# Polynomial and rational matrices with prescribed data

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# Collaborators

The results presented in this talk are based on the following joint works:

- Anguas, Dopico, Hollister, Mackey, Van Dooren's index sum theorem and rational matrices with prescribed structural data, *SIAM Journal on Matrix Analysis and Applications* **40**, (2019), 720–738.
- Baragaña, Dopico, Marcaida, Roca, Polynomial and rational matrices with the invariant rational functions and the four sequences of minimal indices prescribed, *in preparation*.
- De Terán, Dopico, Mackey, Van Dooren, Polynomial zigzag matrices, dual minimal bases, and the realization of completely singular polynomials, *Linear Algebra and its Applications* **488**, (2016), 460–504.
- De Terán, Dopico, Van Dooren, Matrix polynomials with completely prescribed eigenstructure, *SIAM Journal on Matrix Analysis and Applications* **36**, (2015), 302–328.

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# Outline

1. Preliminaries: Which are the data to be prescribed?
2. Goals of the talk
3. The Index Sum Theorems
4. Prescribed complete eigenstructures
5. Prescribed data with minimal indices of row and column spaces
6. Conclusions

# Outline

- In this talk, $\mathbb{F}$ is an algebraically closed field.
- If it helps, consider $\mathbb{F} = \mathbb{C}$.
- However, many of the results hold for arbitrary fields. This will be recalled in some important cases.
- Restricting to algebraically closed fields simplifies the statement of some key results and, I hope, makes it easier to understand the talk.

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# Rational matrices and polynomial matrices

- A rational matrix $R(s)$ is a matrix whose entries are univariate rational functions with coefficients in $\mathbb{F}$.
- A polynomial matrix $P(s)$ is a matrix whose entries are univariate polynomials with coefficients in $\mathbb{F}$.
- Polynomial are particular cases of rational matrices, but they are important on their own. The results in this talk were obtained first for polynomial matrices and will be presented in that order.
- Any rational matrix $R(s)$ can be uniquely expressed as

$$R(s) = P(s) + R_{sp}(s), \quad \text{where}$$

  - $P(s)$ is a polynomial matrix (polynomial part), and
  - the rational matrix $R_{sp}(s)$ is the strictly proper part, whose entries have numerators with smaller degrees than the denominators.
- **Unimodular matrices** are square polynomial matrices with constant nonzero determinant.
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- $\epsilon_1(s) \mid \cdots \mid \epsilon_r(s)$ and $\psi_r(s) \mid \cdots \mid \psi_1(s)$ are scalar monic polynomials,
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## Definition (finite zeros and finite poles)

Given the **Smith-McMillan form** of a rational matrix $R(s) \in \mathbb{F}(s)^{m \times n}$:

$$\text{diag}\left(\frac{\epsilon_1(s)}{\psi_1(s)}, \dots, \frac{\epsilon_r(s)}{\psi_r(s)}, 0_{(m-r)\times(n-r)}\right)$$

The **finite zeros** of $R(s)$ are the roots of the numerators and the **finite poles** are the roots of the denominators.

## Remark

Given any $c \in \mathbb{F}$, one can write for each $i = 1, \dots, r$,

$$\frac{\epsilon_i(s)}{\psi_i(s)} = (s-c)^{\sigma_i(c)} \frac{\widetilde{\epsilon}_i(s)}{\widetilde{\psi}_i(s)}, \qquad \text{with } \widetilde{\epsilon}_i(c) \neq 0, \widetilde{\psi}_i(c) \neq 0.$$

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# Example: invariant rational functions and invariant orders at finite values

The rational matrix

$$R(s) = \begin{bmatrix} \frac{s}{s-1} & & & & & \\ & \frac{1}{s-1} & & & & \\ & & (s-1)^2 & & & \\ & & & 1 & s^2 & \\ & & & & 1 & s^7 \end{bmatrix} \in \mathbb{C}(s)^{5\times 6}$$

has the Smith-McMillan form

$$R(s) \sim \begin{bmatrix} \frac{1}{s-1} & & & & & \\ & \frac{1}{s-1} & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & (s-1)^2 s & 0 \end{bmatrix},$$

and the invariant orders (rank$(R) = 5$)

- $S(R, 1) = (-1, -1, 0, 0, 2)$ (pole and zero),
- $S(R, 0) = (0, 0, 0, 0, 1)$ (zero),
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If $P(s)$ is a polynomial matrix, the denominators of its invariant rational functions are all $1$ and the Smith-McMillan form reduces to the Smith form.

**Definition**

The **Smith form** of a polynomial matrix $P(s) \in \mathbb{F}[s]^{m\times n}$ is the following diagonal matrix obtained under unimodular transformations $U(s)$ and $V(s)$:

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- $\alpha_1(s) \mid \cdots \mid \alpha_r(s)$ are monic scalar polynomials (**invariant factors**).

- The invariant orders at $c \in \mathbb{F}$ are always nonnegative, are called the partial multiplicities at $c$, and the zeros of the invariant factors are called the **finite eigenvalues**.

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# The Smith form of a Polynomial Matrix

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The polynomial matrix

$$P(s) = \begin{bmatrix} 1 & & & \\ & s-1 & & \\ & & (s-1)(s+3)^2 & \\ & & & 0 \end{bmatrix} \in \mathbb{C}[s]^{4\times 4}$$

is already in Smith form.

- Invariant factors of $P(s)$:

$$\alpha_1(s) = 1,\ \alpha_2(s) = (s-1),\ \alpha_3(s) = (s-1)(s+3)^2.$$

- Finite eigenvalues of $P(s)$: $1, -3$.
- Partial multiplicities: $(\text{rank}(P) = 3)$
    - $S(P, 1) = (0, 1, 1)$,
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# Example: invariant factors and partial multiplicities at finite values

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**Definition**

The invariant orders of a rational matrix $R(s)$ at $\infty$ are the invariant orders of $R\left(\frac{1}{s}\right)$ at $s = 0$.

**Proposition: The smallest invariant order at infinity (Amparan, Marcaida, Zaballa, ELA 2015)**

The smallest invariant order of $R(s)$ at infinity is

1. $-$degree (polynomial part of $R(s)$), if this polynomial part is nonzero,
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**Remark:** Though polynomial matrices are rational matrices and their invariant orders at $\infty$ can be defined as in the previous slide, the literature on polynomial matrices uses another equivalent but different definition for the structure at $\infty$ and use another name.

**Definition**

Let

$$P(s) = P_d s^d + P_{d-1} s^{d-1} + \cdots + P_0, \qquad P_d \neq 0,$$

be a **polynomial matrix of degree** $d$. The **reversal** of $P(s)$ is

$$\mathrm{rev} P(s) := s^d P\left(\frac{1}{s}\right) = P_d + P_{d-1}\, s + \cdots + P_0\, s^d\,.$$

**Definition (Eigenvalue and partial mutiplicities at $\infty$)**

1. The partial multiplicities of $P(s)$ at $\infty$ are the partial multiplicities of $\mathrm{rev} P(s)$ at $0$.
2. **$\infty$ is an eigenvalue of $P(s)$ if $0$ is an eigenvalue of $\mathrm{rev} P(s)$.**

**Remark:** Though polynomial matrices are rational matrices and their invariant orders at $\infty$ can be defined as in the previous slide, the literature on polynomial matrices uses another equivalent but different definition for the structure at $\infty$ and use another name.

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$$P(s) = P_d s^d + P_{d-1} s^{d-1} + \cdots + P_0, \qquad P_d \neq 0,$$

be a **polynomial matrix of degree** $d$. The **reversal** of $P(s)$ is

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2. The smallest partial multiplicity of $P(s)$ at $\infty$ is zero.

**Comments**

- The reason of (2) is that $\mathrm{rev}P(0) = P_d \neq 0$.
- The trivial restriction (2) on the smallest partial multiplicity at $\infty$ of a polynomial matrix is a consequence of the definition and will appear in several of the results we will present.

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In this talk:

- $\mathbb{F}[s]$ is the ring of univariate polynomials with coefficients in $\mathbb{F}$.
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A **minimal basis** of a rational subspace $\mathcal{V} \in \mathbb{F}(s)^n$ is a basis

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I have attributed these concepts to Forney (1975) and I have presented them in the way he did but:

- Kailath in p. 460 of *Linear Systems (Prentice Hall, 1980)* wrote
  *"I.C. Gohberg pointed out to the author that the significance of minimal bases was perhaps first realized by J. Plemelj in 1908 and then substantially developed in 1943 by N.I. Mushkelishvili and N.P. Vekua ... These authors were studying the ... Riemann-Hilbert problem, which was ... related to ... Wiener-Hopf integral equations, as described ... in the definitive paper of Gohberg and Krein (1958)",*
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An $m \times n$ rational matrix $R(s) \in \mathbb{F}(s)^{m \times n}$ whose rank $r$ is smaller than $m$ and/or $n$ has non-trivial left and/or right rational null spaces (over the field $\mathbb{F}(s)$ of rational functions):

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Regular polynomial and rational matrices are very important in many applications, but they are not in the center of this talk.

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# Example of left/right minimal bases and indices of a rational matrix

$$P(s) = \begin{bmatrix} 1 & -s^3 & & & \\ & & 1 & -s & \\ & & & 1 & -s \end{bmatrix} \in \mathbb{C}(s)^{3\times 5}$$

$$\mathcal{N}_r(P) = \text{Span}\{ \underbrace{\begin{bmatrix} s^3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ s^2 \\ s \\ 1 \end{bmatrix}}_{u_2} \} = \text{Span}\{ \underbrace{\begin{bmatrix} s^3 \\ 1 \\ s^3 \\ s^2 \\ s \end{bmatrix}}_{w_1}, \underbrace{\begin{bmatrix} s^5 \\ s^2 \\ s^2 \\ s \\ 1 \end{bmatrix}}_{w_2} \}$$

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## Definition

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$$\frac{\epsilon_1(s)}{\psi_1(s)}, \cdots, \frac{\epsilon_r(s)}{\psi_r(s)} \quad \text{(finite pole/zero structure)},$$

**(ii)** the invariant orders at $\infty$ $\quad q_1 \leq \cdots \leq q_r \quad$ (infinite pole/zero structure),

**(iii)** the right minimal indices $\quad d_1 \geq \cdots \geq d_{n-r} \quad$ (right singular structure),

**(iv)** the left minimal indices $\quad v_1 \geq \cdots \geq v_{m-r} \quad$ (left singular structure).

**Remark 1:** the name "eigenstructure" comes from Van Dooren (PhD Thesis, 1979) and has been used by Van Dooren in several papers since then. Other authors use zero, pole and nullspace (or singular) structures (Kailath, 1980).

**Remark 2:** given the complete eigenstructure, one can recover the rank and the size.

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$$\frac{\epsilon_1(s)}{\psi_1(s)}, \cdots, \frac{\epsilon_r(s)}{\psi_r(s)} \quad \text{(finite pole/zero structure)},$$

**(ii)** the invariant orders at $\infty$ $\quad q_1 \leq \cdots \leq q_r \quad$ (infinite pole/zero structure),

**(iii)** the right minimal indices $\quad d_1 \geq \cdots \geq d_{n-r} \quad$ (right singular structure),

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**Remark 2:** given the complete eigenstructure, one can recover the rank and the size.

# Complete eigenstructure of a rational matrix

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## Complete eigenstructure of a polynomial matrix

**Definition**

Given a polynomial matrix $P(s) \in \mathbb{F}[s]^{m\times n}$ with rank $r$, the complete eigenstructure of $P(s)$ consists of the following lists

- **(i)** the invariant factors $\alpha_1(s) \mid \cdots \mid \alpha_r(s)$ (finite eigenvalue structure),
- **(ii)** the partial multiplicities at $\infty$ $f_1 \leq \cdots \leq f_r$ (infinite eigenvalue structure),
- **(iii)** the right minimal indices $d_1 \geq \cdots \geq d_{n-r}$ (right singular structure),
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**Remark 1:** If the degree of $P(s)$ is one, i.e., $P(s)$ is a pencil, then its complete eigenstructure is revealed by the sizes of the blocks of the Kronecker Canonical Form under strict equivalence.

**Remark 2:** Such a form does not exist for rational and polynomial matrices (of degree larger than one), which makes it challenging the problems considered in this talk. No canonical form reveals the complete eigenstructure.

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# More structure: Row and column spaces

- The structural data described above have received a lot of attention in the literature on polynomial and rational matrices,
- but, as we teach in basic Linear Algebra courses, every matrix has four fundamental subspaces
- and, so far, we have only used two: the left and right null spaces.
- Given a rational matrix $R(s) \in \mathbb{F}(s)^{m\times n}$ the other two are

$$\mathcal{R}ow(R) = \{R(s)^T w(s) \, : \, w(s) \in \mathbb{F}(s)^m\} \subseteq \mathbb{F}(s)^n \,,$$
$$\mathcal{C}ol(R) = \{R(s) v(s) \, : \, v(s) \in \mathbb{F}(s)^n\} \subseteq \mathbb{F}(s)^m \,,$$

- which have minimal bases and indices as any other rational subspace.
- Thus a rational matrix has four sequences of minimal indices.
- As far as we know, minimal bases and indices of row and column spaces have not received much attention in the literature, but are fundamental for constructing rank revealing factorizations of polynomial matrices with good properties (Dmytryshyn, D, Van Dooren, submitted, 2023).

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## Example of minimal bases and indices of $\mathcal{R}ow(R)$ and $\mathcal{C}ol(R)$

$$P(s) = \begin{bmatrix} 1 & -s^3 & & & \\ & & 1 & -s & \\ & & & 1 & -s \end{bmatrix} \in \mathbb{C}(s)^{3\times 5}$$

- Since $\operatorname{rank} P(s) = 3$, $\mathcal{C}ol(P) = \mathbb{C}(s)^3$, $\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$ is a minimal basis of $\mathcal{C}ol(P)$ and, so, the **minimal indices of $\mathcal{C}ol(P) = 0, 0, 0$**.
- **Remark: the columns of $I_n$ are a minimal basis of $\mathbb{F}(s)^n$ which, therefore, has $n$ minimal indices equal to $0$.**
- $\mathcal{R}ow(P) = \text{Span}\{ \begin{bmatrix} 1 \\ -s^3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -s \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -s \end{bmatrix} \}$,
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# Outline

1. If a complete eigenstructure is prescribed and **a degree $d$ is also prescribed**, to find necessary and sufficient conditions for the existence of a polynomial matrix with precisely this complete eigenstructure and **this degree**.

2. If a complete eigenstructure is prescribed, to find necessary and sufficient conditions for the existence of a rational matrix with precisely this complete eigenstructure.

3. Consider necessary and sufficient conditions for the existence problems above when the minimal indices of the row and column spaces are prescribed

   1. instead of the minimal indices of the right and left null spaces

   or

   2. in addition to the complete eigenstructure.

# Goals of the talk

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## Remarks

- All these problems are inverse eigenstructure problems.
- In the polynomial case if the prescribed degree is $1$ (pencil case!!), then the first problem is easy: write the prescribed structure (after factorizing the invariant factors as products of irreducible polynomials) in terms of the corresponding blocks of the Kronecker canonical form of pencils and check if they adjust the prescribed rank and size.
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# Other works on prescribed degree inverse polynomial eigen-problems

- Marques de Sá, LAA 1979: regular polys with nonsingular leading coefficient, arbitrary degree.
- Gohberg, Lancaster, Rodman, Matrix Polynomials (book), 1982: regular monic polys, arbitrary degree.
- Lancaster, SIMAX 2007: regular, degree 2, some symmetries, e-vectors.
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**Theorem**

*Let $P(s) \in \mathbb{F}[s]^{m\times n}$ be a polynomial matrix of degree $d$ and normal rank $r$, with*

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**(iii)** *right minimal indices* $d_1 \geq \cdots \geq d_{n-r}$,

**(iv)** *left minimal indices* $v_1 \geq \cdots \geq v_{m-r}$.

*Then,*

$$\sum_{i=1}^{r} \deg(\alpha_i) + \sum_{i=1}^{r} f_i + \sum_{i=1}^{n-r} d_i + \sum_{i=1}^{m-r} v_i = rd.$$

## Theorem

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$$\sum_{i=1}^{r} \deg(\epsilon_i) - \sum_{i=1}^{r} \deg(\psi_i) + \sum_{i=1}^{r} q_i + \sum_{i=1}^{n-r} d_i + \sum_{i=1}^{m-r} v_i = 0.$$

# Remark

- Though both results seem different,
- this is due to the fact that the polynomial one uses partial multiplicities at $\infty$, $f_1 \leq \cdots \leq f_r$,
- and the rational one uses invariant orders at $\infty$, $q_1 \leq \cdots \leq q_r$.
- The Polynomial Index Sum Theorem is a particular case of the Rational Index Sum Theorem, as it can be easily seen from the relation

$$f_i = q_i + d, \quad \text{for } i = 1, \ldots, r.$$

# Outline

**Theorem**

*There exists a polynomial matrix $P(s) \in \mathbb{F}[s]^{m\times n}$ with degree $d$ and normal rank $r \le \min\{m,n\}$, and with*

(i) *invariant factors* $\alpha_1(s) \mid \cdots \mid \alpha_r(s)$,

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2. *and*
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## The polynomial result

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- This result was proved by De Terán, D, Van Dooren, SIMAX 2015, for infinite fields $\mathbb{F}$
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- which uses previous works by Dodig, Stošić, SIMAX 2019 on completions of pencils.
- The proof from 2015 is constructive (though long and nontrivial), as well as the one from 2024, though in this case one has to dig hard into the proofs of previous results in several papers by Dodig, Stošić.
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**Theorem (Anguas, D, Hollister, Mackey, SIMAX 2019)**

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## Comments

- The rational result follows from the polynomial result.
- The key idea to see this is that any rational matrix $R(s)$ can be expressed as

$$R(s) = \frac{1}{\psi_1(s)} P(s),$$

  where $P(s)$ is polynomial and $\psi_1(s)$ is the least common denominator of the entries of $R(s)$, which is also the denominator of its first invariant rational function.
- Then, it is possible to transform the rational prescribed data into polynomial prescribed data, realize such data with a polynomial matrix and obtain the desired rational matrix with the formula above.

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# Outline

**Theorem (consequence of Forney, SIAM J. Control 1975)**

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This immediately implies a...

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## A dual version of the Index Sum Theorem for Polynomial Matrices

### Theorem

*Let $P(s) \in \mathbb{F}[s]^{m\times n}$ be a polynomial matrix of degree $d$ and normal rank $r$, with*

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*Then,*

$$\sum_{i=1}^{r} \deg(\alpha_i) + \sum_{i=1}^{r} f_i + \sum_{i=1}^{r} \ell_i + \sum_{i=1}^{r} k_i = rd.$$

**Remark:** This result hints a neat solution for the problem of the existence of a polynomial matrix with all these data prescribed, but this dream is too beautiful to be true.

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**Remark:** This result hints a neat solution for the problem of the existence of a polynomial matrix with all these data prescribed, but this dream is too beautiful to be true.

## Definition

Let

$$\mathbf{a} = (a_1 \geq \cdots \geq a_m) \quad \text{and} \quad \mathbf{b} = (b_1 \geq \cdots \geq b_m)$$

be two decreasingly ordered sequences of integers.

It is said that $\mathbf{a}$ is *majorized* by $\mathbf{b}$, denoted by

$$\mathbf{a} \prec \mathbf{b},$$

if

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i \quad \text{for } 1 \leq k \leq m-1 \text{ and}$$

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} b_i.$$

**Theorem (Baragaña, D, Marcaida, Roca, in preparation 2024)**

*There exists a polynomial matrix $P(s) \in \mathbb{F}[s]^{m \times n}$ with degree $d$ and normal rank $r < \min\{m, n\}$, with*

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**if and only if**

**(1)**

$$(d - g_r, \ldots, d - g_1) \prec (\deg(\alpha_r) + f_r, \ldots, \deg(\alpha_1) + f_1),$$

*where $g_1 \geq \cdots \geq g_r$ is the decreasing reordering of $k_r + \ell_1, \ldots, k_1 + \ell_r$,*

**(2)** *and $f_1 = 0$.*

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**(2)** *and $f_1 = 0$.*

## Remarks on previous theorem

- The majorization condition amounts in fact to $r$ conditions. The last of such conditions is the Index Sum Theorem, i.e., the unique condition appearing in the result for prescribing the complete eigenstructure.
- If $r = \min\{m, n\}$, we have to add two trivial conditions to (1) and (2):

  **(3)** $\ell_1 = \cdots = \ell_r = 0$ if $r = n$

  (coming from $\mathcal{R}ow(P) = \mathbb{F}(s)^n$ in this case),

  **(4)** $k_1 = \cdots = k_r = 0$ if $r = m$

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- The proof of the necessity if valid over arbitrary fields. The proof of the sufficiency requires algebraically closed fields in an essential way and we have examples over $\mathbb{R}$ for which the theorem does not hold.

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# Prescribed minimal indices of *Row/Col* instead of *right/left Nulls*: Rational

**Theorem (Baragaña, D, Marcaida, Roca, in preparation 2024)**

*There exists a rational matrix $R(s) \in \mathbb{F}(s)^{m \times n}$ with normal rank $r < \min\{m, n\}$, with*

- **(i)** *invariant rational functions* $\frac{\epsilon_1(s)}{\psi_1(s)}, \cdots, \frac{\epsilon_r(s)}{\psi_r(s)}$,
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**if and only if**

$$(-g_r, \ldots, -g_1) \prec (\deg(\epsilon_r) - \deg(\psi_r) + q_r, \ldots, \deg(\epsilon_1) - \deg(\psi_1) + q_1),$$

*where $g_1 \geq \cdots \geq g_r$ is the decreasing reordering of $k_r + \ell_1, \ldots, k_1 + \ell_r$.*

## Theorem (Baragaña, D, Marcaida, Roca, in preparation 2024)

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- **(2)** *and* $\sum_{i=1}^{m-r} v_i = \sum_{i=1}^{r} k_i$ and $\sum_{i=1}^{n-r} d_i = \sum_{i=1}^{r} \ell_i$.

## Remarks

- With respect to the results in the last three slides:
  - If $r = \min\{m, n\}$, we have to add two trivial conditions:
    1. $\ell_1 = \cdots = \ell_r = 0$ if $r = n$

       (coming from $\mathcal{R}ow(P) = \mathbb{F}(s)^n$ in this case),
    2. $k_1 = \cdots = k_r = 0$ if $r = m$

       (coming from $\mathcal{C}ol(P) = \mathbb{F}(s)^m$ in this case).
  - The proof of the necessity if valid over arbitrary fields. The proof of the sufficiency requires algebraically closed fields.
- With respect to the results that prescribe the minimal indices of the row and column spaces **instead of** the minimal indices of the left and right null spaces, the same necessary and sufficient conditions hold if **minimal bases of the row and column spaces are prescribed** and not just their minimal indices.

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# Outline

# Conclusions

- We have provided necessary and sufficient conditions for the existence of polynomial and rational matrices when their **classical complete eigenstructures** are prescribed.
- We have provided necessary and sufficient conditions for the existence of polynomial and rational matrices when **instead of or in addition to** their **classical complete eigenstructures** the minimal indices of their row and column spaces are prescribed.
- The obtained necessary and sufficient conditions are very simple and only require to check some equalities or inequalities of integer numbers related to the prescribed data, the prescribed degree and rank.
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