **minimal bases** of $P(\lambda)$. The **minimal indices** of $P(\lambda)$ are the degrees of the vectors of any minimal basis.

Minimal indices of singular PEPs

- PEPs are **singular** when $\boxed{P(\lambda) = P_d \lambda^d + \cdots + P_1 \lambda + P_0}$ is either **rectangular or square with** $\det P(\lambda) \equiv 0$.
- **Singular PEPs appear in applications**, in particular in Multivariable System Theory and Control Theory.
- In addition to eigenvalues, **singular matrix polynomials have** other "interesting numbers" called **minimal indices**,
- which are related to the fact that a singular $m \times n$ matrix polynomial $P(\lambda)$ has non-trivial left and/or right null-spaces over the field $\mathbb{C}(\lambda)$ of rational functions:

$$\begin{aligned} \mathcal{N}_\ell(P) &:= \left\{ y(\lambda)^T \in \mathbb{F}(\lambda)^{1 \times m} \, : \, y(\lambda)^T P(\lambda) \equiv 0^T \right\}, \\ \mathcal{N}_r(P) &:= \left\{ x(\lambda) \in \mathbb{F}(\lambda)^{n \times 1} \, : \, P(\lambda) x(\lambda) \equiv 0 \right\}. \end{aligned}$$

- They have bases consisting entirely of vector polynomials.
- The polynomial bases with "minimal sum of the degrees" of their vectors are the **minimal bases** of $P(\lambda)$. The **minimal indices** of $P(\lambda)$ are the degrees of the vectors of any minimal basis.

As a consequence of the previous discussion, we define:

Definition

The **complete "eigenstructure"** of a polynomial matrix $P(\lambda)$ is comprised of:

- its finite eigenvalues, together with their partial multiplicities,
- its infinite eigenvalue, together with its partial multiplicities,
- its right minimal indices, and
- its left minimal indices.

Remarks

- The partial multiplicities are defined through the Smith form of $P(\lambda)$ and for matrices and pencils they are just the sizes of the Jordan blocks associated to each eigenvalue.

As a consequence of the previous discussion, we define:

Definition

The **complete "eigenstructure"** of a polynomial matrix $P(\lambda)$ is comprised of:

- its finite eigenvalues, together with their partial multiplicities,
- its infinite eigenvalue, together with its partial multiplicities,
- its right minimal indices, and
- its left minimal indices.

Remarks

- The partial multiplicities are defined through the Smith form of $P(\lambda)$ and for matrices and pencils they are just the sizes of the Jordan blocks associated to each eigenvalue.

Analogously, we define:

Definition

The **complete “eigenstructure”** of a rational matrix $G(\lambda)$ is comprised of:

- its finite zeros and **poles**, together with their partial multiplicities,
- its infinite zeros and **poles**, together with its partial multiplicities,
- its right minimal indices, and
- its left minimal indices.

Remarks

- The partial multiplicities are defined through the Smith-McMillan form of $G(\lambda)$.
- The infinite zeros and poles, together with its partial multiplicities, of $G(\lambda)$ are defined as the zeros and poles at $\lambda = 0$, together with its partial multiplicities, of $G(1/\lambda)$.

The complete "eigenstructure" of a rational matrix

Analogously, we define:

Definition

The **complete "eigenstructure"** of a rational matrix $G(\lambda)$ is comprised of:

- its finite zeros and **poles**, together with their partial multiplicities,
- its infinite zeros and **poles**, together with its partial multiplicities,
- its right minimal indices, and
- its left minimal indices.

Remarks

- The partial multiplicities are defined through the Smith-McMillan form of $G(\lambda)$.
- The infinite zeros and poles, together with its partial multiplicities, of $G(\lambda)$ are defined as the zeros and poles at $\lambda = 0$, together with its partial multiplicities, of $G(1/\lambda)$.

Outline

Definition

- A **linear polynomial matrix (or matrix pencil)** $L(\lambda)$ is a **linearization** of $P(\lambda) = P_d\,\lambda^d + \cdots + P_1\lambda + P_0$ if there exist **unimodular** polynomial matrices $U(\lambda), V(\lambda)$ such that

$$U(\lambda)\,L(\lambda)\,V(\lambda) = \begin{bmatrix} I_s & 0 \\ 0 & P(\lambda) \end{bmatrix}.$$

- $L(\lambda)$ is a **strong linearization** of $P(\lambda)$ if, **in addition**, $\operatorname{rev} L(\lambda)$ is a linearization for $\operatorname{rev} P(\lambda)$, i.e.,

$$\widetilde{U}(\lambda)\,(\operatorname{rev} L(\lambda))\,\widetilde{V}(\lambda) = \begin{bmatrix} I_s & 0 \\ 0 & \operatorname{rev} P(\lambda) \end{bmatrix},$$

with $\widetilde{U}(\lambda)$ and $\widetilde{V}(\lambda)$ unimodular.

Theorem

A matrix pencil $L(\lambda)$ is a linearization *of a polynomial matrix* $P(\lambda)$ **if and only if**

(1) $L(\lambda)$ *and* $P(\lambda)$ **have the same number of right minimal indices**.

(2) $L(\lambda)$ *and* $P(\lambda)$ **have the same number of left minimal indices**.

(3) $L(\lambda)$ *and* $P(\lambda)$ **have the same finite eigenvalues** *with the same partial multiplicities.*

$L(\lambda)$ **is a strong linearization** *of* $P(\lambda)$ **if and only if** *(1), (2), (3) and*

(4) $L(\lambda)$ *and* $P(\lambda)$ **have the same infinite eigenvalues** *with the same partial multiplicities.*

Remark: The **minimal indices** of $L(\lambda)$ **may have arbitrarily different values** from those of $P(\lambda)$, though in the most important classes of linearizations they are easily related.

Theorem

A matrix pencil $L(\lambda)$ is a linearization *of a polynomial matrix $P(\lambda)$* **if and only if**

(1) $L(\lambda)$ *and* $P(\lambda)$ **have the same number of right minimal indices**.

(2) $L(\lambda)$ *and* $P(\lambda)$ **have the same number of left minimal indices**.

(3) $L(\lambda)$ *and* $P(\lambda)$ **have the same finite eigenvalues** *with the same partial multiplicities.*

$L(\lambda)$ **is a strong linearization** *of* $P(\lambda)$ **if and only if** *(1), (2), (3) and*

(4) $L(\lambda)$ *and* $P(\lambda)$ **have the same infinite eigenvalues** *with the same partial multiplicities.*

Remark: The **minimal indices** of $L(\lambda)$ **may have arbitrarily different values** from those of $P(\lambda)$, though in the most important classes of linearizations they are easily related.

Theorem

A matrix pencil $L(\lambda)$ is a linearization *of a polynomial matrix $P(\lambda)$* **if and only if**

(1) $L(\lambda)$ *and* $P(\lambda)$ **have the same number of right minimal indices**.

(2) $L(\lambda)$ *and* $P(\lambda)$ **have the same number of left minimal indices**.

(3) $L(\lambda)$ *and* $P(\lambda)$ **have the same finite eigenvalues** *with the same partial multiplicities.*

$L(\lambda)$ **is a strong linearization** *of* $P(\lambda)$ **if and only if** *(1), (2), (3) and*

(4) $L(\lambda)$ *and* $P(\lambda)$ **have the same infinite eigenvalues** *with the same partial multiplicities.*

Remark: The **minimal indices** of $L(\lambda)$ **may have arbitrarily different values** from those of $P(\lambda)$, though in the most important classes of linearizations they are easily related.

The most famous strong linearization

The classical **Frobenius companion form** of the $m \times n$ matrix polynomial

$$P(\lambda) = P_d \lambda^d + \cdots + P_1 \lambda + P_0$$

is

$$C_1(\lambda) := \begin{bmatrix} \lambda P_d + P_{d-1} & P_{d-2} & \cdots & P_1 & P_0 \\ -I_n & \lambda I_n & & & \\ & \ddots & \ddots & & \\ & & \ddots & \lambda I_n & \\ & & & -I_n & \lambda I_n \end{bmatrix} \in \mathbb{C}[\lambda]^{(m+n(d-1)) \times nd}$$

Theorem ($C_1(\lambda)$ is much more than a strong linearization!!)

(a) *If $0 \leq \varepsilon_1 \leq \cdots \leq \varepsilon_p$ are the right minimal indices of $P(\lambda)$, then the right minimal indices of $C_1(\lambda)$ are $\varepsilon_1 + d - 1 \leq \cdots \leq \varepsilon_p + d - 1$.*

(b) *If $0 \leq \eta_1 \leq \cdots \leq \eta_q$ are the left minimal indices of $P(\lambda)$, then the left minimal indices of $C_1(\lambda)$ are $\eta_1 \leq \cdots \leq \eta_q$.*

The classical **Frobenius companion form** of the $m \times n$ matrix polynomial

$$P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0$$

is

$$C_1(\lambda) := \begin{bmatrix} \lambda P_d + P_{d-1} & P_{d-2} & \cdots & P_1 & P_0 \\ -I_n & \lambda I_n & & & \\ & \ddots & \ddots & & \\ & & \ddots & \lambda I_n & \\ & & & -I_n & \lambda I_n \end{bmatrix} \in \mathbb{C}[\lambda]^{(m+n(d-1))\times nd}$$

Theorem ($C_1(\lambda)$ is much more than a strong linearization!!)

(a) *If* $0 \leq \varepsilon_1 \leq \cdots \leq \varepsilon_p$ *are the right minimal indices of* $P(\lambda)$*, then the right minimal indices of* $C_1(\lambda)$ *are* $\varepsilon_1 + d - 1 \leq \cdots \leq \varepsilon_p + d - 1$.

(b) *If* $0 \leq \eta_1 \leq \cdots \leq \eta_q$ *are the left minimal indices of* $P(\lambda)$*, then the left minimal indices of* $C_1(\lambda)$ *are* $\eta_1 \leq \cdots \leq \eta_q$.

Some comments on linearizations of rational matrices

- For brevity, I will not present the definition of linearizations and strong linearizations of rational matrices.
- In contrast with the polynomial case, there is no agreement in the community on the definition of **(strong) linearization** of a rational matrix.
- Pioneering works on linearizations of rational matrices where developed by Van Dooren and Verghese in late 70s & early 80s though they did not give a general definition.
- Amparan, D, Marcaida, and Zaballa, *Strong linearizations of rational matrices*, SIMAX 2018 introduced a definition of strong linearization of any rational matrix $R(\lambda)$ that reduces to the one for polynomials when $R(\lambda)$ is a polynomial matrix. They also constructed explicitly many of such linearizations.
- Another closely related approach for defining linearizations of rational matrices was initiated by Alam and Behera, *Linearizations for rational matrix functions and Rosenbrock system polynomials*, SIMAX 2016 and followed by other students of Alam.

Some comments on linearizations of rational matrices

- For brevity, I will not present the definition of linearizations and strong linearizations of rational matrices.
- In contrast with the polynomial case, there is no agreement in the community on the definition of **(strong) linearization** of a rational matrix.
- Pioneering works on linearizations of rational matrices where developed by Van Dooren and Verghese in late 70s & early 80s though they did not give a general definition.
- Amparan, D, Marcaida, and Zaballa, *Strong linearizations of rational matrices*, SIMAX 2018 introduced a definition of strong linearization of any rational matrix $R(\lambda)$ that reduces to the one for polynomials when $R(\lambda)$ is a polynomial matrix. They also constructed explicitly many of such linearizations.
- Another closely related approach for defining linearizations of rational matrices was initiated by Alam and Behera, *Linearizations for rational matrix functions and Rosenbrock system polynomials*, SIMAX 2016 and followed by other students of Alam.

Some comments on linearizations of rational matrices

- For brevity, I will not present the definition of linearizations and strong linearizations of rational matrices.
- In contrast with the polynomial case, there is no agreement in the community on the definition of **(strong) linearization** of a rational matrix.
- Pioneering works on linearizations of rational matrices where developed by Van Dooren and Verghese in late 70s & early 80s though they did not give a general definition.
- Amparan, D, Marcaida, and Zaballa, *Strong linearizations of rational matrices*, SIMAX 2018 introduced a definition of strong linearization of any rational matrix $R(\lambda)$ that reduces to the one for polynomials when $R(\lambda)$ is a polynomial matrix. They also constructed explicitly many of such linearizations.
- Another closely related approach for defining linearizations of rational matrices was initiated by Alam and Behera, *Linearizations for rational matrix functions and Rosenbrock system polynomials*, SIMAX 2016 and followed by other students of Alam.

Some comments on linearizations of rational matrices

- For brevity, I will not present the definition of linearizations and strong linearizations of rational matrices.
- In contrast with the polynomial case, there is no agreement in the community on the definition of **(strong) linearization** of a rational matrix.
- Pioneering works on linearizations of rational matrices where developed by Van Dooren and Verghese in late 70s & early 80s though they did not give a general definition.
- Amparan, D, Marcaida, and Zaballa, *Strong linearizations of rational matrices*, SIMAX 2018 introduced a definition of strong linearization of any rational matrix $R(\lambda)$ that reduces to the one for polynomials when $R(\lambda)$ is a polynomial matrix. They also constructed explicitly many of such linearizations.
- Another closely related approach for defining linearizations of rational matrices was initiated by Alam and Behera, *Linearizations for rational matrix functions and Rosenbrock system polynomials*, SIMAX 2016 and followed by other students of Alam.

Some comments on linearizations of rational matrices

- For brevity, I will not present the definition of linearizations and strong linearizations of rational matrices.
- In contrast with the polynomial case, there is no agreement in the community on the definition of **(strong) linearization** of a rational matrix.
- Pioneering works on linearizations of rational matrices where developed by Van Dooren and Verghese in late 70s & early 80s though they did not give a general definition.
- Amparan, D, Marcaida, and Zaballa, *Strong linearizations of rational matrices*, SIMAX 2018 introduced a definition of strong linearization of any rational matrix $R(\lambda)$ that reduces to the one for polynomials when $R(\lambda)$ is a polynomial matrix. They also constructed explicitly many of such linearizations.
- Another closely related approach for defining linearizations of rational matrices was initiated by Alam and Behera, *Linearizations for rational matrix functions and Rosenbrock system polynomials*, SIMAX 2016 and followed by other students of Alam.

Some comments on linearizations of rational matrices (II)

- In simple words, a linearization $L(\lambda)$ of a rational matrix $R(\lambda)$ is a matrix pencil (i.e., a matrix polynomial of degree 1) whose eigenvalues are the finite zeros of $R(\lambda)$ and such that the eigenvalues of a certain square and nonsingular submatrix of $L(\lambda)$ are the finite poles of $R(\lambda)$,
- with equal partial multiplicities for zeros and poles.
- If $L(\lambda)$ contains also the pole-zero information of $R(\lambda)$ at infinity, then it is said to be a strong linearization of $R(\lambda)$.
- There are many families of strong linearizations of polynomial and rational matrices.
- The family considered in this talk includes (modulo permutations) many of the linearizations available in the literature.

Some comments on linearizations of rational matrices (II)

- In simple words, a linearization $L(\lambda)$ of a rational matrix $R(\lambda)$ is a matrix pencil (i.e., a matrix polynomial of degree 1) whose eigenvalues are the finite zeros of $R(\lambda)$ and such that the eigenvalues of a certain square and nonsingular submatrix of $L(\lambda)$ are the finite poles of $R(\lambda)$,
- with equal partial multiplicities for zeros and poles.
- If $L(\lambda)$ contains also the pole-zero information of $R(\lambda)$ at infinity, then it is said to be a strong linearization of $R(\lambda)$.
- There are many families of strong linearizations of polynomial and rational matrices.
- The family considered in this talk includes (modulo permutations) many of the linearizations available in the literature.

Some comments on linearizations of rational matrices (II)

- In simple words, a linearization $L(\lambda)$ of a rational matrix $R(\lambda)$ is a matrix pencil (i.e., a matrix polynomial of degree 1) whose eigenvalues are the finite zeros of $R(\lambda)$ and such that the eigenvalues of a certain square and nonsingular submatrix of $L(\lambda)$ are the finite poles of $R(\lambda)$,
- with equal partial multiplicities for zeros and poles.
- If $L(\lambda)$ contains also the pole-zero information of $R(\lambda)$ at infinity, then it is said to be a strong linearization of $R(\lambda)$.
- There are many families of strong linearizations of polynomial and rational matrices.
- The family considered in this talk includes (modulo permutations) many of the linearizations available in the literature.

Some comments on linearizations of rational matrices (II)

- In simple words, a linearization $L(\lambda)$ of a rational matrix $R(\lambda)$ is a matrix pencil (i.e., a matrix polynomial of degree 1) whose eigenvalues are the finite zeros of $R(\lambda)$ and such that the eigenvalues of a certain square and nonsingular submatrix of $L(\lambda)$ are the finite poles of $R(\lambda)$,
- with equal partial multiplicities for zeros and poles.
- If $L(\lambda)$ contains also the pole-zero information of $R(\lambda)$ at infinity, then it is said to be a strong linearization of $R(\lambda)$.
- There are many families of strong linearizations of polynomial and rational matrices.
- The family considered in this talk includes (modulo permutations) many of the linearizations available in the literature.

Some comments on linearizations of rational matrices (II)

- In simple words, a linearization $L(\lambda)$ of a rational matrix $R(\lambda)$ is a matrix pencil (i.e., a matrix polynomial of degree 1) whose eigenvalues are the finite zeros of $R(\lambda)$ and such that the eigenvalues of a certain square and nonsingular submatrix of $L(\lambda)$ are the finite poles of $R(\lambda)$,
- with equal partial multiplicities for zeros and poles.
- If $L(\lambda)$ contains also the pole-zero information of $R(\lambda)$ at infinity, then it is said to be a strong linearization of $R(\lambda)$.
- There are many families of strong linearizations of polynomial and rational matrices.
- The family considered in this talk includes (modulo permutations) many of the linearizations available in the literature.

Two fundamental auxiliary polynomial matrices in the rest of the talk are

$$L_k(\lambda) := \begin{bmatrix} -1 & \lambda & & & \\ & -1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & -1 & \lambda \end{bmatrix} \in \mathbb{C}[\lambda]^{k\times(k+1)},$$

$$\Lambda_k(\lambda)^T := \begin{bmatrix} \lambda^k & \lambda^{k-1} & \cdots & \lambda & 1 \end{bmatrix} \in \mathbb{C}[\lambda]^{1\times(k+1)},$$

and their Kronecker products by identities

$$L_k(\lambda)\otimes I_n := \begin{bmatrix} -I_n & \lambda I_n & & & \\ & -I_n & \lambda I_n & & \\ & & \ddots & \ddots & \\ & & & -I_n & \lambda I_n \end{bmatrix} \in \mathbb{C}[\lambda]^{nk\times n(k+1)},$$

$$\Lambda_k(\lambda)^T\otimes I_n := \begin{bmatrix} \lambda^k I_n & \lambda^{k-1} I_n & \cdots & \lambda I_n & I_n \end{bmatrix} \in \mathbb{C}[\lambda]^{n\times n(k+1)}.$$

Auxiliary polynomial matrices

Two fundamental auxiliary polynomial matrices in the rest of the talk are

$$L_k(\lambda) := \begin{bmatrix} -1 & \lambda & & & \\ & -1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & -1 & \lambda \end{bmatrix} \in \mathbb{C}[\lambda]^{k\times(k+1)},$$

$$\Lambda_k(\lambda)^T := \begin{bmatrix} \lambda^k & \lambda^{k-1} & \cdots & \lambda & 1 \end{bmatrix} \in \mathbb{C}[\lambda]^{1\times(k+1)},$$

and their Kronecker products by identities

$$L_k(\lambda)\otimes I_n := \begin{bmatrix} -I_n & \lambda I_n & & & \\ & -I_n & \lambda I_n & & \\ & & \ddots & \ddots & \\ & & & -I_n & \lambda I_n \end{bmatrix} \in \mathbb{C}[\lambda]^{nk\times n(k+1)},$$

$$\Lambda_k(\lambda)^T\otimes I_n := \begin{bmatrix} \lambda^k I_n & \lambda^{k-1} I_n & \cdots & \lambda I_n & I_n \end{bmatrix} \in \mathbb{C}[\lambda]^{n\times n(k+1)}.$$

We have seen one of these matrices before

in the Frobenius companion form of the $m \times n$ matrix polynomial $P(\lambda) = P_d \lambda^d + \cdots + P_1 \lambda + P_0$

$$C_1(\lambda) := \begin{bmatrix} \lambda P_d + P_{d-1} & P_{d-2} & \cdots & P_1 & P_0 \\ -I_n & \lambda I_n & & & \\ & \ddots & \ddots & & \\ & & \ddots & \lambda I_n & \\ & & & -I_n & \lambda I_n \end{bmatrix},$$

which can be compactly written with the polynomials defined above as

$$C_1(\lambda) := \left[\begin{array}{c} \lambda P_d + P_{d-1} \quad P_{d-2} \quad \cdots \quad P_1 \quad P_0 \\ \hline L_{d-1}(\lambda) \otimes I_n \end{array} \right].$$

Observe also that

$$P(\lambda) = \begin{bmatrix} \lambda P_d + P_{d-1} & P_{d-2} & \cdots & P_1 & P_0 \end{bmatrix} (\Lambda_{d-1}(\lambda) \otimes I_n).$$

We have seen one of these matrices before

in the Frobenius companion form of the $m \times n$ matrix polynomial $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0$

$$C_1(\lambda) := \begin{bmatrix} \lambda P_d + P_{d-1} & P_{d-2} & \cdots & P_1 & P_0 \\ -I_n & \lambda I_n & & & \\ & \ddots & \ddots & & \\ & & \ddots & \lambda I_n & \\ & & & -I_n & \lambda I_n \end{bmatrix},$$

which can be compactly written with the polynomials defined above as

$$C_1(\lambda) := \left[\begin{array}{c} \lambda P_d + P_{d-1} \quad P_{d-2} \quad \cdots \quad P_1 \quad P_0 \\ \hline L_{d-1}(\lambda) \otimes I_n \end{array} \right].$$

Observe also that

$$P(\lambda) = \begin{bmatrix} \lambda P_d + P_{d-1} & P_{d-2} & \cdots & P_1 & P_0 \end{bmatrix} (\Lambda_{d-1}(\lambda) \otimes I_n).$$

We have seen one of these matrices before

in the Frobenius companion form of the $m \times n$ matrix polynomial $P(\lambda) = P_d \lambda^d + \cdots + P_1 \lambda + P_0$

$$C_1(\lambda) := \begin{bmatrix} \lambda P_d + P_{d-1} & P_{d-2} & \cdots & P_1 & P_0 \\ -I_n & \lambda I_n & & & \\ & \ddots & \ddots & & \\ & & \ddots & \lambda I_n & \\ & & & -I_n & \lambda I_n \end{bmatrix},$$

which can be compactly written with the polynomials defined above as

$$C_1(\lambda) := \left[\begin{array}{c} \lambda P_d + P_{d-1} \quad P_{d-2} \quad \cdots \quad P_1 \quad P_0 \\ \hline L_{d-1}(\lambda) \otimes I_n \end{array} \right].$$

Observe also that

$$P(\lambda) = \begin{bmatrix} \lambda P_d + P_{d-1} & P_{d-2} & \cdots & P_1 & P_0 \end{bmatrix} (\Lambda_{d-1}(\lambda) \otimes I_n).$$

Definition (D, Lawrence, Pérez, Van Dooren, Numer. Math., 2018)

Let $M(\lambda)$ be an arbitrary pencil. Then any pencil of the form

$$\mathcal{L}(\lambda) = \begin{array}{c} \left[\begin{array}{c|c} M(\lambda) & L_\eta(\lambda)^T \otimes I_m \\ \hline L_\varepsilon(\lambda) \otimes I_n & 0 \end{array}\right] \\ \underbrace{\hphantom{L_\varepsilon(\lambda) \otimes I_n}}_{(\varepsilon+1)n} \quad \underbrace{\hphantom{L_\eta(\lambda)^T \otimes I_m}}_{\eta m} \end{array} \begin{array}{l} \}\,(\eta+1)m \\ \}\,\varepsilon n \end{array} ,$$

is called a block Kronecker pencil (one-block row and column cases included).

Theorem (D, Lawrence, Pérez, Van Dooren, Numer. Math., 2018)

Any block Kronecker pencil $\mathcal{L}(\lambda)$ is a strong linearization of the matrix polynomial

$$Q(\lambda) := (\Lambda_\eta(\lambda)^T \otimes I_m) M(\lambda) (\Lambda_\varepsilon(\lambda) \otimes I_n) \in \mathbb{C}[\lambda]^{m\times n},$$

the right minimal indices of $\mathcal{L}(\lambda)$ are those of $Q(\lambda)$ plus ε, and the left minimal indices of $\mathcal{L}(\lambda)$ are those of $Q(\lambda)$ plus η.

Definition (D, Lawrence, Pérez, Van Dooren, Numer. Math., 2018)

Let $M(\lambda)$ be an arbitrary pencil. Then any pencil of the form

$$\mathcal{L}(\lambda) = \begin{array}{c} \left[\begin{array}{c|c} M(\lambda) & L_\eta(\lambda)^T \otimes I_m \\ \hline L_\varepsilon(\lambda) \otimes I_n & 0 \end{array}\right] \\ \begin{array}{cc} \underbrace{\hphantom{L_\varepsilon(\lambda) \otimes I_n}}_{(\varepsilon+1)n} & \underbrace{\hphantom{L_\eta(\lambda)^T \otimes I_m}}_{\eta m} \end{array} \end{array} \begin{array}{l} \}\,{\scriptstyle (\eta+1)m} \\ \}\,{\scriptstyle \varepsilon n} \\ \end{array},$$

is called a block Kronecker pencil (one-block row and column cases included).

Theorem (D, Lawrence, Pérez, Van Dooren, Numer. Math., 2018)

Any block Kronecker pencil $\mathcal{L}(\lambda)$ is a strong linearization of the matrix polynomial

$$Q(\lambda) := (\Lambda_\eta(\lambda)^T \otimes I_m) M(\lambda) (\Lambda_\varepsilon(\lambda) \otimes I_n) \in \mathbb{C}[\lambda]^{m \times n},$$

the right minimal indices of $\mathcal{L}(\lambda)$ are those of $Q(\lambda)$ plus ε, and the left minimal indices of $\mathcal{L}(\lambda)$ are those of $Q(\lambda)$ plus η.

Examples of block Kronecker linearizations of polynomial matrices (I)

(apart from the Frobenius companion form!!!)

$$P(\lambda) = \lambda^5 P_5 + \lambda^4 P_4 + \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0 \in \mathbb{C}[\lambda]^{m \times n}$$

$$\left[\begin{array}{ccc|cc} \lambda P_5 + P_4 & 0 & 0 & -I_m & 0 \\ 0 & \lambda P_3 + P_2 & 0 & \lambda I_m & -I_m \\ 0 & 0 & \lambda P_1 + P_0 & 0 & \lambda I_m \\ \hline -I_n & \lambda I_n & 0 & 0 & 0 \\ 0 & -I_n & \lambda I_n & 0 & 0 \end{array}\right]$$

Examples of block Kronecker linerizations of polynomial matrices (II)

$$P(\lambda) = \lambda^5 P_5 + \lambda^4 P_4 + \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0 \in \mathbb{C}[\lambda]^{m\times n}$$

$$\left[\begin{array}{ccc|cc} \lambda P_5 & \lambda P_4 & \lambda P_3 & -I_m & 0 \\ 0 & 0 & \lambda P_2 & \lambda I_m & -I_m \\ 0 & 0 & \lambda P_1 + P_0 & 0 & \lambda I_m \\ \hline -I_n & \lambda I_n & 0 & 0 & 0 \\ 0 & -I_n & \lambda I_n & 0 & 0 \end{array}\right]$$

Outline

- Rational matrices can be represented in different forms.
- In this talk, we consider that the rational matrix is represented as

$$R(\lambda) = R_p(\lambda) + D(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i \lambda^i \in \mathbb{C}(\lambda)^{m\times n},$$

  where the triple $\{A, B, C\}$ is a minimal state-space realization of the strictly proper part $R_p(\lambda)$, and d is the degree of the polynomial part.
- This minimality means that $\begin{bmatrix} \lambda_0 I_\ell - A \\ C \end{bmatrix}$ and $\begin{bmatrix} \lambda_0 I_\ell - A & B \end{bmatrix}$ have full column and row ranks, respectively, for any $\lambda_0 \in \mathbb{C}$.
- **Any rational matrix can be represented in this form, which is one of the most standard representations in linear systems theory.**

- Rational matrices can be represented in different forms.
- In this talk, we consider that the rational matrix is represented as

$$R(\lambda) = R_p(\lambda) + D(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i \lambda^i \in \mathbb{C}(\lambda)^{m\times n},$$

where the triple $\{A, B, C\}$ is a minimal state-space realization of the strictly proper part $R_p(\lambda)$, and d is the degree of the polynomial part.

- This minimality means that $\begin{bmatrix} \lambda_0 I_\ell - A \\ C \end{bmatrix}$ and $\begin{bmatrix} \lambda_0 I_\ell - A & B \end{bmatrix}$ have full column and row ranks, respectively, for any $\lambda_0 \in \mathbb{C}$.
- **Any rational matrix can be represented in this form, which is one of the most standard representations in linear systems theory.**

- Rational matrices can be represented in different forms.
- In this talk, we consider that the rational matrix is represented as

$$R(\lambda) = R_p(\lambda) + D(\lambda) = C(\lambda I_\ell - A)^{-1} B + \sum_{i=0}^{d} D_i \lambda^i \in \mathbb{C}(\lambda)^{m \times n},$$

where the triple $\{A, B, C\}$ is a minimal state-space realization of the strictly proper part $R_p(\lambda)$, and d is the degree of the polynomial part.

- This minimality means that $\begin{bmatrix} \lambda_0 I_\ell - A \\ C \end{bmatrix}$ and $\begin{bmatrix} \lambda_0 I_\ell - A & B \end{bmatrix}$ have full column and row ranks, respectively, for any $\lambda_0 \in \mathbb{C}$.
- **Any rational matrix can be represented in this form, which is one of the most standard representations in linear systems theory.**

- Rational matrices can be represented in different forms.
- In this talk, we consider that the rational matrix is represented as

$$\boxed{R(\lambda) = R_p(\lambda) + D(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i \lambda^i \in \mathbb{C}(\lambda)^{m\times n}},$$

where the triple $\{A, B, C\}$ is a minimal state-space realization of the strictly proper part $R_p(\lambda)$, and d is the degree of the polynomial part.

- This minimality means that $\begin{bmatrix} \lambda_0 I_\ell - A \\ C \end{bmatrix}$ and $\begin{bmatrix} \lambda_0 I_\ell - A & B \end{bmatrix}$ have full column and row ranks, respectively, for any $\lambda_0 \in \mathbb{C}$.
- **Any rational matrix can be represented in this form, which is one of the most standard representations in linear systems theory**.

This representation captures many rational matrices coming from NLEPs

- Loaded elastic string (Betcke et al., NLEVP, (2013); Solov'ëv (2006)):

$$R(\lambda) = A - \lambda B + \frac{\lambda}{\lambda - \sigma} E = (A + E) - \lambda B + \frac{\sigma}{\lambda - \sigma} E \in \mathbb{C}(\lambda)^{n \times n}.$$

- Damped vibration of a viscoelastic structure (Mehrmann & Voss, (2004)):

$$R(\lambda) = \lambda^2 M + K - \sum_{i=1}^{k} \frac{1}{1 + b_i \lambda} \Delta G_i \in \mathbb{C}(\lambda)^{n \times n}.$$

- El-Guide, Miedlar, Saad (2020) consider for approximating some NLEPs

$$R(\lambda) = -B_0 + \lambda A_0 + \frac{B_1}{\lambda - \sigma_1} + \cdots + \frac{B_s}{\lambda - \sigma_s} \in \mathbb{C}(\lambda)^{p \times p},$$

$$= -B_0 + \lambda A_0 + \begin{bmatrix} B_1 & \cdots & B_s \end{bmatrix} \begin{bmatrix} (\lambda - \sigma_1) I_p & & \\ & \ddots & \\ & & (\lambda - \sigma_s) I_p \end{bmatrix}^{-1} \begin{bmatrix} I_p \\ \vdots \\ I_p \end{bmatrix}$$

This representation captures many rational matrices coming from NLEPs

- Loaded elastic string (Betcke et al., NLEVP, (2013); Solov'ëv (2006)):

$$R(\lambda) = A - \lambda B + \frac{\lambda}{\lambda - \sigma} E = (A + E) - \lambda B + \frac{\sigma}{\lambda - \sigma} E \in \mathbb{C}(\lambda)^{n \times n}.$$

- Damped vibration of a viscoelastic structure (Mehrmann & Voss, (2004)):

$$R(\lambda) = \lambda^2 M + K - \sum_{i=1}^{k} \frac{1}{1 + b_i \lambda} \Delta G_i \in \mathbb{C}(\lambda)^{n \times n}.$$

- El-Guide, Miedlar, Saad (2020) consider for approximating some NLEPs

$$R(\lambda) = -B_0 + \lambda A_0 + \frac{B_1}{\lambda - \sigma_1} + \cdots + \frac{B_s}{\lambda - \sigma_s} \in \mathbb{C}(\lambda)^{p \times p},$$

$$= -B_0 + \lambda A_0 + \begin{bmatrix} B_1 & \cdots & B_s \end{bmatrix} \begin{bmatrix} (\lambda - \sigma_1) I_p & & \\ & \ddots & \\ & & (\lambda - \sigma_s) I_p \end{bmatrix}^{-1} \begin{bmatrix} I_p \\ \vdots \\ I_p \end{bmatrix}$$

This representation captures many rational matrices coming from NLEPs

- Loaded elastic string (Betcke et al., NLEVP, (2013); Solov'ëv (2006)):

$$R(\lambda) = A - \lambda B + \frac{\lambda}{\lambda - \sigma} E = (A + E) - \lambda B + \frac{\sigma}{\lambda - \sigma} E \in \mathbb{C}(\lambda)^{n\times n}.$$

- Damped vibration of a viscoelastic structure (Mehrmann & Voss, (2004)):

$$R(\lambda) = \lambda^2 M + K - \sum_{i=1}^{k} \frac{1}{1 + b_i\lambda} \Delta G_i \in \mathbb{C}(\lambda)^{n\times n}.$$

- El-Guide, Miedlar, Saad (2020) consider for approximating some NLEPs

$$\begin{aligned} R(\lambda) &= -B_0 + \lambda A_0 + \frac{B_1}{\lambda - \sigma_1} + \cdots + \frac{B_s}{\lambda - \sigma_s} \in \mathbb{C}(\lambda)^{p\times p}, \\ &= -B_0 + \lambda A_0 + \begin{bmatrix} B_1 & \cdots & B_s \end{bmatrix} \begin{bmatrix} (\lambda - \sigma_1) I_p & & \\ & \ddots & \\ & & (\lambda - \sigma_s) I_p \end{bmatrix}^{-1} \begin{bmatrix} I_p \\ \vdots \\ I_p \end{bmatrix} \end{aligned}$$

This representation captures many rational matrices coming from NLEPs

- Loaded elastic string (Betcke et al., NLEVP, (2013); Solov'ëv (2006)):

$$R(\lambda) = A - \lambda B + \frac{\lambda}{\lambda - \sigma} E = (A + E) - \lambda B + \frac{\sigma}{\lambda - \sigma} E \in \mathbb{C}(\lambda)^{n \times n}.$$

- Damped vibration of a viscoelastic structure (Mehrmann & Voss, (2004)):

$$R(\lambda) = \lambda^2 M + K - \sum_{i=1}^{k} \frac{1}{1 + b_i \lambda} \Delta G_i \in \mathbb{C}(\lambda)^{n \times n}.$$

- El-Guide, Miedlar, Saad (2020) consider for approximating some NLEPs

$$R(\lambda) = -B_0 + \lambda A_0 + \frac{B_1}{\lambda - \sigma_1} + \cdots + \frac{B_s}{\lambda - \sigma_s} \in \mathbb{C}(\lambda)^{p \times p},$$

$$= -B_0 + \lambda A_0 + \begin{bmatrix} B_1 & \cdots & B_s \end{bmatrix} \begin{bmatrix} (\lambda - \sigma_1) I_p & & \\ & \ddots & \\ & & (\lambda - \sigma_s) I_p \end{bmatrix}^{-1} \begin{bmatrix} I_p \\ \vdots \\ I_p \end{bmatrix}$$

... but does not capture all interesting cases

- NLEIGS-Rational Eigenvalue Problems (REPs) coming from linear rational interpolation of NLEPs (Güttel, Van Beeumen, Meerbergen, Michiels (2014)).
- REPs appearing in "Automatic rational approximation and linearization of nonlinear eigenvalue problems" (2022) by Lietaert, Meerbergen, Pérez, Vandereycken.

- NLEIGS-Rational Eigenvalue Problems (REPs) coming from linear rational interpolation of NLEPs (Güttel, Van Beeumen, Meerbergen, Michiels (2014)).
- REPs appearing in "Automatic rational approximation and linearization of nonlinear eigenvalue problems" (2022) by Lietaert, Meerbergen, Pérez, Vandereycken.

More auxiliary pencils

We have used so far

$$L_k(\lambda) := \begin{bmatrix} 1 & -\lambda & & & \\ & 1 & -\lambda & & \\ & & \ddots & \ddots & \\ & & & 1 & -\lambda \end{bmatrix} \in \mathbb{C}[\lambda]^{k\times(k+1)},$$

$$\Lambda_k(\lambda)^T := \begin{bmatrix} \lambda^k & \lambda^{k-1} & \cdots & \lambda & 1 \end{bmatrix} \in \mathbb{C}[\lambda]^{1\times(k+1)}.$$

and their Kronecker products by identities

$$K_1(\lambda) := L_\epsilon(\lambda) \otimes I_n,$$
$$K_2(\lambda) := L_\eta(\lambda) \otimes I_m.$$

Now, we introduce

$$\widehat{K}_1 := \mathbf{e}_{\epsilon+1}^T \otimes I_n = \underbrace{\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}}_{\epsilon+1} \otimes I_n,$$

$$\widehat{K}_2 := \mathbf{e}_{\eta+1}^T \otimes I_m = \underbrace{\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}}_{\eta+1} \otimes I_m.$$

More auxiliary pencils

We have used so far

$$L_k(\lambda) := \begin{bmatrix} 1 & -\lambda & & & \\ & 1 & -\lambda & & \\ & & \ddots & \ddots & \\ & & & 1 & -\lambda \end{bmatrix} \in \mathbb{C}[\lambda]^{k\times(k+1)},$$

$$\Lambda_k(\lambda)^T := \begin{bmatrix} \lambda^k & \lambda^{k-1} & \cdots & \lambda & 1 \end{bmatrix} \in \mathbb{C}[\lambda]^{1\times(k+1)}.$$

and their Kronecker products by identities

$$K_1(\lambda) := L_\epsilon(\lambda) \otimes I_n,$$
$$K_2(\lambda) := L_\eta(\lambda) \otimes I_m.$$

Now, we introduce

$$\widehat{K}_1 := \mathbf{e}_{\epsilon+1}^T \otimes I_n = \underbrace{\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}}_{\epsilon+1} \otimes I_n,$$

$$\widehat{K}_2 := \mathbf{e}_{\eta+1}^T \otimes I_m = \underbrace{\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}}_{\eta+1} \otimes I_m.$$

More auxiliary pencils

We have used so far

$$L_k(\lambda) := \begin{bmatrix} 1 & -\lambda & & & \\ & 1 & -\lambda & & \\ & & \ddots & \ddots & \\ & & & 1 & -\lambda \end{bmatrix} \in \mathbb{C}[\lambda]^{k\times(k+1)},$$

$$\Lambda_k(\lambda)^T := \begin{bmatrix} \lambda^k & \lambda^{k-1} & \cdots & \lambda & 1 \end{bmatrix} \in \mathbb{C}[\lambda]^{1\times(k+1)}.$$

and their Kronecker products by identities

$$K_1(\lambda) := L_\epsilon(\lambda) \otimes I_n,$$
$$K_2(\lambda) := L_\eta(\lambda) \otimes I_m.$$

Now, we introduce

$$\widehat{K}_1 := \mathbf{e}_{\epsilon+1}^T \otimes I_n = \underbrace{\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}}_{\epsilon+1} \otimes I_n,$$

$$\widehat{K}_2 := \mathbf{e}_{\eta+1}^T \otimes I_m = \underbrace{\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}}_{\eta+1} \otimes I_m\,.$$

Theorem (Amparan, D, Marcaida, Zaballa, SIMAX, 2018)

Let

- $A \in \mathbb{C}^{\ell \times \ell}$, $B \in \mathbb{C}^{\ell \times n}$, $C \in \mathbb{C}^{m \times \ell}$ *be arbitrary constant matrices and* $M(\lambda)$ *be an arbitrary pencil of adequate size, and*
- $K_1(\lambda)$, $K_2(\lambda)$, $\widehat{K}_1$, $\widehat{K}_2$ *be the pencils and matrices in the previous slide.*

Let us consider the pencil

$$S(\lambda) = \left[\begin{array}{c|cc} A - \lambda I_\ell & B\widehat{K}_1 & 0 \\ \hline \widehat{K}_2^T C & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array} \right],$$

and the rational matrix

$$R(\lambda) = \underbrace{(\Lambda_\eta(\lambda)^T \otimes I_m) M(\lambda) (\Lambda_\epsilon(\lambda) \otimes I_n)}_{D(\lambda) := \text{poly. part}} + \underbrace{C(\lambda I_\ell - A)^{-1} B}_{\text{strict. proper. part}}.$$

If $\{A, B, C\}$ *is a minimal state-space realization,* **then** $S(\lambda)$ **is a strong linearization of** $R(\lambda)$.

Theorem (Amparan, D, Marcaida, Zaballa, SIMAX, 2018)

Let

- $A \in \mathbb{C}^{\ell \times \ell}$, $B \in \mathbb{C}^{\ell \times n}$, $C \in \mathbb{C}^{m \times \ell}$ *be arbitrary constant matrices and* $M(\lambda)$ *be an arbitrary pencil of adequate size, and*
- $K_1(\lambda)$, $K_2(\lambda)$, $\widehat{K}_1$, $\widehat{K}_2$ *be the pencils and matrices in the previous slide.*

Let us consider the pencil

$$S(\lambda) = \left[\begin{array}{c|cc} A - \lambda I_\ell & B\widehat{K}_1 & 0 \\ \hline \widehat{K}_2^T C & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array}\right],$$

and the rational matrix

$$R(\lambda) = \underbrace{(\Lambda_\eta(\lambda)^T \otimes I_m) M(\lambda) (\Lambda_\epsilon(\lambda) \otimes I_n)}_{D(\lambda) := \text{poly. part}} + \underbrace{C(\lambda I_\ell - A)^{-1} B}_{\text{strict. proper. part}}.$$

If $\{A, B, C\}$ *is a minimal state-space realization,* **then $S(\lambda)$ is a strong linearization of $R(\lambda)$.**

Theorem (Amparan, D, Marcaida, Zaballa, SIMAX, 2018)

Let

- $A \in \mathbb{C}^{\ell\times\ell}$, $B \in \mathbb{C}^{\ell\times n}$, $C \in \mathbb{C}^{m\times\ell}$ *be arbitrary constant matrices and* $M(\lambda)$ *be an arbitrary pencil of adequate size, and*
- $K_1(\lambda)$, $K_2(\lambda)$, $\widehat{K}_1$, $\widehat{K}_2$ *be the pencils and matrices in the previous slide.*

Let us consider the pencil

$$S(\lambda) = \left[\begin{array}{c|cc} A-\lambda I_\ell & B\widehat{K}_1 & 0 \\ \hline \widehat{K}_2^T C & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array}\right] \to \begin{bmatrix} M(\lambda) & \widehat{K}_2^T C & K_2(\lambda)^T \\ B\widehat{K}_1 & A-\lambda I_\ell & 0 \\ K_1(\lambda) & 0 & 0 \end{bmatrix},$$

and the rational matrix

$$R(\lambda) = \underbrace{(\Lambda_\eta(\lambda)^T \otimes I_m) M(\lambda) (\Lambda_\epsilon(\lambda) \otimes I_n)}_{D(\lambda) := \text{poly. part}} + \underbrace{C(\lambda I_\ell - A)^{-1} B}_{\text{strict. proper. part}}.$$

If $\{A, B, C\}$ *is a minimal state-space realization,* **then $S(\lambda)$ is a strong linearization of $R(\lambda)$.**

Example of block Kronecker lin: Van Dooren & De Wilde, LAA, (1983)

- Given the rational matrix:

$$R(\lambda) = C(\lambda I_\ell - A)^{-1} B + \sum_{i=0}^{d} D_i \lambda^i \in \mathbb{C}(\lambda)^{m\times n},$$

where the triple $\{A, B, C\}$ is a minimal state-space realization,

- the following pencil

$$L(\lambda) = \left[\begin{array}{c|ccccc} A-\lambda I_\ell & & & & & B \\ \hline & I_n & -\lambda I_n & & & \\ & & I_n & \ddots & & \\ & & & \ddots & -\lambda I_n & \\ & & & & I_n & -\lambda I_n \\ C & \lambda D_d & \dots & \dots & \lambda D_2 & \lambda D_1 + D_0 \end{array}\right]$$

- is a (permuted) Block Kronecker (strong) linearization of $R(\lambda)$.

- Given the rational matrix:

$$R(\lambda) = C(\lambda I_\ell - A)^{-1} B + \sum_{i=0}^{d} D_i \lambda^i \in \mathbb{C}(\lambda)^{m\times n},$$

where the triple $\{A, B, C\}$ is a minimal state-space realization,

- the following pencil

$$L(\lambda) = \left[\begin{array}{c|ccccc} A - \lambda I_\ell & & & & & B \\ \hline & I_n & -\lambda I_n & & & \\ & & I_n & \ddots & & \\ & & & \ddots & -\lambda I_n & \\ & & & & I_n & -\lambda I_n \\ C & \lambda D_d & \dots & \dots & \lambda D_2 & \lambda D_1 + D_0 \end{array}\right]$$

- is a (permuted) Block Kronecker (strong) linearization of $R(\lambda)$.

- Given the rational matrix:

$$R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i \lambda^i \in \mathbb{C}(\lambda)^{m\times n},$$

where the triple $\{A, B, C\}$ is a minimal state-space realization,

- the following pencil

$$L(\lambda) = \left[\begin{array}{c|cccccc} A-\lambda I_\ell & & & & & B \\ \hline C & \lambda D_d + D_{d-1} & D_{d-2} & \cdots & D_1 & D_0 \\ & I_n & -\lambda I_n & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & -\lambda I_n & \\ & & & & I_n & -\lambda I_n \end{array}\right],$$

- is a Block Kronecker (strong) linearization of $R(\lambda)$.

Example of block Kronecker lin: Su & Bai, SIMAX, (2011)

- Given the rational matrix:

$$R(\lambda) = C(\lambda I_\ell - A)^{-1} B + \sum_{i=0}^{d} D_i \lambda^i \in \mathbb{C}(\lambda)^{m\times n},$$

where the triple $\{A, B, C\}$ is a minimal state-space realization,

- the following pencil

$$L(\lambda) = \left[\begin{array}{c|cccccc} A-\lambda I_\ell & & & & & B \\ \hline C & \lambda D_d + D_{d-1} & D_{d-2} & \cdots & D_1 & D_0 \\ & I_n & -\lambda I_n & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & -\lambda I_n & \\ & & & & I_n & -\lambda I_n \end{array}\right],$$

- is a Block Kronecker (strong) linearization of $R(\lambda)$.

- Given the rational matrix:

$$R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i \lambda^i \in \mathbb{C}(\lambda)^{m\times n},$$

where the triple $\{A, B, C\}$ is a minimal state-space realization,

- the following pencil

$$L(\lambda) = \left[\begin{array}{c|cccccc} A-\lambda I_\ell & & & & & & B \\ \hline C & \lambda D_d + D_{d-1} & D_{d-2} & \cdots & & D_1 & D_0 \\ & I_n & -\lambda I_n & & & & \\ & & \ddots & \ddots & & & \\ & & & \ddots & & -\lambda I_n & \\ & & & & & I_n & -\lambda I_n \end{array}\right],$$

- is a Block Kronecker (strong) linearization of $R(\lambda)$.

Example of block Kronecker lin: Amparan, D, Marcaida, Zaballa, SIMAX, (2018)

- Given the rational matrix:

$$R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{5} D_i \lambda^i \in \mathbb{C}(\lambda)^{m\times n},$$

where the triple $\{A, B, C\}$ is a minimal state-space realization,

- the following "symmetrizable" pencil

$$L(\lambda) = \left[\begin{array}{c|ccc|cc} A-\lambda I_\ell & 0 & 0 & B & 0 & 0 \\ \hline 0 & \lambda D_5 + D_4 & 0 & 0 & I_m & 0 \\ 0 & 0 & \lambda D_3 + D_2 & 0 & -\lambda I_m & I_m \\ C & 0 & 0 & \lambda D_1 + D_0 & 0 & -\lambda I_m \\ \hline 0 & I_n & -\lambda I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & -\lambda I_n & 0 & 0 \end{array}\right]$$

- is a Block Kronecker (strong) linearization of $R(\lambda)$.

Example of block Kronecker lin: Amparan, D, Marcaida, Zaballa, SIMAX, (2018)

- Given the rational matrix:

$$R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{5} D_i \lambda^i \in \mathbb{C}(\lambda)^{m\times n},$$

where the triple $\{A, B, C\}$ is a minimal state-space realization,

- the following "symmetrizable" pencil

$$L(\lambda) = \left[\begin{array}{c|ccc|cc} A-\lambda I_\ell & 0 & 0 & B & 0 & 0 \\ \hline 0 & \lambda D_5 + D_4 & 0 & 0 & I_m & 0 \\ 0 & 0 & \lambda D_3 + D_2 & 0 & -\lambda I_m & I_m \\ C & 0 & 0 & \lambda D_1 + D_0 & 0 & -\lambda I_m \\ \hline 0 & I_n & -\lambda I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & -\lambda I_n & 0 & 0 \end{array}\right]$$

- is a Block Kronecker (strong) linearization of $R(\lambda)$.

Example of block Kronecker lin: Amparan, D, Marcaida, Zaballa, SIMAX, (2018)

- Given the rational matrix:

$$R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{5} D_i \lambda^i \in \mathbb{C}(\lambda)^{m\times n},$$

  where the triple $\{A, B, C\}$ is a minimal state-space realization,
- the following "symmetrizable" pencil

$$L(\lambda) = \left[\begin{array}{c|ccc|cc} A-\lambda I_\ell & 0 & 0 & B & 0 & 0 \\ \hline 0 & \lambda D_5 + D_4 & 0 & 0 & I_m & 0 \\ 0 & 0 & \lambda D_3 + D_2 & 0 & -\lambda I_m & I_m \\ C & 0 & 0 & \lambda D_1 + D_0 & 0 & -\lambda I_m \\ \hline 0 & I_n & -\lambda I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & -\lambda I_n & 0 & 0 \end{array}\right]$$

- is a Block Kronecker (strong) linearization of $R(\lambda)$.

Outline

1. Brief reminder of “Eigenstructures” of PEPs and REPs
2. Linearizations of polynomial and rational matrices
3. Block Kronecker linearizations of polynomial matrices
4. Block Kronecker linearizations of rational matrices
5. **Global backward stability of PEPs solved with linearizations**
6. Global backward stability of REPs solved with linearizations
7. Conclusions

The Setting

- We consider a **general** $m \times n$ **polynomial matrix** of degree d

$$\boxed{P(\lambda) = P_d \lambda^d + \cdots + P_1 \lambda + P_0}, \qquad P_i \in \mathbb{C}^{m \times n},$$

- and we assume that its **complete eigenstructure**
- has been computed by applying a **backward stable algorithm** **(QZ for regular, Staircase for singular)**
- to a **strong linearization** $\mathcal{L}(\lambda)$ **in the class of block Kronecker linearizations** of $P(\lambda)$.

- We consider a **general** $m \times n$ **polynomial matrix** of degree d

$$\boxed{P(\lambda) = P_d \lambda^d + \cdots + P_1 \lambda + P_0}, \qquad P_i \in \mathbb{C}^{m \times n},$$

- and we assume that its **complete eigenstructure**
- has been computed by applying a **backward stable algorithm** **(QZ for regular, Staircase for singular)**
- to a **strong linearization** $\mathcal{L}(\lambda)$ **in the class of block Kronecker linearizations** of $P(\lambda)$.

- We consider a **general** $m \times n$ **polynomial matrix** of degree d

$$P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0 \quad , \qquad P_i \in \mathbb{C}^{m\times n},$$

- and we assume that its **complete eigenstructure**
- has been computed by applying a **backward stable algorithm** **(QZ for regular, Staircase for singular)**
- to a **strong linearization** $\mathcal{L}(\lambda)$ **in the class of block Kronecker linearizations** of $P(\lambda)$.

- We consider a **general $m \times n$ polynomial matrix** of degree d

$$\boxed{P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0}, \qquad P_i \in \mathbb{C}^{m\times n},$$

- and we assume that its **complete eigenstructure**
- has been computed by applying a **backward stable algorithm** **(QZ for regular, Staircase for singular)**
- to a **strong linearization $\mathcal{L}(\lambda)$ in the class of block Kronecker linearizations** of $P(\lambda)$.

- *The computed complete eigenstructure of $\mathcal{L}(\lambda)$ is the exact complete eigenstructure of a matrix pencil $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ such that*

$$\frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F} = O(\mathbf{u}),$$

  where $\mathbf{u} \approx 10^{-16}$ is the unit roundoff and

- $\|\cdot\|_F$ is the Frobenius norm, i.e., for any matrix polynomial

$$\|Q_k\lambda^k + \cdots + Q_1\lambda + Q_0\|_F = \sqrt{\|Q_k\|_F^2 + \cdots + \|Q_1\|_F^2 + \|Q_0\|_F^2}.$$

- But, does this imply that the computed complete eigenstructure of $P(\lambda)$ is the exact complete eigenstructure of a polynomial matrix of the same degree $P(\lambda) + \Delta P(\lambda)$ such that

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} = O(\mathbf{u}) \text{ ??}$$

Backward stable algorithms on strong linearizations and question

- *The computed complete eigenstructure of $\mathcal{L}(\lambda)$ is the exact complete eigenstructure of a matrix pencil $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ such that*

$$\frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F} = O(\mathbf{u}),$$

where $\mathbf{u} \approx 10^{-16}$ is the unit roundoff and

- $\|\cdot\|_F$ is the Frobenius norm, i.e., for any matrix polynomial

$$\boxed{\|Q_k\lambda^k + \cdots + Q_1\lambda + Q_0\|_F = \sqrt{\|Q_k\|_F^2 + \cdots + \|Q_1\|_F^2 + \|Q_0\|_F^2}}\,.$$

- But, does this imply that the computed complete eigenstructure of $P(\lambda)$ is the exact complete eigenstructure of a polynomial matrix of the same degree $P(\lambda) + \Delta P(\lambda)$ such that

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} = O(\mathbf{u})\ ??$$

Backward stable algorithms on strong linearizations and question

- *The computed complete eigenstructure of $\mathcal{L}(\lambda)$ is the exact complete eigenstructure of a matrix pencil $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ such that*

$$\frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F} = O(\mathbf{u}),$$

  where $\mathbf{u} \approx 10^{-16}$ is the unit roundoff and

- $\|\cdot\|_F$ is the Frobenius norm, i.e., for any matrix polynomial

$$\boxed{\|Q_k\lambda^k + \cdots + Q_1\lambda + Q_0\|_F = \sqrt{\|Q_k\|_F^2 + \cdots + \|Q_1\|_F^2 + \|Q_0\|_F^2}}\ .$$

- But, does this imply that the computed complete eigenstructure of $P(\lambda)$ is the exact complete eigenstructure of a polynomial matrix of the same degree $P(\lambda) + \Delta P(\lambda)$ such that

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} = O(\mathbf{u})\ ??$$

Why is not obvious to answer this question?

because block Kronecker linearizations are highly structured pencils and perturbations destroy the structure!!

Example: The Frobenius Companion Form

$$C_1(\lambda) := \begin{bmatrix} \lambda P_d + P_{d-1} & P_{d-2} & \cdots & P_1 & P_0 \\ -I_n & \lambda I_n & & & \\ & \ddots & \ddots & & \\ & & \ddots & \lambda I_n & \\ & & & -I_n & \lambda I_n \end{bmatrix}$$

$C_1(\lambda) + \Delta\mathcal{L}(\lambda) =$

$$\begin{bmatrix} \lambda(P_d + E_{11}) + (P_{d-1} + F_{11}) & \lambda E_{12} + P_{d-2} + F_{12} & \cdots & \lambda E_{1,d-1} + P_1 + F_{1,d-1} & \cdots \\ \lambda E_{21} - I_n + F_{21} & \lambda(I_n + E_{22}) + F_{22} & \lambda E_{23} + F_{23} & & \\ \lambda E_{31} + F_{31} & \lambda E_{32} + F_{32} & \ddots & & \\ \vdots & \vdots & \ddots & \lambda(I_n + E_{d-1,d-1}) + F_{d-1,d-1} & \\ \lambda E_{d1} + F_{d1} & \lambda E_{d2} + F_{d2} & & \lambda E_{d,d-1} + F_{d,d-1} - I_n & \cdots \end{bmatrix}$$

because block Kronecker linearizations are highly structured pencils and perturbations destroy the structure!!

Example: The Frobenius Companion Form

$$
C_1(\lambda) := \begin{bmatrix} \lambda P_d + P_{d-1} & P_{d-2} & \cdots & P_1 & P_0 \\ -I_n & \lambda I_n & & & \\ & \ddots & \ddots & & \\ & & \ddots & \lambda I_n & \\ & & & -I_n & \lambda I_n \end{bmatrix}
$$

$C_1(\lambda) + \Delta\mathcal{L}(\lambda) =$

$$
\left[\begin{matrix} \lambda(P_d + E_{11}) + (P_{d-1} + F_{11}) & \lambda E_{12} + P_{d-2} + F_{12} & \cdots & \lambda E_{1,d-1} + P_1 + F_{1,d-1} & \cdots \\ \lambda E_{21} - I_n + F_{21} & \lambda(I_n + E_{22}) + F_{22} & \lambda E_{23} + F_{23} & & \\ \lambda E_{31} + F_{31} & \lambda E_{32} + F_{32} & \ddots & & \\ \vdots & \vdots & \ddots & \lambda(I_n + E_{d-1,d-1}) + F_{d-1,d-1} & \\ \lambda E_{d1} + F_{d1} & \lambda E_{d2} + F_{d2} & & \lambda E_{d,d-1} + F_{d,d-1} - I_n & \cdots \end{matrix}\right.
$$

The matrix perturbation problems to be solved

- **Problem 1:** To establish conditions on $\|\Delta\mathcal{L}(\lambda)\|_F$ such that $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ is a strong linearization for some polynomial matrix $P(\lambda) + \Delta P(\lambda)$ of degree d.
- **Problem 2:** To prove a perturbation bound

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq C_{P,\mathcal{L}} \frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F},$$

  with $C_{P,\mathcal{L}}$ a number depending on $P(\lambda)$ and $\mathcal{L}(\lambda)$.
- For those $P(\lambda)$ and $\mathcal{L}(\lambda)$ s.t. $C_{P,\mathcal{L}}$ is moderate, to use global backward stable algorithms on $\mathcal{L}(\lambda)$ gives global backward stability for $P(\lambda)$.

- **Problem 1:** To establish conditions on $\|\Delta\mathcal{L}(\lambda)\|_F$ such that $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ is a strong linearization for some polynomial matrix $P(\lambda) + \Delta P(\lambda)$ of degree d.
- **Problem 2:** To prove a perturbation bound

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq C_{P,\mathcal{L}} \frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F},$$

with $C_{P,\mathcal{L}}$ a number depending on $P(\lambda)$ and $\mathcal{L}(\lambda)$.

- For those $P(\lambda)$ and $\mathcal{L}(\lambda)$ s.t. $C_{P,\mathcal{L}}$ is moderate, to use global backward stable algorithms on $\mathcal{L}(\lambda)$ gives global backward stability for $P(\lambda)$.

The matrix perturbation problems to be solved

- **Problem 1:** To establish conditions on $\|\Delta\mathcal{L}(\lambda)\|_F$ such that $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ is a strong linearization for some polynomial matrix $P(\lambda) + \Delta P(\lambda)$ of degree d.
- **Problem 2:** To prove a perturbation bound

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq C_{P,\mathcal{L}} \, \frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F},$$

  with $C_{P,\mathcal{L}}$ a number depending on $P(\lambda)$ and $\mathcal{L}(\lambda)$.
- For those $P(\lambda)$ and $\mathcal{L}(\lambda)$ s.t. $C_{P,\mathcal{L}}$ is moderate, to use global backward stable algorithms on $\mathcal{L}(\lambda)$ gives global backward stability for $P(\lambda)$.

The main perturbation theorem for polynomial matrices

Theorem (D, Lawrence, Pérez, Van Dooren, Numer. Math., 2018)

Let $\mathcal{L}(\lambda)$ be a block Kronecker pencil for $P(\lambda) = \sum_{i=0}^{d} P_i \lambda^i \in \mathbb{C}[\lambda]^{m\times n}$, i.e.,

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M(\lambda) & L_\eta(\lambda)^T \otimes I_m \\ \hline L_\varepsilon(\lambda) \otimes I_n & 0 \end{array}\right].$$

If $\Delta\mathcal{L}(\lambda)$ is any pencil with the same size as $\mathcal{L}(\lambda)$ and such that

$$\|\Delta\mathcal{L}(\lambda)\|_F < \frac{(\sqrt{2}-1)^2}{d^{5/2}} \frac{1}{1+\|M(\lambda)\|_F},$$

then $\mathcal{L}(\lambda) + \Delta\mathcal{L}(\lambda)$ is a strong linearization of a polynomial matrix $P(\lambda) + \Delta P(\lambda)$ with grade d and such that

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq 14\, d^{5/2} \frac{\|\mathcal{L}(\lambda)\|_F}{\|P(\lambda)\|_F} \left(1 + \|M(\lambda)\|_F + \|M(\lambda)\|_F^2\right) \frac{\|\Delta\mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}.$$

Discussion of the perturbation bound for block Kronecker pencils

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M(\lambda) & L_\eta(\lambda)^T \otimes I_m \\ \hline L_\varepsilon(\lambda) \otimes I_n & 0 \end{array}\right].$$

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq \underbrace{14\, d^{5/2} \frac{\|\mathcal{L}(\lambda)\|_F}{\|P(\lambda)\|_F} \left(1 + \|M(\lambda)\|_F + \|M(\lambda)\|_F^2\right)}_{C_{P,\mathcal{L}}} \frac{\|\Delta \mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}.$$

- It can be proved that if $\|P(\lambda)\|_F \ll 1$ or $\|P(\lambda)\|_F \gg 1$, then $C_{P,\mathcal{L}} \gg 1$,
- and that, if $\|M(\lambda)\|_F \gg 1$, then $C_{P,\mathcal{L}} \gg 1$.
- Therefore, **for getting "backward stability" from Block Kronecker linearizations**, one needs to normalize the matrix poly $\|P(\lambda)\|_F = 1$ and to use pencils such that $\|M(\lambda)\|_F \approx \|P(\lambda)\|_F$, then

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \lesssim d^3 \sqrt{m+n}\, \frac{\|\Delta \mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}.$$

For Fiedler, Frobenius, etc linearizations $\|M(\lambda)\|_F = \|P(\lambda)\|_F$.

Discussion of the perturbation bound for block Kronecker pencils

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M(\lambda) & L_\eta(\lambda)^T \otimes I_m \\ \hline L_\varepsilon(\lambda) \otimes I_n & 0 \end{array}\right].$$

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq \underbrace{14\, d^{5/2} \frac{\|\mathcal{L}(\lambda)\|_F}{\|P(\lambda)\|_F} \left(1 + \|M(\lambda)\|_F + \|M(\lambda)\|_F^2\right)}_{C_{P,\mathcal{L}}} \frac{\|\Delta \mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}.$$

- It can be proved that if $\|P(\lambda)\|_F \ll 1$ or $\|P(\lambda)\|_F \gg 1$, then $C_{P,\mathcal{L}} \gg 1$,
- and that, if $\|M(\lambda)\|_F \gg 1$, then $C_{P,\mathcal{L}} \gg 1$.
- Therefore, **for getting "backward stability" from Block Kronecker linearizations**, one needs to normalize the matrix poly $\|P(\lambda)\|_F = 1$ and to use pencils such that $\|M(\lambda)\|_F \approx \|P(\lambda)\|_F$, then

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \lesssim d^3 \sqrt{m+n}\, \frac{\|\Delta \mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}.$$

For Fiedler, Frobenius, etc linearizations $\|M(\lambda)\|_F = \|P(\lambda)\|_F$.

Discussion of the perturbation bound for block Kronecker pencils

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M(\lambda) & L_\eta(\lambda)^T \otimes I_m \\ \hline L_\varepsilon(\lambda) \otimes I_n & 0 \end{array}\right].$$

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq \underbrace{14\, d^{5/2} \frac{\|\mathcal{L}(\lambda)\|_F}{\|P(\lambda)\|_F} \left(1 + \|M(\lambda)\|_F + \|M(\lambda)\|_F^2\right)}_{C_{P,\mathcal{L}}} \frac{\|\Delta \mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}.$$

- It can be proved that if $\|P(\lambda)\|_F \ll 1$ or $\|P(\lambda)\|_F \gg 1$, then $C_{P,\mathcal{L}} \gg 1$,
- and that, if $\|M(\lambda)\|_F \gg 1$, then $C_{P,\mathcal{L}} \gg 1$.
- Therefore, **for getting "backward stability" from Block Kronecker linearizations**, one needs to normalize the matrix poly $\|P(\lambda)\|_F = 1$ and to use pencils such that $\|M(\lambda)\|_F \approx \|P(\lambda)\|_F$, then

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \lesssim d^3 \sqrt{m+n}\, \frac{\|\Delta \mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}.$$

For Fiedler, Frobenius, etc linearizations $\|M(\lambda)\|_F = \|P(\lambda)\|_F$.

Discussion of the perturbation bound for block Kronecker pencils

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M(\lambda) & L_\eta(\lambda)^T \otimes I_m \\ \hline L_\varepsilon(\lambda) \otimes I_n & 0 \end{array}\right].$$

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq \underbrace{14\, d^{5/2} \frac{\|\mathcal{L}(\lambda)\|_F}{\|P(\lambda)\|_F} \left(1 + \|M(\lambda)\|_F + \|M(\lambda)\|_F^2\right)}_{C_{P,\mathcal{L}}} \frac{\|\Delta \mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}.$$

- It can be proved that if $\|P(\lambda)\|_F \ll 1$ or $\|P(\lambda)\|_F \gg 1$, then $C_{P,\mathcal{L}} \gg 1$,
- and that, if $\|M(\lambda)\|_F \gg 1$, then $C_{P,\mathcal{L}} \gg 1$.
- Therefore, **for getting "backward stability" from Block Kronecker linearizations**, one needs to normalize the matrix poly $\|P(\lambda)\|_F = 1$ and to use pencils such that $\|M(\lambda)\|_F \approx \|P(\lambda)\|_F$, then

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \lesssim d^3 \sqrt{m+n}\, \frac{\|\Delta \mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}.$$

For Fiedler, Frobenius, etc linearizations $\|M(\lambda)\|_F = \|P(\lambda)\|_F$.

Discussion of the perturbation bound for block Kronecker pencils

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M(\lambda) & L_\eta(\lambda)^T \otimes I_m \\ \hline L_\varepsilon(\lambda) \otimes I_n & 0 \end{array}\right].$$

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq \underbrace{14\, d^{5/2} \frac{\|\mathcal{L}(\lambda)\|_F}{\|P(\lambda)\|_F} \left(1 + \|M(\lambda)\|_F + \|M(\lambda)\|_F^2\right)}_{C_{P,\mathcal{L}}} \frac{\|\Delta \mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}.$$

- It can be proved that if $\|P(\lambda)\|_F \ll 1$ or $\|P(\lambda)\|_F \gg 1$, then $C_{P,\mathcal{L}} \gg 1$,
- and that, if $\|M(\lambda)\|_F \gg 1$, then $C_{P,\mathcal{L}} \gg 1$.
- Therefore, **for getting "backward stability" from Block Kronecker linearizations**, one needs to normalize the matrix poly $\|P(\lambda)\|_F = 1$ and to use pencils such that $\|M(\lambda)\|_F \approx \|P(\lambda)\|_F$, then

$$\boxed{\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \lesssim d^3 \sqrt{m+n}\, \frac{\|\Delta \mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}}.$$

For Fiedler, Frobenius, etc linearizations $\|M(\lambda)\|_F = \|P(\lambda)\|_F$.

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M(\lambda) & L_\eta(\lambda)^T \otimes I_m \\ \hline L_\varepsilon(\lambda) \otimes I_n & 0 \end{array}\right].$$

$$\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \leq \underbrace{14\, d^{5/2} \frac{\|\mathcal{L}(\lambda)\|_F}{\|P(\lambda)\|_F} \left(1 + \|M(\lambda)\|_F + \|M(\lambda)\|_F^2\right)}_{C_{P,\mathcal{L}}} \frac{\|\Delta \mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}.$$

- It can be proved that if $\|P(\lambda)\|_F \ll 1$ or $\|P(\lambda)\|_F \gg 1$, then $C_{P,\mathcal{L}} \gg 1$,
- and that, if $\|M(\lambda)\|_F \gg 1$, then $C_{P,\mathcal{L}} \gg 1$.
- Therefore, **for getting "backward stability" from Block Kronecker linearizations**, one needs to normalize the matrix poly $\|P(\lambda)\|_F = 1$ and to use pencils such that $\|M(\lambda)\|_F \approx \|P(\lambda)\|_F$, then

$$\boxed{\frac{\|\Delta P(\lambda)\|_F}{\|P(\lambda)\|_F} \lesssim d^3 \sqrt{m+n}\, \frac{\|\Delta \mathcal{L}(\lambda)\|_F}{\|\mathcal{L}(\lambda)\|_F}}.$$

For Fiedler, Frobenius, etc linearizations $\|M(\lambda)\|_F = \|P(\lambda)\|_F$.

Outline

We need some additional norms....

- **We have already used the Frobenius norm of a matrix polynomial.** For $D(\lambda) = \sum_{i=0}^{d} D_i \lambda^i$, we define

$$\|D(\lambda)\|_F := \sqrt{\sum_{i=0}^{d} \|D_i\|_F^2}.$$

- **"Norm" of a rational matrix represented as**

$$R(\lambda) = C(\lambda I_\ell - A)^{-1} B + \sum_{i=0}^{d} D_i \lambda^i ,$$

$$\|R(\lambda)\|_F := \left\| \begin{bmatrix} A - \lambda I_\ell & B \\ C & \sum_{i=0}^{d} D_i \lambda^i \end{bmatrix} \right\|_F = \sqrt{\ell + \|A\|_F^2 + \|B\|_F^2 + \|C\|_F^2 + \sum_{i=0}^{d} \|D_i\|_F^2}$$

- **Two-norm of a matrix pencil.**

$$\|S(\lambda)\|_2 = \|S_1 \lambda + S_0\|_2 := \| \begin{bmatrix} S_1 & S_0 \end{bmatrix} \|_2$$

- **We have already used the Frobenius norm of a matrix polynomial.** For $D(\lambda)=\sum_{i=0}^{d} D_i\lambda^i$, we define

$$\|D(\lambda)\|_F := \sqrt{\sum_{i=0}^{d} \|D_i\|_F^2}.$$

- **"Norm" of a rational matrix represented as**

$$R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i\lambda^i\,,$$

$$\|R(\lambda)\|_F := \left\| \begin{bmatrix} A-\lambda I_\ell & B \\ C & \sum_{i=0}^{d} D_i\lambda^i \end{bmatrix} \right\|_F = \sqrt{\ell + \|A\|_F^2 + \|B\|_F^2 + \|C\|_F^2 + \sum_{i=0}^{d} \|D_i\|_F^2}$$

- **Two-norm of a matrix pencil.**

$$\|S(\lambda)\|_2 = \|S_1\lambda + S_0\|_2 := \left\| \begin{bmatrix} S_1 & S_0 \end{bmatrix} \right\|_2$$

We need some additional norms....

- **We have already used the Frobenius norm of a matrix polynomial.** For $D(\lambda) = \sum_{i=0}^{d} D_i \lambda^i$, we define

$$\|D(\lambda)\|_F := \sqrt{\sum_{i=0}^{d} \|D_i\|_F^2}.$$

- **"Norm" of a rational matrix represented as**

$$R(\lambda) = C(\lambda I_\ell - A)^{-1} B + \sum_{i=0}^{d} D_i \lambda^i ,$$

$$\|R(\lambda)\|_F := \left\| \begin{bmatrix} A - \lambda I_\ell & B \\ C & \sum_{i=0}^{d} D_i \lambda^i \end{bmatrix} \right\|_F = \sqrt{\ell + \|A\|_F^2 + \|B\|_F^2 + \|C\|_F^2 + \sum_{i=0}^{d} \|D_i\|_F^2}$$

- **Two-norm of a matrix pencil.**

$$\|S(\lambda)\|_2 = \|S_1 \lambda + S_0\|_2 := \| \begin{bmatrix} S_1 & S_0 \end{bmatrix} \|_2$$

Statement of the problem (I)

- Given a rational matrix represented as

$$R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i \lambda^i ,$$

- its zeros (and minimal indices) are computed by applying a backward stable algorithm (QZ for regular, Staircase for singular) for computing the eigenvalues (and minimal indices) of its block Kronecker linearization

$$S(\lambda) = \begin{bmatrix} M(\lambda) & \widehat{K}_2^T C & K_2(\lambda)^T \\ B\widehat{K}_1 & A - \lambda I_\ell & 0 \\ K_1(\lambda) & 0 & 0 \end{bmatrix},$$

where $\sum_{i=0}^{d} D_i \lambda^i = (\Lambda_\eta(\lambda)^T \otimes I_m) M(\lambda) (\Lambda_\epsilon(\lambda) \otimes I_n)$.

- This means that **we have computed the exact eigenvalues (and minimal indices) of a pencil**

$$\widehat{S}(\lambda) := S(\lambda) + \Delta_S(\lambda), \quad \text{with} \quad \frac{\|\Delta_S(\lambda)\|_F}{\|S(\lambda)\|_F} = O(\mathbf{u}),$$

with $\mathbf{u} \approx 10^{-16}$ the unit roundoff of the computer.

- Given a rational matrix represented as

$$R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i \lambda^i ,$$

- its zeros (and minimal indices) are computed by applying a backward stable algorithm (QZ for regular, Staircase for singular) for computing the eigenvalues (and minimal indices) of its block Kronecker linearization

$$S(\lambda) = \begin{bmatrix} M(\lambda) & \widehat{K}_2^T C & K_2(\lambda)^T \\ B\widehat{K}_1 & A - \lambda I_\ell & 0 \\ K_1(\lambda) & 0 & 0 \end{bmatrix},$$

where $\sum_{i=0}^{d} D_i \lambda^i = (\Lambda_\eta(\lambda)^T \otimes I_m) M(\lambda) (\Lambda_\epsilon(\lambda) \otimes I_n)$.

- This means that **we have computed the exact eigenvalues (and minimal indices) of a pencil**

$$\widehat{S}(\lambda) := S(\lambda) + \Delta_S(\lambda), \quad \text{with} \quad \frac{\|\Delta_S(\lambda)\|_F}{\|S(\lambda)\|_F} = O(\mathbf{u}),$$

with $\mathbf{u} \approx 10^{-16}$ the unit roundoff of the computer.

Statement of the problem (I)

- Given a rational matrix represented as

$$R(\lambda) = C(\lambda I_\ell - A)^{-1} B + \sum_{i=0}^{d} D_i \lambda^i \,,$$

- its zeros (and minimal indices) are computed by applying a backward stable algorithm (QZ for regular, Staircase for singular) for computing the eigenvalues (and minimal indices) of its block Kronecker linearization

$$S(\lambda) = \begin{bmatrix} M(\lambda) & \widehat{K}_2^T C & K_2(\lambda)^T \\ B\widehat{K}_1 & A - \lambda I_\ell & 0 \\ K_1(\lambda) & 0 & 0 \end{bmatrix},$$

  where $\sum_{i=0}^{d} D_i \lambda^i = (\Lambda_\eta(\lambda)^T \otimes I_m) M(\lambda) (\Lambda_\epsilon(\lambda) \otimes I_n)\,.$

- This means that **we have computed the exact eigenvalues (and minimal indices) of a pencil**

$$\widehat{S}(\lambda) := S(\lambda) + \Delta_S(\lambda), \quad \text{with} \quad \frac{\|\Delta_S(\lambda)\|_F}{\|S(\lambda)\|_F} = O(\mathbf{u}),$$

  with $\mathbf{u} \approx 10^{-16}$ the unit roundoff of the computer.

Statement of the problem (II)

- But, **does this imply that we have computed the exact zeros of a nearby rational matrix?**
- **Nearby in the following structural sense:**

$$\widetilde{R}(\lambda) = (C + \Delta C)(\lambda I_\ell - (A + \Delta A))^{-1}(B + \Delta B) + \sum_{i=0}^{d} (D_i + \Delta D_i)\lambda^i ,$$

with

$$\frac{\|\Delta_R\|_F}{\|R(\lambda)\|_F} := \frac{\sqrt{\|\Delta A\|_F^2 + \|\Delta B\|_F^2 + \|\Delta C\|_F^2 + \sum_{i=0}^{d} \|\Delta D_i\|_F^2}}{\|R(\lambda)\|_F} = O(\mathbf{u}) \;??$$

- Again, this problem is not easy because **the perturbation pencil $\Delta_S(\lambda)$ destroys completely the delicate structure of $S(\lambda)$**.
- In fact, **this problem is considerably harder than in the polynomial case**, because the corresponding block Kronecker pencils are more complicated and have additional structures, and requires additional techniques.

Statement of the problem (II)

- But, **does this imply that we have computed the exact zeros of a nearby rational matrix?**
- **Nearby in the following structural sense:**

$$\widetilde{R}(\lambda) = (C + \Delta C)(\lambda I_\ell - (A + \Delta A))^{-1}(B + \Delta B) + \sum_{i=0}^{d} (D_i + \Delta D_i)\lambda^i ,$$

with

$$\frac{\|\Delta_R\|_F}{\|R(\lambda)\|_F} := \frac{\sqrt{\|\Delta A\|_F^2 + \|\Delta B\|_F^2 + \|\Delta C\|_F^2 + \sum_{i=0}^{d} \|\Delta D_i\|_F^2}}{\|R(\lambda)\|_F} = O(\mathbf{u}) \text{ ??}$$

- Again, this problem is not easy because **the perturbation pencil $\Delta_S(\lambda)$ destroys completely the delicate structure of $S(\lambda)$**.
- In fact, **this problem is considerably harder than in the polynomial case**, because the corresponding block Kronecker pencils are more complicated and have additional structures, and requires additional techniques.

Statement of the problem (II)

- But, **does this imply that we have computed the exact zeros of a nearby rational matrix?**
- **Nearby in the following structural sense:**

$$\widetilde{R}(\lambda) = (C + \Delta C)(\lambda I_\ell - (A + \Delta A))^{-1}(B + \Delta B) + \sum_{i=0}^{d}(D_i + \Delta D_i)\lambda^i ,$$

with

$$\frac{\|\Delta_R\|_F}{\|R(\lambda)\|_F} := \frac{\sqrt{\|\Delta A\|_F^2 + \|\Delta B\|_F^2 + \|\Delta C\|_F^2 + \sum_{i=0}^{d}\|\Delta D_i\|_F^2}}{\|R(\lambda)\|_F} = O(\mathbf{u}) \text{ ??}$$

- Again, this problem is not easy because **the perturbation pencil $\Delta_S(\lambda)$ destroys completely the delicate structure of $S(\lambda)$**.
- In fact, **this problem is considerably harder than in the polynomial case**, because the corresponding block Kronecker pencils are more complicated and have additional structures, and requires additional techniques.

Statement of the problem (II)

- But, **does this imply that we have computed the exact zeros of a nearby rational matrix?**
- **Nearby in the following structural sense:**

$$\widetilde{R}(\lambda) = (C + \Delta C)(\lambda I_\ell - (A + \Delta A))^{-1}(B + \Delta B) + \sum_{i=0}^{d}(D_i + \Delta D_i)\lambda^i \, ,$$

with

$$\frac{\|\Delta_R\|_F}{\|R(\lambda)\|_F} := \frac{\sqrt{\|\Delta A\|_F^2 + \|\Delta B\|_F^2 + \|\Delta C\|_F^2 + \sum_{i=0}^{d} \|\Delta D_i\|_F^2}}{\|R(\lambda)\|_F} = O(\mathbf{u}) \text{ ??}$$

- Again, this problem is not easy because **the perturbation pencil $\Delta_S(\lambda)$ destroys completely the delicate structure of $S(\lambda)$**.
- In fact, **this problem is considerably harder than in the polynomial case**, because the corresponding block Kronecker pencils are more complicated and have additional structures, and requires additional techniques.

Theorem (D, Quintana, Van Dooren, Calcolo, 2023)

Given $R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i\lambda^i$, *let* $S(\lambda)$ *be any of its block Kronecker linearizations, and let us perturb it with a pencil to get*

$$\widehat{S}(\lambda) = S(\lambda) + \Delta_S(\lambda)$$

If $\|\Delta_S(\lambda)\|_F$ *is sufficiently small, then the eigenvalues and min. indices of* $\widehat{S}(\lambda)$ *are the zeros and shifted min. indices of*

$$\widetilde{R}(\lambda) = (C + \Delta C)(\lambda I_\ell - (A + \Delta A))^{-1}(B + \Delta B) + \sum_{i=0}^{d}(D_i + \Delta D_i)\lambda^i$$

such that, to first order in $\|\Delta_S(\lambda)\|_F$,

$$\frac{\sqrt{\|\Delta A\|_F^2 + \|\Delta B\|_F^2 + \|\Delta C\|_F^2 + \sum_{i=0}^{d}\|\Delta D_i\|_F^2}}{\|R(\lambda)\|_F} \le C_{S,R}\,\frac{\|\Delta_S(\lambda)\|_F}{\|S(\lambda)\|_F},$$

where

$$C_{S,R} = \sqrt{2\min(\varepsilon+1, \eta+1)}\,(1 + f_1\|S(\lambda)\|_2)(1 + f_2\|S(\lambda)\|_2)(1 + f_3\|S(\lambda)\|_2)\,\frac{\|S(\lambda)\|_F}{\|R(\lambda)\|_F}$$

Theorem (continuation)

where

$$\begin{aligned}
\alpha &:= 1 + 2\varepsilon \max(1, \|A\|_2^{\varepsilon}),\\
\beta &:= 1 + 2\eta \max(1, \|A\|_2^{\eta}),\\
\gamma &:= \frac{\varepsilon + \eta}{2\sqrt{2}},\\
s &:= \max(\alpha, \beta, \gamma) + \gamma(\beta\|B\|_2 + \alpha\|C\|_2)
\end{aligned}$$

and

$$\begin{aligned}
f_1 &:= \frac{4\sqrt{2}s}{2-\sqrt{3}},\\
f_2 &:= \frac{\sqrt{2}\,(4\max(\varepsilon,\eta)-1)}{3},\\
f_3 &:= \sqrt{2}\,[1 + 2\max(\eta,\varepsilon)\max(1, \|A\|_2^{\max(\eta,\varepsilon)})]
\end{aligned}$$

only depend on the initial data.

The previous bound does not guarantee structural backward stability

- We should have

$$C_{S,R} = \sqrt{2\min(\varepsilon+1,\eta+1)}\,(1+f_1\|S(\lambda)\|_2)(1+f_2\|S(\lambda)\|_2)(1+f_3\|S(\lambda)\|_2)\,\frac{\|S(\lambda)\|_F}{\|R(\lambda)\|_F}$$

moderate for guaranteeing structural backward stability.

- However this magnitude can be very large if $\|S(\lambda)\|_2$ is large or, under the weaker condition that $\|R(\lambda)\|_F$ is large,
- and we have performed numerical experiments that confirm that under such conditions the structured backward errors can be indeed very large.
- Thus, additional conditions are needed in order to guarantee structural backward stability.

The previous bound does not guarantee structural backward stability

- We should have

$$C_{S,R} = \sqrt{2\min(\varepsilon+1,\eta+1)}\,(1+f_1\|S(\lambda)\|_2)(1+f_2\|S(\lambda)\|_2)(1+f_3\|S(\lambda)\|_2)\,\frac{\|S(\lambda)\|_F}{\|R(\lambda)\|_F}$$

moderate for guaranteeing structural backward stability.

- However this magnitude can be very large if $\|S(\lambda)\|_2$ is large or, under the weaker condition that $\|R(\lambda)\|_F$ is large,
- and we have performed numerical experiments that confirm that under such conditions the structured backward errors can be indeed very large.
- Thus, additional conditions are needed in order to guarantee structural backward stability.

The previous bound does not guarantee structural backward stability

- We should have

$$C_{S,R} = \sqrt{2\min(\varepsilon+1,\eta+1)}\,(1+f_1\|S(\lambda)\|_2)(1+f_2\|S(\lambda)\|_2)(1+f_3\|S(\lambda)\|_2)\frac{\|S(\lambda)\|_F}{\|R(\lambda)\|_F}$$

  moderate for guaranteeing structural backward stability.

- However this magnitude can be very large if $\|S(\lambda)\|_2$ is large or, under the weaker condition that $\|R(\lambda)\|_F$ is large,
- and we have performed numerical experiments that confirm that under such conditions the structured backward errors can be indeed very large.
- Thus, additional conditions are needed in order to guarantee structural backward stability.

The previous bound does not guarantee structural backward stability

- We should have

$$C_{S,R} = \sqrt{2\min(\varepsilon+1,\eta+1)}\,(1+f_1\|S(\lambda)\|_2)(1+f_2\|S(\lambda)\|_2)(1+f_3\|S(\lambda)\|_2)\frac{\|S(\lambda)\|_F}{\|R(\lambda)\|_F}$$

  moderate for guaranteeing structural backward stability.

- However this magnitude can be very large if $\|S(\lambda)\|_2$ is large or, under the weaker condition that $\|R(\lambda)\|_F$ is large,
- and we have performed numerical experiments that confirm that under such conditions the structured backward errors can be indeed very large.
- Thus, additional conditions are needed in order to guarantee structural backward stability.

Sufficient conditions for structural backward stability

Theorem (D, Quintana, Van Dooren, Calcolo, 2023)

Let $R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i \lambda^i \in \mathbb{C}(\lambda)^{m\times n}$ *and* $S(\lambda)$ *be any of its block Kronecker linearizations. If*

$$\max(\|A\|_F, \|B\|_F, \|C\|_F, \|D(\lambda)\|_F) \leq 1 \quad \textit{and} \quad \|M(\lambda)\|_F \approx \|D(\lambda)\|_F$$

then

$$C_{S,R} \leq g\, d^5 \sqrt{m+n}\,,$$

where g *is a moderate number (a constant that does not depend on* d, m, n, ℓ*) and, to first order in* $\|\Delta_S(\lambda)\|_F$*,*

$$\frac{\sqrt{\|\Delta A\|_F^2 + \|\Delta B\|_F^2 + \|\Delta C\|_F^2 + \sum_{i=0}^{d} \|\Delta D_i\|_F^2}}{\|R(\lambda)\|_F} \lesssim d^5 \sqrt{m+n}\, \frac{\|\Delta_S(\lambda)\|_F}{\|S(\lambda)\|_F}\,.$$

- **There are rational matrices for which**

$$\max(\|A\|_F, \|B\|_F, \|C\|_F, \|D(\lambda)\|_F) \leq 1$$

do not hold.

- **In such cases, we can scale the original rational matrix $R(\lambda)$** as

$$\widehat{R}(\widehat{\lambda}) := \widehat{D}(\widehat{\lambda}) + \widehat{C}(\widehat{\lambda} I_\ell - \widehat{A})^{-1}\widehat{B} := d_R R(\widehat{\lambda}/d_\lambda)$$

where

$$\widehat{A} := d_\lambda T^{-1} A T, \quad \widehat{B} := \sqrt{d_\lambda d_R}\, T^{-1} B, \quad \widehat{C} := \sqrt{d_\lambda d_R}\, C T, \quad \widehat{D}_i := d_R d_\lambda^{-i} D_i,$$

$i = 0, 1, \ldots, d$, and $T = \mathrm{diag}(d_1, \ldots, d_\ell)$,

- **to obtain a rational matrix $\widehat{R}(\widehat{\lambda})$ that satisfies the desired conditions.**
- Moreover, this can be done without errors with parameters that are integer powers of 2.
- However, **this requires to scale also the variable λ** in contrast with the polynomial case.

What to do if these conditions do not hold? SCALING!!!

- **There are rational matrices for which**

$$\max(\|A\|_F, \|B\|_F, \|C\|_F, \|D(\lambda)\|_F) \leq 1$$

do not hold.

- **In such cases, we can scale the original rational matrix $R(\lambda)$** as

$$\widehat{R}(\widehat{\lambda}) := \widehat{D}(\widehat{\lambda}) + \widehat{C}(\widehat{\lambda} I_\ell - \widehat{A})^{-1}\widehat{B} := d_R R(\widehat{\lambda}/d_\lambda)$$

where

$$\widehat{A} := d_\lambda T^{-1} A T, \;\; \widehat{B} := \sqrt{d_\lambda d_R}\, T^{-1} B, \;\; \widehat{C} := \sqrt{d_\lambda d_R}\, C T, \;\; \widehat{D}_i := d_R d_\lambda^{-i} D_i,$$

$i = 0, 1, \ldots, d$, and $T = \mathrm{diag}(d_1, \ldots, d_\ell)$,

- **to obtain a rational matrix $\widehat{R}(\widehat{\lambda})$ that satisfies the desired conditions.**
- Moreover, this can be done without errors with parameters that are integer powers of 2.
- However, **this requires to scale also the variable λ** in contrast with the polynomial case.

What to do if these conditions do not hold? SCALING!!!

- **There are rational matrices for which**

$$\max(\|A\|_F, \|B\|_F, \|C\|_F, \|D(\lambda)\|_F) \leq 1$$

  do not hold.

- **In such cases, we can scale the original rational matrix $R(\lambda)$** as

$$\widehat{R}(\widehat{\lambda}) := \widehat{D}(\widehat{\lambda}) + \widehat{C}(\widehat{\lambda} I_\ell - \widehat{A})^{-1}\widehat{B} := d_R R(\widehat{\lambda}/d_\lambda)$$

  where

$$\widehat{A} := d_\lambda T^{-1} A T, \;\; \widehat{B} := \sqrt{d_\lambda d_R}\, T^{-1} B, \;\; \widehat{C} := \sqrt{d_\lambda d_R}\, C T, \;\; \widehat{D}_i := d_R d_\lambda^{-i} D_i,$$

  $i = 0, 1, \ldots, d$, and $T = \mathrm{diag}(d_1, \ldots, d_\ell)$,

- **to obtain a rational matrix $\widehat{R}(\widehat{\lambda})$ that satisfies the desired conditions.**
- Moreover, this can be done without errors with parameters that are integer powers of 2.
- However, **this requires to scale also the variable** λ in contrast with the polynomial case.

What to do if these conditions do not hold? SCALING!!!

- **There are rational matrices for which**

$$\max(\|A\|_F, \|B\|_F, \|C\|_F, \|D(\lambda)\|_F) \leq 1$$

  do not hold.

- **In such cases, we can scale the original rational matrix $R(\lambda)$** as

$$\widehat{R}(\widehat{\lambda}) := \widehat{D}(\widehat{\lambda}) + \widehat{C}(\widehat{\lambda} I_\ell - \widehat{A})^{-1}\widehat{B} := d_R R(\widehat{\lambda}/d_\lambda)$$

  where

$$\widehat{A} := d_\lambda T^{-1} A T, \quad \widehat{B} := \sqrt{d_\lambda d_R}\, T^{-1} B, \quad \widehat{C} := \sqrt{d_\lambda d_R}\, C T, \quad \widehat{D}_i := d_R d_\lambda^{-i} D_i,$$

  $i = 0, 1, \ldots, d$, and $T = \mathrm{diag}(d_1, \ldots, d_\ell)$,

- **to obtain a rational matrix $\widehat{R}(\widehat{\lambda})$ that satisfies the desired conditions.**
- Moreover, this can be done without errors with parameters that are integer powers of 2.
- However, **this requires to scale also the variable λ** in contrast with the polynomial case.

What to do if these conditions do not hold? SCALING!!!

- **There are rational matrices for which**

$$\max(\|A\|_F, \|B\|_F, \|C\|_F, \|D(\lambda)\|_F) \leq 1$$

  do not hold.

- **In such cases, we can scale the original rational matrix $R(\lambda)$** as

$$\widehat{R}(\widehat{\lambda}) := \widehat{D}(\widehat{\lambda}) + \widehat{C}(\widehat{\lambda} I_\ell - \widehat{A})^{-1}\widehat{B} := d_R R(\widehat{\lambda}/d_\lambda)$$

  where

$$\widehat{A} := d_\lambda T^{-1} A T, \quad \widehat{B} := \sqrt{d_\lambda d_R}\, T^{-1} B, \quad \widehat{C} := \sqrt{d_\lambda d_R}\, C T, \quad \widehat{D}_i := d_R d_\lambda^{-i} D_i,$$

  $i = 0, 1, \ldots, d$, and $T = \mathrm{diag}(d_1, \ldots, d_\ell)$,

- **to obtain a rational matrix $\widehat{R}(\widehat{\lambda})$ that satisfies the desired conditions.**

- Moreover, this can be done without errors with parameters that are integer powers of 2.

- However, **this requires to scale also the variable λ** in contrast with the polynomial case.

Outline

1. Brief reminder of “Eigenstructures” of PEPs and REPs
2. Linearizations of polynomial and rational matrices
3. Block Kronecker linearizations of polynomial matrices
4. Block Kronecker linearizations of rational matrices
5. Global backward stability of PEPs solved with linearizations
6. Global backward stability of REPs solved with linearizations
7. **Conclusions**

- We have proved that the computation of the complete eigenstructure of a matrix polynomial $P(\lambda)$ (regular or singular) via block Kronecker linearizations is backward stable from the polynomial point of view
- if the polynomial is multiplied by a number in such a way that $\|P(\lambda)\|_F = 1$
- and the linearization is chosen to satisfy $\|M(\lambda)\|_F \approx \|P(\lambda)\|_F$, which happens for many of the linearizations used in the literature.
- We have obtained similar results for rational matrices expressed as $R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i\lambda^i \in \mathbb{C}(\lambda)^{m\times n}$
- though in this case a delicate scaling of the rational matrix, which requires to scale the variable, is needed.
- We have also analyzed (D, Pérez, Van Dooren, Math. Comp., 2019) structured polynomial matrices (Hermitian, alternating, palindromic, ...) with structure preserving backward errors and we have obtained similar positive results.

Conclusions

- We have proved that the computation of the complete eigenstructure of a matrix polynomial $P(\lambda)$ (regular or singular) via block Kronecker linearizations is backward stable from the polynomial point of view
- if the polynomial is multiplied by a number in such a way that $\|P(\lambda)\|_F = 1$
- and the linearization is chosen to satisfy $\|M(\lambda)\|_F \approx \|P(\lambda)\|_F$, which happens for many of the linearizations used in the literature.
- We have obtained similar results for rational matrices expressed as $R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i \lambda^i \in \mathbb{C}(\lambda)^{m\times n}$
- though in this case a delicate scaling of the rational matrix, which requires to scale the variable, is needed.
- We have also analyzed (D, Pérez, Van Dooren, Math. Comp., 2019) structured polynomial matrices (Hermitian, alternating, palindromic, ...) with structure preserving backward errors and we have obtained similar positive results.

Conclusions

- We have proved that the computation of the complete eigenstructure of a matrix polynomial $P(\lambda)$ (regular or singular) via block Kronecker linearizations is backward stable from the polynomial point of view
- if the polynomial is multiplied by a number in such a way that $\|P(\lambda)\|_F = 1$
- and the linearization is chosen to satisfy $\|M(\lambda)\|_F \approx \|P(\lambda)\|_F$, which happens for many of the linearizations used in the literature.
- We have obtained similar results for rational matrices expressed as $R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i \lambda^i \in \mathbb{C}(\lambda)^{m\times n}$
- though in this case a delicate scaling of the rational matrix, which requires to scale the variable, is needed.
- We have also analyzed (D, Pérez, Van Dooren, Math. Comp., 2019) structured polynomial matrices (Hermitian, alternating, palindromic, ...) with structure preserving backward errors and we have obtained similar positive results.

Conclusions

- We have proved that the computation of the complete eigenstructure of a matrix polynomial $P(\lambda)$ (regular or singular) via block Kronecker linearizations is backward stable from the polynomial point of view
- if the polynomial is multiplied by a number in such a way that $\|P(\lambda)\|_F = 1$
- and the linearization is chosen to satisfy $\|M(\lambda)\|_F \approx \|P(\lambda)\|_F$, which happens for many of the linearizations used in the literature.
- We have obtained similar results for rational matrices expressed as $R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i\lambda^i \in \mathbb{C}(\lambda)^{m\times n}$
- though in this case a delicate scaling of the rational matrix, which requires to scale the variable, is needed.
- We have also analyzed (D, Pérez, Van Dooren, Math. Comp., 2019) structured polynomial matrices (Hermitian, alternating, palindromic, ...) with structure preserving backward errors and we have obtained similar positive results.

Conclusions

- We have proved that the computation of the complete eigenstructure of a matrix polynomial $P(\lambda)$ (regular or singular) via block Kronecker linearizations is backward stable from the polynomial point of view
- if the polynomial is multiplied by a number in such a way that $\|P(\lambda)\|_F = 1$
- and the linearization is chosen to satisfy $\|M(\lambda)\|_F \approx \|P(\lambda)\|_F$, which happens for many of the linearizations used in the literature.
- We have obtained similar results for rational matrices expressed as $R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i \lambda^i \in \mathbb{C}(\lambda)^{m\times n}$
- though in this case a delicate scaling of the rational matrix, which requires to scale the variable, is needed.
- We have also analyzed (D, Pérez, Van Dooren, Math. Comp., 2019) structured polynomial matrices (Hermitian, alternating, palindromic, ...) with structure preserving backward errors and we have obtained similar positive results.

Conclusions

- We have proved that the computation of the complete eigenstructure of a matrix polynomial $P(\lambda)$ (regular or singular) via block Kronecker linearizations is backward stable from the polynomial point of view
- if the polynomial is multiplied by a number in such a way that $\|P(\lambda)\|_F = 1$
- and the linearization is chosen to satisfy $\|M(\lambda)\|_F \approx \|P(\lambda)\|_F$, which happens for many of the linearizations used in the literature.
- We have obtained similar results for rational matrices expressed as $R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i\lambda^i \in \mathbb{C}(\lambda)^{m\times n}$
- though in this case a delicate scaling of the rational matrix, which requires to scale the variable, is needed.
- We have also analyzed (D, Pérez, Van Dooren, Math. Comp., 2019) structured polynomial matrices (Hermitian, alternating, palindromic, ...) with structure preserving backward errors and we have obtained similar positive results.