Beyond Rosenbrock's Theorem

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joint work with **Vanni Noferini** (Aalto University, Finland)
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Minisymposium Developments in Structured Matrices

On the occasion of Steve Mackey's 70th birthday

SIAM Conference on Applied Linear Algebra (LA24)

Sorbonne Université. Paris, France. May 13-17, 2024

GOBIERNO DE ESPAÑA MINISTERIO DE CIENCIA E INNOVACIÓN

AGENCIA ESTATAL DE INVESTIGACIÓN

uc3m | Universidad Carlos III de Madrid

Happy (anticipated) 70th Birthday Steve!! (July 7th, San Fermín Day!!)

At Householder Symposium XVII on Numerical Linear Algebra, Zeuthen, Germany, 1-6 June 2008.

Thanks a lot for many years of a great collaboration and friendship

- We met in person at the Joint GAMM-SIAM Conference on Applied Linear Algebra, Düsseldorf, Germany, 24-27 July 2006.
- Nick Higham introduced each other during the reception in Düsseldorf and we had some interesting discussions during that conference.
- Then, I invited Steve and Nil (Kamela also joined some time) to visit UC3M for one month in June-July 2007.
- Since then:
  - 9 published joint papers, 7 of them with Fernando, others with some of our PhD Students (Vasilije, Luismi, Richard),
  - 8 more visits of Steve to UC3M (+ 1 more the next two weeks),
  - 1 visit from Froilán to Kalamazoo (2016),
  - many long and thoughtful emails,
  - several papers in preparation,
- **and, more important, many good moments together.**

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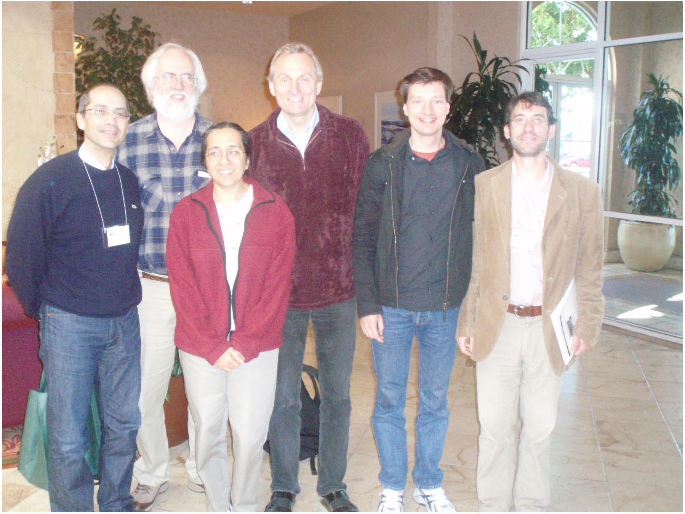
SIAM Conference on Applied Linear Algebra, Monterey, California. 26-29 October 2009.



Visiting the Great Gatsby's Mansion

18th Conference of the International Linear Algebra Society, Providence, Rhode Island, 3-7 June 2013.

Working together in Steve and Nil's living room at Kalamazoo and visiting Lake Michigan (2016)

Visit to Kalamazoo, 6-18 May 2016 ...

and much more in the past and much more to come!! Thanks, Steve!

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$\mathfrak{R}$ is an EDD and $\mathbb{F}$ its field of fractions.

Theorem (Rosenbrock's Theorem over EDDs)

Let $A \in \mathfrak{R}^{n\times n}$, $B \in \mathfrak{R}^{n\times m}$, $C \in \mathfrak{R}^{p\times n}$ *and* $D \in \mathfrak{R}^{p\times m}$ *with* $\det A \neq 0$. *Let*

$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{R}^{(n+p)\times(n+m)}, \qquad G = D - CA^{-1}B \in \mathbb{F}^{p\times m}, \qquad r = \operatorname{rank} G.$$

Assume that A *and* B *are left coprime and that* A *and* C *are right coprime. If the Smith-McMillan form of* G *is*

$$S_G \doteq \operatorname{Diag}\left(\frac{\varepsilon_1}{\psi_1}, \dots, \frac{\varepsilon_r}{\psi_r}\right) \oplus 0_{(p-r)\times(m-r)} \in \mathbb{F}^{p\times m},$$

and g *is the largest index in* $\{1, \dots, r\}$ *such that* $\psi_g \notin U(\mathfrak{R})$, *then the Smith forms of* P *and* A *are, respectively,*

$$S_P \doteq I_n \oplus \operatorname{Diag}(\varepsilon_1, \dots, \varepsilon_r) \oplus 0_{(p-r)\times(m-r)} \in \mathfrak{R}^{(n+p)\times(n+m)},$$

and

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Assume that A and B are left coprime and that A and C are right coprime. *If the Smith-McMillan form of G is*

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- **Goal: What happens when A and B are not left coprime or A and C are not right coprime?** We want to investigate the relations between the Smith-McMillan form of G and the Smith forms of A and P when the coprimeness assumptions do not hold.
- **Motivation:**
  - In general, it is not always easy to check if the coprimeness conditions hold.
  - Some works about the numerical solution of Nonlinear Eigenvalue Problems have used linear polynomial system matrices without guarateeing the coprimeness conditions.
- **Nomenclature:** System matrices for which the coprimeness conditions hold are said to be *minimal*, or *of least order*, or *irreducible*.

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Theorem

Let $G_1 \in \mathfrak{R}^{p\times m}$ *and* $G_2 \in \mathfrak{R}^{q\times m}$, $p+q \geq m$. *The following are equivalent:*

i) G_1 *and* G_2 *are right coprime in* $\mathfrak{R}$, *i.e., every common right divisor is unimodular.*

ii) *The Smith form over* $\mathfrak{R}$ *of* $\left[\begin{smallmatrix} G_1 \\ G_2 \end{smallmatrix}\right]$ *is* $\left[\begin{smallmatrix} I_m \\ 0 \end{smallmatrix}\right]$.

iii) *There exists a unimodular matrix* $U \in \mathfrak{R}^{(p+q)\times(p+q)}$ *such that* $U\left[\begin{smallmatrix} G_1 \\ G_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} I_m \\ 0 \end{smallmatrix}\right]$.

iv) *There exist matrices* $C \in \mathfrak{R}^{p\times(p+q-m)}$, $D \in \mathfrak{R}^{q\times(p+q-m)}$ *such that* $\left[\begin{smallmatrix} G_1 & C \\ G_2 & D \end{smallmatrix}\right]$ *is unimodular.*

v) *There exist matrices* $X \in \mathfrak{R}^{m\times p}$, $Y \in \mathfrak{R}^{m\times q}$ *such that* $XG_1 + YG_2 = I_m$.

The polynomial matrices $G_1(z) \in \mathbb{C}[z]^{p\times m}, G_2(z) \in \mathbb{C}[z]^{q\times m}$ are right coprime if and only if

$$\operatorname{rank}\begin{bmatrix} G_1(z_0) \\ G_2(z_0) \end{bmatrix} = m, \quad \forall z_0 \in \mathbb{C}.$$

Necessary and sufficient conditions for coprimeness

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Key auxiliary result: from reducible to irreducible system matrices

Theorem (D, Noferini, Zaballa, 2024)

Let $A \in \mathfrak{R}^{n\times n}$, $B \in \mathfrak{R}^{n\times m}$, $C \in \mathfrak{R}^{p\times n}$ *and* $D \in \mathfrak{R}^{p\times m}$ *with* $\det A \neq 0$. *If* A *and* B **are not left coprime** *or* A *and* C **are not right coprime**, *then there exist matrices* $A_0 \in \mathfrak{R}^{n\times n}$, *with* $\det A_0 \neq 0$, $B_0 \in \mathfrak{R}^{n\times m}$, $C_0 \in \mathfrak{R}^{p\times n}$, $E \in \mathfrak{R}^{n\times n}$ *and* $F \in \mathfrak{R}^{n\times n}$ *such that*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} E & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} A_0 & B_0 \\ C_0 & D \end{bmatrix} \begin{bmatrix} F & 0 \\ 0 & I_m \end{bmatrix}$$

and

i) A_0 *and* B_0 **are left coprime** *and* A_0 *and* C_0 **are right coprime**;

ii) $\det E \neq 0$, $\det F \neq 0$, *and at least one of these determinants is not a unit of* $\mathfrak{R}$;

iii) $D - CA^{-1}B = D - C_0 A_0^{-1} B_0$, **i.e., Schur complement does not change!!**

Essential idea

Extract the "largest" possible **nonunimodular** common left and right divisors E and F.

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This factorization can be combined with

1. The fact that Rosenbrock's Theorem holds for $\begin{bmatrix} A_0 & B_0 \\ C_0 & D \end{bmatrix}$.

2. **Proposition.** *Let* $A_1 \in \mathfrak{R}^{m\times n}$, $A_2 \in \mathfrak{R}^{n\times p}$ *and let* $A = A_1 A_2$. *Let* $\alpha_1^{(1)} \mid \cdots \mid \alpha_{r_1}^{(1)}$, $\alpha_1^{(2)} \mid \cdots \mid \alpha_{r_2}^{(2)}$ *and* $\alpha_1 \mid \cdots \mid \alpha_r$ *be the invariant factors of* A_1, A_2 *and* A, *respectively. Then* $\alpha_k^{(j)} \mid \alpha_k$ *for* $j = 1, 2$ *and* $k = 1, \ldots, r$.

   In words: Invariant factors of matrix factors divide the invariant factors of the product.

3. The classical expression of the minors of the Schur complement in terms of the minors of the whole matrix and $\det A$.

for proving ...

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and **assume that A and B are not left coprime or that A and C are not right coprime**. *Let*

$$S_G \doteq \mathrm{Diag}\left(\frac{\varepsilon_1}{\psi_1}, \ldots, \frac{\varepsilon_r}{\psi_r}\right) \oplus 0_{(p-r)\times(m-r)} \in \mathbb{F}^{p\times m},$$

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be the Smith-McMillan form of G and the Smith forms of A and P, respectively. Let g be the largest index in $\{1, \ldots, r\}$ such that $\psi_g \notin U(\mathfrak{R})$. Then

i) $n \geq g$ *and* $\psi_i \mid \widetilde{\psi}_i$, *for* $i = 1, \ldots, g$;

ii) $\dfrac{\widetilde{\psi}_n \cdots \widetilde{\psi}_2\, \widetilde{\psi}_1}{\psi_g \cdots \psi_2\, \psi_1} \notin U(\mathfrak{R})$;

Main Theorem (II)

Theorem (continuation)

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v) *if G and P are square and nonsingular, then*

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Remark

In general, $\dfrac{\widetilde{\psi}_n\cdots\widetilde{\psi}_2\,\widetilde{\psi}_1}{\psi_g\cdots\psi_2\,\psi_1}$ can be any element in the ring!!!, but in practice ...

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- One of the reasons why it is not easy to work on general EDDs is because they are not, in general, Unique Factorization Domains (UFD),
- i.e., we cannot assume that their elements have a unique factorization into prime elements.
- In particular, the invariant factors of the Smith forms of matrices over EDDs cannot be uniquely factorized into prime elements and "elementary divisors" cannot be defined.
- Thus, for matrices in general EDDs, we loose one of the fundamental concepts/tools of matrix polynomials: the elementary divisors.
- Moreover, not every UFD is an EDD,
- but if $\mathfrak{R}$ is a PID, then it is simultaneously an EDD and a UFD.
- PIDs include the ring of integers and rings of polynomials in one variable with coefficients in a field.

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- Moreover, not every UFD is an EDD,
- but if $\mathfrak{R}$ is a PID, then it is simultaneously an EDD and a UFD.
- PIDs include the ring of integers and rings of polynomials in one variable with coefficients in a field.

Improving the "numerator" part over Principal Ideal Domains (PIDs)

- One of the reasons why it is not easy to work on general EDDs is because they are not, in general, Unique Factorization Domains (UFD),
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Reminder: Elementary Divisors of Matrices over a PID $\mathfrak{R}$

Let $A \in \mathfrak{R}^{p\times m}$ with Smith form

$$S_A \doteq \mathrm{Diag}\,(\alpha_1, \dots, \alpha_r) \oplus 0_{(p-r)\times(m-r)} \in \mathfrak{R}^{p\times m}.$$

We can write

$$\begin{aligned} \alpha_1 &= \beta_1^{e_{11}} \beta_2^{e_{12}} \cdots \beta_\ell^{e_{1\ell}}, \\ \alpha_2 &= \beta_1^{e_{21}} \beta_2^{e_{22}} \cdots \beta_\ell^{e_{2\ell}}, \\ &\vdots \qquad\qquad \vdots \\ \alpha_r &= \beta_1^{e_{r1}} \beta_2^{e_{r2}} \cdots \beta_\ell^{e_{r\ell}}, \end{aligned}$$

- where $\beta_1, \dots, \beta_\ell$ are prime elements of $\mathfrak{R}$ and e_{ij} are nonnegative integers that satisfy $0 \leq e_{1j} \leq e_{2j} \leq \cdots \leq e_{rj}, \quad j = 1, \dots, \ell.$
- The factors $\beta_j^{e_{ij}}$ with $e_{ij} > 0$ are called the elementary divisors of A.
- The sequence of *partial multiplicities* of A at any prime $\pi \in \mathfrak{R}$ is the sequence of the **positive integers** t_i such that $\alpha_i = \pi^{t_i} \gamma_i$ with $\gamma_i \in \mathfrak{R}$, and $\gcd(\pi, \gamma_i) \doteq 1$ for $i = 1, \dots, r$.
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Theorem (D, Noferini, Zaballa, 2024)

Let $\mathfrak{R}$ be a PID and $\mathbb{F}$ its field of fractions. Let $A \in \mathfrak{R}^{n\times n}$, $B \in \mathfrak{R}^{n\times m}$, $C \in \mathfrak{R}^{p\times n}$ and $D \in \mathfrak{R}^{p\times m}$ with $\det A \neq 0$,

$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{R}^{(n+p)\times(n+m)}, \quad \text{and} \quad G = D - CA^{-1}B \in \mathbb{F}^{p\times m}.$$

Let

$$S_G \doteq \operatorname{Diag}\left(\frac{\varepsilon_1}{\psi_1}, \ldots, \frac{\varepsilon_r}{\psi_r}\right) \oplus 0_{(p-r)\times(m-r)} \in \mathbb{F}^{p\times m}$$

be the Smith-McMillan form of G and g be the largest index in $\{1, \ldots, r\}$ such that $\psi_g \notin U(\mathfrak{R})$. If $\pi \in \mathfrak{R}$ is prime and

$$\gcd\left(\pi, \frac{\det A}{\psi_g \cdots \psi_2 \psi_1}\right) \doteq 1,$$

then the sequence of the partial multiplicities of P at π is equal to the sequence of the partial multiplicities of $\operatorname{Diag}(\varepsilon_1, \ldots, \varepsilon_r)$ at π.

Remarks

- This result holds under the more restrictive (but easier to verify) condition $\gcd(\pi\,,\,\det A) \doteq 1,$
- since this implies that $\gcd\left(\pi\,,\,\dfrac{\det A}{\psi_g\cdots\psi_2\,\psi_1}\right) \doteq 1.$
- Thus, if (1) we know the prime divisors of $\det A$ and (2) we are not interested in the possible elementary divisors of the Smith-McMillan numerators of G at that primes, then
- using non-minimal system matrices is safe.
- This was in fact the case in S. Güttel, R. Van Beeumen, K. Meerbergen, W. Michiels, "NLEIGS: a class of fully rational Krylov methods for nonlinear eigenvalue problems", *SIAM J. Sci. Comput.*, (2014).

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- We have also investigated the relations between the Smith-McMillan form of G and the Smith-McMillan forms of A and P.
- We have obtained results in the same spirit of Rosenbrock's Theorem, though they require some additional hypotheses, in addition to the coprimeness, and are more cumbersome.
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Moving beyond Rosenbrock's Theorem in another direction

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