Strongly minimal linear polynomial system matrices of structured rational matrices

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Rational matrices and linear polynomial system matrices

- A rational matrix $R(t) \in \mathbb{C}(t)^{p\times m}$ is a matrix whose entries are univariate rational functions with coefficients in $\mathbb{C}$.
- Rational matrices play a fundamental role in systems and control theory, where they typically represent transfer functions of linear time invariant systems.
- Recently they have been also applied in the numerical solution of nonlinear eigenvalue problems, since they are used to approximate other more general nonlinear matrix functions.
- Relevant quantities of rational matrices, as their pole, zero and null space structures, are usually studied/computed trough linear polynomial system matrices related to them, i.e., through special pencils which are often called linearizations.
- Rational matrices appearing in applications often have particular structures. We consider in this talk some of such structures and linearizations that "try" to preserve these structures.

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Outline

1. Preliminaries on rational matrices
2. Strongly minimal linearizations of rational matrices
3. Structured rational matrices
4. Strongly minimal linearizations for (skew) Hermitian, even and odd
5. Strongly minimal linearizations related to para-(skew)-Hermitian
6. Conclusions

Outline

Local Smith-McMillan form

Let $\lambda_0 \in \mathbb{C}$. Any rational matrix $R(t) \in \mathbb{C}(t)^{p\times m}$ is **equivalent at λ_0** to a diagonal rational matrix of the form

$$\left[\begin{array}{ccc|c} (t-\lambda_0)^{\nu_1} & & & \\ & \ddots & & 0_{r\times(m-r)} \\ & & (t-\lambda_0)^{\nu_r} & \\ \hline & 0_{(p-r)\times r} & & 0_{(p-r)\times(m-r)} \end{array}\right] = U(t)R(t)V(t).$$

- $U(t)$ and $V(t)$ are rational matrices invertible at λ_0.
- The integers $\nu_1 \leq \cdots \leq \nu_r$ are the **invariant orders** at λ_0 of $R(t)$.
- The diagonal matrix is the **local Smith–McMillan form** of $R(t)$ at λ_0.
- $r = \operatorname{rank} R(t)$.

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Finite poles and zeros: Let $R(t) \in \mathbb{C}(t)^{p\times m}$ and $\lambda_0 \in \mathbb{C}$. Let

$$\nu_1 \leq \cdots \leq \nu_k < 0 = \nu_{k+1} = \cdots = \nu_{u-1} < \nu_u \leq \cdots \leq \nu_r$$

be the invariant orders at λ_0 of $R(t)$. Then λ_0 is

- a **pole** of $R(t)$ with **partial multiplicities** $-\nu_k, \ldots, -\nu_1$, if $k \geq 1$,
- a **zero** of $R(t)$ with **partial multiplicities** $\nu_u, \ldots, \nu_r$, if $u \leq r$.

Pole, zero and partial multiplicities at ∞ of $R(t)$ are those at 0 of $R\left(\frac{1}{t}\right)$.

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- **The pole structure** of $R(t)$ is the set of its poles together with their partial multiplicities.
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Poles and zeros of a rational matrix

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- A rational matrix $R(t)$ is **singular** when is either **rectangular or square with** $\det R(t) \equiv 0$.
- In addition to poles and zeros, **singular rational matrices have** other "important numbers" called **minimal indices**,
- which are related to the fact that a singular $R(t) \in \mathbb{C}(t)^{p\times m}$ has non-trivial left and/or right null spaces over the field $\mathbb{C}(t)$ of rational functions:

$$\begin{aligned} \mathcal{N}_\ell(R) &:= \left\{ y(t) \in \mathbb{C}(t)^p \; : \; y(t)^T R(t) = 0 \right\}, \\ \mathcal{N}_r(R) &:= \left\{ x(t) \in \mathbb{C}(t)^m \; : \; R(t)\, x(t) = 0 \right\}. \end{aligned}$$

- We skip the detailed definition of these concepts for brevity.

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Polynomial system matrices (Rosenbrock, 1970)

- Any rational matrix $R(t) \in \mathbb{C}(t)^{p\times m}$ can be written as

$$R(t) = D(t) + C(t)A(t)^{-1}B(t)$$

for some polynomial matrices $A(t) \in \mathbb{C}[t]^{n\times n}$, $B(t) \in \mathbb{C}[t]^{n\times m}$, $C(t) \in \mathbb{C}[t]^{p\times n}$ and $D(t) \in \mathbb{C}[t]^{p\times m}$ with $A(t)$ nonsingular.

- The polynomial matrix

$$S(t) = \begin{bmatrix} A(t) & B(t) \\ -C(t) & D(t) \end{bmatrix} \in \mathbb{C}[t]^{(n+p)\times(n+m)}$$

is called a **polynomial system matrix** of $R(t)$, i.e., $R(t)$ is the Schur complement of $A(t)$ in $S(t)$.

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Minimal polynomial system matrices

The polynomial system matrix

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is **minimal**, if the matrices

$$\begin{bmatrix} A(\lambda_0) & B(\lambda_0) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A(\lambda_0) \\ -C(\lambda_0) \end{bmatrix}$$

have, respectively, full row and column rank for all $\lambda_0 \in \mathbb{C}$.

Theorem (Rosenbrock, 1970)

Let

$$R(t) = D(t) + C(t)A(t)^{-1}B(t)$$

be the transfer function matrix of $S(t)$. If $S(t)$ is minimal, then

- *the finite pole structure of $R(t)$ = the finite zero structure of $A(t)$,*
- *the finite zero structure of $R(t)$ = the finite zero structure of $S(t)$.*

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Outline

Definition (D, Marcaida, Quintana, Van Dooren, LAA 2020)

Consider a linear polynomial system matrix

$$L(t) := \begin{bmatrix} tA_1 - A_0 & tB_1 - B_0 \\ -tC_1 + C_0 & tD_1 - D_0 \end{bmatrix} =: \begin{bmatrix} A(t) & B(t) \\ -C(t) & D(t) \end{bmatrix}.$$

such that

1. $L(t)$ is **minimal**, that is,

$$\begin{bmatrix} \lambda_0 A_1 - A_0 & \lambda_0 B_1 - B_0 \end{bmatrix}$$

and

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have, respectively, full row and column rank for all $\lambda_0 \in \mathbb{C}$, and,

2. $L(t)$ is **minimal at ∞**, that is,

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$L(t)$ is a **strongly minimal linearization** of $R(t) := D(t) + C(t)A(t)^{-1}B(t)$.

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Properties of strongly minimal linearizations

Theorem (D, Quintana, Van Dooren, SIMAX 2022)

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be a ***strongly minimal linearization*** *of*

$$R(t) := D(t) + C(t)A(t)^{-1}B(t).$$

Then

1. *The finite zero structure of $R(t)$ = the finite zero structure of $L(t)$.*
2. *The finite pole structure of $R(t)$ = the finite zero structure of $A(t)$.*
3. *The infinite pole and zero structure of $R(t)$ can be recovered from the infinite pole and zero structures of $L(t)$ and $A(t)$.*
4. *The left and right minimal indices of $R(t)$ and $L(t)$ are the same.*
5. *The eigenvectors and minimal bases of $R(t)$ can be easily recovered from those of $L(t)$ through block extraction.*

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Structured rational matrices: Poles, zeros, minimal indices symmetries

Symmetry with respect to the real line $\mathbb{R}$: $(\lambda, \overline{\lambda})$	Hermitian: $R^*(x) = R(x)$	Skew-Hermitian: $R^*(x) = -R(x)$
Symmetry with respect to the imaginary axis $i\mathbb{R}$: $(\lambda, -\overline{\lambda})$	Even: $R^*(s) = R(-s)$	Odd: $R^*(s) = -R(-s)$
Symmetry with respect to the unit circle S^1 : $(\lambda, 1/\overline{\lambda})$	Para-Hermitian: $R^*(z) = R(1/z)$	Para-skew-Hermitian: $R^*(z) = -R(1/z)$

- $R^*(t) = (R(\bar{t}))^*$.
- The symmetries of poles and zeros include partial multiplicities.
- For all these rational matrices the left minimal indices are equal to the right ones.

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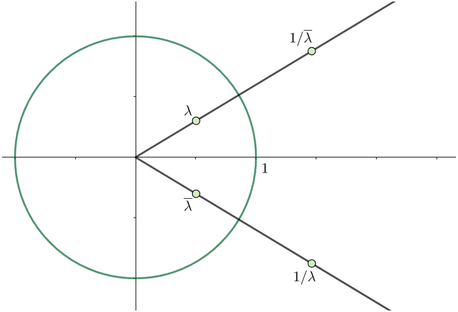

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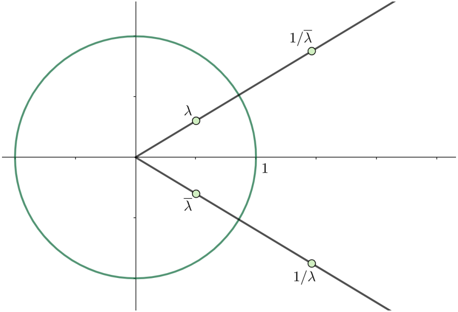

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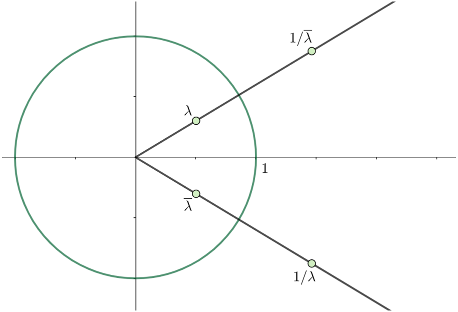
Symmetry with respect to the imaginary axis $i\mathbb{R}$: $(\lambda, -\overline{\lambda})$	Even: $R^*(s) = R(-s)$	Odd: $R^*(s) = -R(-s)$
Symmetry with respect to the unit circle S^1 : $(\lambda, 1/\overline{\lambda})$	Para-Hermitian: $R^*(z) = R(1/z)$	Para-skew-Hermitian: $R^*(z) = -R(1/z)$



- $R^*(t) = (R(\overline{t}))^*$.
- The symmetries of poles and zeros include partial multiplicities.
- For all these rational matrices the left minimal indices are equal to the right ones.

Preserving poles, zeros and minimal indices symmetries

- These **symmetries** are very important and should be **preserved** when computing the poles, zeros and minimal indices of a structured rational matrix $R(t)$.
- Such special structures occur in **numerous applications** in engineering, mechanics, control, ... For instance, para-Hermitian rational matrices are relevant in signal processing.
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Outline

Theorem (D, Quintana, Van Dooren, SIMAX 2022)

A rational matrix $R(t)$ is Hermitian (resp. skew-Hermitian or even or odd) if and only if there exists a strongly minimal Hermitian (resp. skew-Hermitian or even or odd) linearization of $R(t)$.

Moreover such **strongly minimal linearizations can be computed by performing unitary transformations on constant matrices**

- constructed from the coefficients of the Laurent expansion of $R(t)$ around the point at infinity or
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- **First obstacle:** Para-Hermitian (nonconstant) rational matrices $R^*(z) = R(1/z)$ do not have strongly minimal para-Hermitian linearizations, because there are no para-Hermitian pencils $L(z) = zL_1 + L_0$.
- **Possible solution:** Look for a class of structured pencils whose eigenvalues and minimal indices have the same symmetries of the poles, zeros and minimal indices of para-Hermitian matrices and try to linearize $R(z)$ with a pencil of this class.

Definition (Mackey, Mackey, Mehl, Mehrmann, SIMAX 2006)

A polynomial matrix $P(z)$ of degree d is **palindromic** if it satisfies

$$z^d P(1/z) = P^*(z).$$

In particular a pencil is palindromic if and only if $L(z) = zF + F^*$.

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The eigenvalues of palindromic polynomial matrices appear in pairs $(\lambda, 1/\overline{\lambda})$, i.e., they are symmetric with respect to S^1, and their left minimal indices are equal to the right ones.

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2^{nd} obstacle: para-Hermitian matrix cannot have palindromic system matrix

Theorem (D, Noferini, Quintana, Van Dooren, 2024)

The transfer function $H(z)$ of a palindromic linear system matrix $L(z)$ satisfies

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$$z\,H\,(1/z) = H^*(z) \iff \frac{1}{1+z}H(z) \text{ is para-Hermitian.}$$

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- Given a para-Hermitian rational matrix $R(z)$
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Invariant orders and minimal indices relations: $R(z)$ vs. $H(z) = (1+z)R(z)$

- For any finite $\lambda \neq -1$ the invariant orders of $H(z)$ and $R(z)$ at λ are the same.
- $\nu_1 \leq \cdots \leq \nu_r$ are the invariant orders at -1 of $H(z)$ if and only if $\nu_1 - 1 \leq \cdots \leq \nu_r - 1$ are the invariant orders at -1 of $R(z)$.
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We solve the problem via the following Möbius transform T and its inverse T^{-1}:

$$T: \quad x \longmapsto z = \frac{i-x}{i+x} \qquad \text{and} \quad T^{-1}: \quad z \longmapsto x = i\frac{1-z}{1+z}.$$

Remark: T maps $x \in \mathbb{R}$ to $T(x) \in S^1$ and T^{-1} maps $z \in S^1$ to $T^{-1}(z) \in \mathbb{R}$.

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Remarks on previous theorem

- The proof is constructive but due to the Möbius transform does not operate directly on constant matrices.
- For rational **para-Hermitian matrices $R(z)$ without poles on the unit circle S^1**, we see in the next slides how to construct strongly minimal palindromic linearizations of $(1+z)R(z)$ **without using Möbius transforms**.
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$$R(t) = R_{in}(t) + R_{out}(t) + R_{S^1}(t) + R_0,$$

- $R_{in}(t)$ *is a strictly proper rational matrix that has all its poles inside* S^1 *(stable part);*
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Constructing strongly minimal linearizations if no poles on the unit circle

If $R(z)$ **is para-Hermitian and has no poles on the unit circle**:

$R(z) = R_{in}(z) + R_{out}(z) + R_0$, with $R_{in}^*(z) = R_{out}(1/z)$ and $R_0^* = R_0$.

Theorem (D, Noferini, Quintana, Van Dooren, 2024)

Let $R(z)$ be a para-Hermitian rational matrix having no poles on the unit circle. Consider a minimal generalized state-space realization of $R_{in}(z)$:

$$R_{in}(z) = B(zA_1 - A_0)^{-1}C,$$

with A_1 invertible. Then,

$$R_{out}(z) = zC^*(A_1^* - zA_0^*)^{-1}B^*$$

is a minimal generalized state-space realization of $R_{out}(z)$, and $L(z)$ is a strongly minimal palindromic linearization of $(1+z)R(z)$:

$$L(z) = \left[\begin{array}{cc|c} 0 & A_0 - zA_1 & C \\ zA_0^* - A_1^* & 0 & B^*(1+z) \\ \hline zC^* & B(1+z) & R_0(1+z) \end{array}\right].$$

Constructing strongly minimal linearizations if no poles on the unit circle

If $R(z)$ **is para-Hermitian and has no poles on the unit circle**:

$$R(z) = R_{in}(z) + R_{out}(z) + R_0, \quad \text{with } R_{in}^*(z) = R_{out}(1/z) \text{ and } R_0^* = R_0.$$

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Outline

Conclusions

- We have shown that for Hermitian, skew-Hermitian, even and odd rational matrices, it is always possible to construct strongly minimal linearizations that preserve such structures.
- We have seen that for para-Hermitian (resp. para-skew-Hermitian) rational matrices $R(z)$ some unavoidable obstructions arise that make it impossible to construct strongly minimal linearizations that preserve such structure, but
- we have shown that it is always possible to construct strongly minimal palindromic (resp. anti-palindromic) linearizations of $(1+z)R(z)$.

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