Strongly minimal self-conjugate linearizations for polynomial and rational matrices

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joint work with **María C. Quintana** (Aalto University, Finland)
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AGENCIA ESTATAL DE INVESTIGACIÓN

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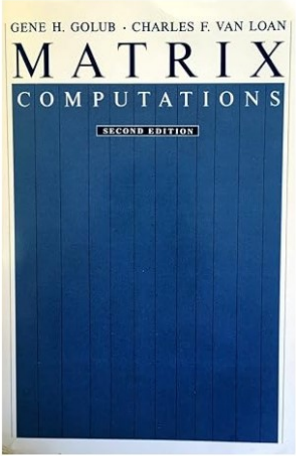
The first time I heard about David Watkins ... Autumn 1994 (I)



I was preparing a graduate course on "Numerical Linear Algebra", that I had to teach in the second semester. I had no experience on the topic and somebody recommended me Golub & Van Loan's book as the main reference for that topic. To be honest, to understand clearly and fully the QR-algorithm was not easy for me, and I was discouraged but ...

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UNDERSTANDING THE *QR* ALGORITHM*

DAVID S. WATKINS†

Abstract. The *QR* algorithm is currently the most popular method for finding all eigenvalues of a full matrix. While *QR* is now well understood by specialists in eigenvalue computations, this understanding is not being conveyed effectively to the mathematical public. Many accounts present Wilkinson's 1965 convergence proof. Others establish some of the connections between the *QR* algorithm, the power method and inverse iteration. Usually much emphasis is (rightly) placed on the refinements, such as shifts of origin, which are required to make the algorithm competitive. But practically all accounts fail to explain the basic meaning of *QR* iterations. As a consequence, the *QR* algorithm is widely thought to be difficult to understand. The purpose of this paper is to try to convince the reader that the opposite is true. In fact, the *QR* algorithm is neither more nor less than a clever implementation of simultaneous iteration, which is itself a natural, easily understood extension of the power method. This point of view deserves pre-eminence because it shows exactly what *QR* iterations are and evokes a clear geometric picture of the *QR* process. Furthermore, it provides a framework within which the rapid convergence associated with shifts of origin may be explained. No reference to inverse iteration is necessary. Inverse iteration has not, however, been banished from the paper—one section is devoted to an explanation of the interplay between inverse iteration, direct iteration and the *QR* algorithm. The key result of that section is a duality theorem which shows that whenever direct iteration takes place, inverse iteration automatically takes place at the same time.

1. Introduction. The *QR* algorithm is currently the most popular method for calculating the complete set of eigenvalues of a full (i.e., small) matrix. A descendant of Rutishauser's (1955), (1958) *LR* algorithm, it was discovered independently by Francis (1961), (1962) and Kublanovskaya (1961). The basic algorithm is as follows: Given a matrix A whose eigenvalues are desired, let $A_0 = A$. Then, given A_{m-1}, find unitary Q_m and upper triangular R_m such that $A_{m-1} = Q_m R_m$. Finally, define $A_m = R_m Q_m$. Thus

$$A_{m-1} = Q_m R_m, \qquad A_m = R_m Q_m.$$

One's first reaction on seeing this procedure is likely to be, "What does this have to do with eigenvalues?" or "What do these manipulations accomplish?" Most accounts answer these questions by presenting Wilkinson's (1965, p. 517) proof that, under suitable conditions, the sequence of (unitarily similar) matrices A_m converges to upper triangular form. That proof has its merits. For one, it is relatively brief and elementary. Also, it was the best available in the sixties. Unfortunately, it does not show what goes on in a *QR* iteration—the reader is shown that the method works, but is left wondering why. This author believes that the best way to explain what *QR* iterations are is to first introduce and discuss simultaneous iteration, an easily understood, multivector generalization of the power method, then show that the *QR* algorithm is just a clever way to do simultaneous iteration

in Golub & Van Loan's p. 360, I read "Deeper insight into the convergence of the QR algorithm can be attained by reading ... " and, encouraged by the abstract, I did it! I was really impressed by the clarity and the depth of the exposition and, then, I looked for more information about the author. Of course, I discovered the first edition of Watkins, "Fundamentals of Matrix Computations" (1991) and since then I have read and used David's books for my courses and for my research.

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SIAG/LA-SIMUMAT INTERNATIONAL SUMMER SCHOOL ON NUMERICAL LINEAR ALGEBRA, July 21-25, 2008, Castro-Urdiales, Spain

David’s course was “Structured eigenvalue problems: modern theory and computational practice”.

Thank you very much for writing so many wonderful books and papers, and I hope to share with you many great moments in the future.
Happy birthday!!

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2. **GEP**: $(\lambda B - A)\, v = 0$!!!!
3. **PEP**: $(P_d \lambda^d + \cdots + P_1 \lambda + P_0) v = 0$!!!!
4. **REP**: $G(\lambda) v = 0$!!!!
5. **NEP**: $F(\lambda) v = 0$

- We focus on **PEPs and REPs**.
- **Key idea: PEPs and REPs can be solved by transforming the problem into a GEP** via a process known as **LINEARIZATION**.
- This transformation is **exact**: the obtained **GEP** contains (or allows us to easily extract) exactly all the eigen-information of the original **PEP** or **REP**.
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Matrix eigenvalue problems and linearizations

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The goals of the talk

- So far, the linearizations used in the literature for **PEPs** fit into the classical definition of **Gohberg-Lancaster-Rodman (GLR)**,
- and the ones for **REPs** fit into combining the **GLR-approach with Rosenbrock's** polynomial system matrices (Alam, Behera (SIMAX, 2016); Amparan, D, Marcaida, Zaballa (SIMAX, 2018)).
- We will introduce a new unified definition of particular linearizations of PEPs and REPs (**strongly minimal linearizations**) that guarantee stronger properties than those of GLR-linearizarions.
- Moreover, we will show how to construct such linearizations for any polynomial or rational matrix in such a way that
- for structured PEPs and REPs (Hermitian, skew-Hermitian, alternating odd and even) **always preserve such structures**,
- which is not always possible for GLR-linearizations,
- in particular for polynomial matrices of even-degree (Mackey, Mackey, Mehl, Mehrmann, De Terán, D).

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Outline

1. Brief reminder of “Eigenstructures” of PEPs and REPs
2. Gohberg-Lancaster-Rodman linearizations of PEPs (and REPs)
3. Strongly minimal linearizations of polynomial and rational matrices
4. Constructing strongly minimal linearizations of polynomial matrices
5. Constructing strongly minimal linearizations of rational matrices
6. Conclusions

GEPs-PEPs-REPs have more spectral "structural" data than BEPs

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- So far, we only consider **finite eigenvalues**, but
- **GEPs, PEPs, REPs** may have also **infinite eigenvalues**.
- **GEPs, PEPs, REPs** may be **singular**, i.e., rectangular or square with identically zero determinant, (BEPs are always regular) and to have, in addition to eigenvalues, **minimal indices**.
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Given $P(\lambda) = P_d \lambda^d + \cdots + P_1 \lambda + P_0 \in \mathbb{C}[\lambda]^{m \times n}$,

- $\lambda_0 \in \mathbb{C}$ is a **finite eigenvalue** of $P(\lambda)$ if

$$\operatorname{rank} P(\lambda_0) < \max_{\lambda \in \mathbb{C}} \operatorname{rank} P(\lambda)$$

- The infinite eigenvalue of $P(\lambda)$ is defined through **the reversal polynomial**.
- The reversal of $P(\lambda)$ is

$$\operatorname{rev} P(\lambda) := \lambda^d P\left(\frac{1}{\lambda}\right) = P_0 \lambda^d + \cdots + P_{d-1} \lambda + P_d \; .$$

- Then the **infinite eigenvalue** (and its mutiplicities) of $P(\lambda)$ correspond to the **zero eigenvalue** (and its mutiplicities) of $\operatorname{rev} P(\lambda)$.

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Minimal indices of singular PEPs

- PEPs are **singular** when $P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0$ is either **rectangular or square with** $\det P(\lambda) \equiv 0$.
- In addition to eigenvalues, **singular matrix polynomials have** other "interesting numbers" called **minimal indices**,
- which are related to the fact that a singular $m \times n$ matrix polynomial $P(\lambda)$ has non-trivial left and/or right null-spaces over the field $\mathbb{C}(\lambda)$ of rational functions:

$$\begin{aligned} \mathcal{N}_\ell(P) &:= \left\{ y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times m} \; : \; y(\lambda)^T P(\lambda) \equiv 0^T \right\}, \\ \mathcal{N}_r(P) &:= \left\{ x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} \; : \; P(\lambda)x(\lambda) \equiv 0 \right\}. \end{aligned}$$

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- which are related to the fact that a singular $m \times n$ matrix polynomial $P(\lambda)$ has non-trivial left and/or right null-spaces over the field $\mathbb{C}(\lambda)$ of rational functions:

$$\begin{aligned} \mathcal{N}_\ell(P) &:= \left\{ y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times m} \; : \; y(\lambda)^T P(\lambda) \equiv 0^T \right\}, \\ \mathcal{N}_r(P) &:= \left\{ x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} \; : \; P(\lambda) x(\lambda) \equiv 0 \right\}. \end{aligned}$$

- They have bases consisting entirely of vector polynomials.
- The polynomial bases with "minimal sum of the degrees" of their vectors are the **minimal bases** of $P(\lambda)$. The **minimal indices** of $P(\lambda)$ are the degrees of the vectors of any minimal basis.

Definition

The **complete “eigenstructure”** of a polynomial matrix $P(\lambda)$ is comprised of:

- its finite eigenvalues, together with their partial multiplicities,
- its infinite eigenvalue, together with its partial multiplicities,
- its right minimal indices, and
- its left minimal indices.

Analogously, we define:

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The **complete "eigenstructure"** of a rational matrix $G(\lambda)$ is comprised of:

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Remarks

- The infinite zeros and poles, together with its partial multiplicities, of $G(\lambda)$ are defined as the zeros and poles at $\lambda = 0$, together with its partial multiplicities, of $G(1/\lambda)$.

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Outline

Definition: GLR strong linearizations of polynomial matrices

Gohberg, Lancaster, Rodman, *Matrix Polynomials*, 1982 and Gohberg, Kaashoek, Lancaster, Integr. Eq. Operator Theory (1988).

Definition

- A **linear polynomial matrix (or matrix pencil)** $L(\lambda)$ is a **(GLR) linearization** of $P(\lambda) = P_d \lambda^d + \cdots + P_1 \lambda + P_0$ if there exist **unimodular** polynomial matrices $U(\lambda), V(\lambda)$ such that

$$U(\lambda)\, L(\lambda)\, V(\lambda) = \begin{bmatrix} I_s & 0 \\ 0 & P(\lambda) \end{bmatrix}.$$

- $L(\lambda)$ is a **(GLR) strong linearization** of $P(\lambda)$ if, **in addition**, $\mathrm{rev}\, L(\lambda)$ is a linearization for $\mathrm{rev}\, P(\lambda)$, i.e.,

$$\widetilde{U}(\lambda)\, (\mathrm{rev}\, L(\lambda))\, \widetilde{V}(\lambda) = \begin{bmatrix} I_s & 0 \\ 0 & \mathrm{rev}\, P(\lambda) \end{bmatrix},$$

with $\widetilde{U}(\lambda)$ and $\widetilde{V}(\lambda)$ unimodular.

Spectral characterization of linearizations of polynomial matrices

Theorem

A matrix pencil $L(\lambda)$ is a (GLR) linearization *of a polynomial matrix $P(\lambda)$* **if and only if**

(1) $L(\lambda)$ *and* $P(\lambda)$ **have the same number of right minimal indices**.

(2) $L(\lambda)$ *and* $P(\lambda)$ **have the same number of left minimal indices**.

(3) $L(\lambda)$ *and* $P(\lambda)$ **have the same finite eigenvalues** *with the same partial multiplicities.*

$L(\lambda)$ **is a (GLR) strong linearization** *of* $P(\lambda)$ **if and only if** *(1), (2), (3) and*

(4) $L(\lambda)$ *and* $P(\lambda)$ **have the same infinite eigenvalues** *with the same partial multiplicities.*

Remark: The **minimal indices** of $L(\lambda)$ **may have arbitrarily different values** from those of $P(\lambda)$, though in the most important classes of (GLR) linearizations they are easily related.

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The most famous strong linearization

The classical **Frobenius companion form** of the $m \times n$ matrix polynomial

$$P(\lambda) = P_d \lambda^d + \cdots + P_1 \lambda + P_0$$

is

$$C_1(\lambda) := \begin{bmatrix} \lambda P_d + P_{d-1} & P_{d-2} & \cdots & P_1 & P_0 \\ -I_n & \lambda I_n & & & \\ & \ddots & \ddots & & \\ & & \ddots & \lambda I_n & \\ & & & -I_n & \lambda I_n \end{bmatrix} \in \mathbb{C}[\lambda]^{(m+n(d-1)) \times nd}$$

Some comments on (GLR + Rosenbrock) linearizations of REPs

- For brevity, I will not present a definition of (GLR + Rosenbrock)-based linearizations and strong linearizations of rational matrices.
- In contrast with the polynomial case, there is no agreement in the community for a unique definition of **(strong) linearization** of a rational matrix.
- Amparan, D, Marcaida, and Zaballa, *Strong linearizations of rational matrices*, SIMAX 2018 gave a definition of strong linearization of any rational matrix $R(\lambda)$ that reduces to GLR when $R(\lambda)$ is a polynomial matrix.
- Another related approach was initiated by Alam and Behera, *Linearizations for rational matrix functions and Rosenbrock system polynomials*, SIMAX 2016.
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Outline

Definition of strongly minimal linearizations

Since polynomial matrices are also rational matrices the next definition applies to both. $R(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ denotes that $R(\lambda)$ is a $m \times n$ rational matrix.

Definition (D, Quintana, Van Dooren, SIMAX, 2022)

A **strongly minimal linearization** of $R(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ is a matrix pencil

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & -(B_1\lambda + B_0) \\ C_1\lambda + C_0 & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{C}[\lambda]^{(p+m)\times(p+n)}$$

such that:

(a) $R(\lambda) = (D_1\lambda + D_0) + (C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0)$,

(b) $\begin{bmatrix} A_1\lambda + A_0 & -(B_1\lambda + B_0) \end{bmatrix}$ and $\begin{bmatrix} A_1\lambda + A_0 \\ C_1\lambda + C_0 \end{bmatrix}$ have full row and column rank for all $\lambda_0 \in \mathbb{C}$, respectively, and

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is a strongly minimal linearization of $R(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ then:

- *The finite eigenvalue structure of $L(\lambda)$ coincides exactly with the finite zero structure of $R(\lambda)$.*
- *The finite eigenvalue structure of $A_1\lambda + A_0$ coincides exactly with the finite pole structure of $R(\lambda)$.*
- *The infinite eigenvalue structure of $L(\lambda)$ and $A_1\lambda + A_0$ allows us to recover exactly the infinite zero/pole structure of $R(\lambda)$.*
- *$L(\lambda)$ and $R(\lambda)$* **have the same left and right minimal indices**.

Properties of strongly minimal linearizations (II)

Recovery of eigenvectors and minimal bases

The eigenvectors and minimal bases of $R(\lambda)$ can be recovered from those of $L(\lambda)$ simply by removing the first p entries.

Relation with GLR linearizations

- Strongly minimal linearizations are GLR-linearizations.
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Outline

A famous pencil by Lancaster (which is not in general a linearization)

For any

$$P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0 \in \mathbb{C}[\lambda]^{m\times n}$$

we define

$$L_s(\lambda) = \left[\begin{array}{cccc|c} & & & -P_d & \lambda P_d \\ & & \iddots & \lambda P_d - P_{d-1} & \vdots \\ & -P_d & \iddots & \vdots & \vdots \\ -P_d & \lambda P_d - P_{d-1} & \dots & \lambda P_3 - P_2 & \lambda P_2 \\ \hline \lambda P_d & \dots & \dots & \lambda P_2 & \lambda P_1 + P_0 \end{array}\right]$$

- It was proposed by Lancaster for regular polynomial matrices with P_d invertible in Lancaster, *Lambda-Matrices and Vibrating Systems*, 1966.
- If P_d is invertible, then $L_s(\lambda)$ is a GLR strong linearization of $P(\lambda)$. **If P_d is NOT invertible, $L_s(\lambda)$ is not a GLR-linearization**.
- $L_s(\lambda)$ is one of the famous $\mathbb{DL}(P)$ pencils introduced by Mackey, Mackey, Mehl and Mehrmann (SIMAX, 2006).

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A rank revealing factorization of a constant matrix associated to $L_s(\lambda)$

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we define

$$T = \begin{bmatrix} & & & P_d \\ & & \iddots & P_{d-1} \\ & P_d & \iddots & \vdots \\ P_d & P_{d-1} & \dots & P_2 \end{bmatrix}$$

and consider a rank-revealing factorization of T, for instance a SVD,

$$U^*TV = \begin{bmatrix} 0 & 0 \\ 0 & \widehat{T} \end{bmatrix},$$

where U, V, and $\widehat{T} \in \mathbb{C}^{r \times r}$ are invertible.

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A strongly minimal linearization for $P(\lambda)$

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Theorem (D, Quintana, Van Dooren, SIMAX, 2022)

$$\left[\begin{array}{c|c} U^* & \\ \hline & I_m \end{array}\right] \left[\begin{array}{cccc|c} & & & -P_d & \lambda P_d \\ & & \cdot^{\cdot^{\cdot}} & \lambda P_d - P_{d-1} & \vdots \\ & -P_d & \cdot^{\cdot^{\cdot}} & \vdots & \vdots \\ -P_d & \lambda P_d - P_{d-1} & \dots & \lambda P_3 - P_2 & \lambda P_2 \\ \hline \lambda P_d & \dots & \dots & \lambda P_2 & \lambda P_1 + P_0 \end{array}\right] \left[\begin{array}{c|c} V & \\ \hline & I_n \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & \widehat{A}_s(\lambda) & -\widehat{B}_s(\lambda) \\ \hline 0 & \widehat{C}_s(\lambda) & \widehat{D}_s(\lambda) \end{array}\right],$$

where $\widehat{A}_s(\lambda) \in \mathbb{C}[\lambda]^{r\times r}$ *is regular*

and

$$\widehat{L}_s(\lambda) = \left[\begin{array}{c|c} \widehat{A}_s(\lambda) & -\widehat{B}_s(\lambda) \\ \hline \widehat{C}_s(\lambda) & \widehat{D}_s(\lambda) \end{array}\right]$$

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Comments on preservation of structures

$$L_s(\lambda) = \left[\begin{array}{cccc|c} & & & -P_d & \lambda P_d \\ & & \iddots & \lambda P_d - P_{d-1} & \vdots \\ & -P_d & \iddots & \vdots & \vdots \\ -P_d & \lambda P_d - P_{d-1} & \dots & \lambda P_3 - P_2 & \lambda P_2 \\ \hline \lambda P_d & \dots & \dots & \lambda P_2 & \lambda P_1 + P_0 \end{array}\right]$$

- is Hermitian (resp. skew-Hermitian) if $P(\lambda)$ is.
- Moreover, the rank-revealing factorization of T can be chosen to preserve the Hermitian (resp. skew-Hermitian) structure and, so, to get a
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Quadratic polynomial matrices (with low rank leading coefficient)

$$P(\lambda) = P_0 + \lambda P_1 + \lambda^2 P_2 \in \mathbb{C}[\lambda]^{m\times n}$$

- The Lancaster pencil is very simple in the quadratic case

$$L_s(\lambda) = \left[\begin{array}{c|c} -P_2 & \lambda P_2 \\ \hline \lambda P_2 & \lambda P_1 + P_0 \end{array}\right] \in \mathbb{C}[\lambda]^{2m\times 2n} \quad \text{and} \quad T = P_2.$$

- If $P_2 = U_2 \widehat{T} V_2^*$, with $\widehat{T} \in \mathbb{C}^{r_2\times r_2}$ invertible and $U_2 \in \mathbb{C}^{m\times r_2}, V_2 \in \mathbb{C}^{n\times r_2}$ with orthornormal columns. Then

$$\widehat{L}_s(\lambda) = \left[\begin{array}{c|c} -\widehat{T} & \lambda \widehat{T} V_2^* \\ \hline \lambda U_2 \widehat{T} & \lambda P_1 + P_0 \end{array}\right] \in \mathbb{C}[\lambda]^{(r_2+m)\times(r_2+n)}$$

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Outline

Polynomial and strictly proper parts of a rational matrix

Any rational matrix $R(\lambda)$ can be uniquely expressed as

$$R(\lambda)=P(\lambda)+R_{sp}(\lambda),$$

where

1. $P(\lambda)$ is a polynomial matrix (polynomial part of $R(\lambda)$), and
2. the rational matrix $R_{sp}(\lambda)$ is **strictly proper** (strictly proper part of $R(\lambda)$), i.e., $\lim_{\lambda\to\infty} R_{sp}(\lambda)=0$.

For strictly proper rational matrices $R_{sp}(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$, we represent them via a Laurent expansion around the point at infinity

$$R_{sp}(\lambda) := R_{-1}\lambda^{-1} + R_{-2}\lambda^{-2} + R_{-3}\lambda^{-3} + \dots$$

and consider the block Hankel matrix H and shifted block Hankel matrix H_σ:

$$H := \begin{bmatrix} R_{-1} & R_{-2} & \dots & R_{-k} \\ R_{-2} & & \iddots & R_{-k-1} \\ \vdots & \iddots & \iddots & \vdots \\ R_{-k} & R_{-k-1} & \dots & R_{-2k+1} \end{bmatrix}, \quad H_\sigma := \begin{bmatrix} R_{-2} & R_{-3} & \dots & R_{-k-1} \\ R_{-3} & & \iddots & R_{-k-2} \\ \vdots & \iddots & \iddots & \vdots \\ R_{-k-1} & R_{-k-2} & \dots & R_{-2k} \end{bmatrix}.$$

For sufficiently large k the rank r_f of H equals the total polar degree of the finite poles and does not increase more with k.

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Strongly minimal linearizations for strictly proper rational matrices (II)

Theorem (D, Quintana, Van Dooren, SIMAX, 2022)

Let $R_{sp}(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ be a strictly proper rational matrix. Let H and H_σ be the block Hankel matrices and $r_f := \operatorname{rank} H$. Let $U := \begin{bmatrix} U_1 & U_2 \end{bmatrix}$ and $V := \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ be unitary matrices such that

$$U^* H V = \begin{bmatrix} \widehat{H} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} U_1^* H V_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where $\widehat{H}$ is $r_f \times r_f$ and invertible. Partition the matrices U_1 and V_1 as

$$U_1 = \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix}, \quad \text{and} \quad V_1 = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix},$$

where the matrices U_{11} and V_{11} have dimension $m \times r_f$ and $n \times r_f$. Then

$$L_{sp}(\lambda) := \left[\begin{array}{c|c} U_1^* H_\sigma V_1 - \lambda \widehat{H} & \widehat{H} V_{11}^* \\ \hline U_{11} \widehat{H} & 0 \end{array} \right]$$

is a strongly minimal linearization for $R_{sp}(\lambda)$. Consider $\boldsymbol{U} = \boldsymbol{V}$ if $R_{sp}(\lambda)$ is Hermitian or skew-Hermitian.

Theorem (D, Quintana, Van Dooren, SIMAX, 2022)

Let $R(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ *be an arbitrary (resp. structured) rational matrix. Let*

$$R(\lambda) = P(\lambda) + R_{sp}(\lambda),$$

with $P(\lambda)$ *polynomial and* $R_{sp}(\lambda)$ *strictly proper. Let*

$$\widehat{L}_s(\lambda) := \left[\begin{array}{c|c} \widehat{A}_s(\lambda) & \widehat{B}_s(\lambda) \\ \hline -\widehat{C}_s(\lambda) & \widehat{D}_s(\lambda) \end{array}\right] \quad \text{and} \quad L_{sp}(\lambda) := \left[\begin{array}{c|c} A_{sp}(\lambda) & B_{sp}(\lambda) \\ \hline -C_{sp}(\lambda) & 0 \end{array}\right]$$

be (resp. structured) strongly minimal linearizations of $P(\lambda)$ *and* $R_{sp}(\lambda)$*, respectively. Then*

$$L(\lambda) := \left[\begin{array}{cc|c} \widehat{A}_s(\lambda) & 0 & \widehat{B}_s(\lambda) \\ 0 & A_{sp}(\lambda) & B_{sp}(\lambda) \\ \hline -\widehat{C}_s(\lambda) & -C_{sp}(\lambda) & \widehat{D}_s(\lambda) \end{array}\right]$$

is a (structured) strongly minimal linearization of $R(\lambda)$*.*

Outline

Conclusions

- We have introduced the new definition of strongly minimal linearizations.
- It is simultaneously valid for polynomial and rational matrices.
- We have proved that they have excellent recovery properties.
- We showed that they exist for any rational matrix
- and we have shown how to construct them via stable rank-revealing decompositions of constant matrices (SVD, for instance).
- Our constructions always preserve the Hermitian, skew-Hermitian, and alternating structures,
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