Polynomial and rational matrices with the invariant rational functions and the four sequences of minimal indices prescribed

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Outline

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Rational matrices and polynomial matrices

- A rational matrix $R(s)$ is a matrix whose entries are univariate rational functions with coefficients in an arbitrary field $\mathbb{F}$.
- A polynomial matrix $P(s)$ is a matrix whose entries are univariate polynomials with coefficients in $\mathbb{F}$.
- Polynomial are particular cases of rational matrices, but they are important on their own. The results in this talk were obtained first for polynomial matrices, but, for brevity, we focus on rational matrices.
- Any rational matrix $R(s)$ can be uniquely expressed as

$$R(s) = P(s) + R_{sp}(s), \quad \text{where}$$

  - $P(s)$ is a polynomial matrix (polynomial part), and
  - the rational matrix $R_{sp}(s)$ is the strictly proper part, whose entries have numerators with smaller degrees than the denominators.
- **Unimodular matrices** are square polynomial matrices with constant nonzero determinant.
- Polynomial and rational matrices arise in many applications.

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Definition

The **Smith-McMillan form** of a rational matrix $R(s) \in \mathbb{F}(s)^{m\times n}$ is the following **diagonal matrix** obtained under unimodular transformations $U(s)$ and $V(s)$:

$$U(s)R(s)V(s) = \left[\begin{array}{ccc|c} \frac{\epsilon_1(s)}{\psi_1(s)} & & & \\ & \ddots & & 0_{r\times(n-r)} \\ & & \frac{\epsilon_r(s)}{\psi_r(s)} & \\ \hline & 0_{(m-r)\times r} & & 0_{(m-r)\times(n-r)} \end{array}\right].$$

- $\epsilon_1(s) \mid \cdots \mid \epsilon_r(s)$ and $\psi_r(s) \mid \cdots \mid \psi_1(s)$ are scalar monic polynomials,
- the fractions $\frac{\epsilon_i(s)}{\psi_i(s)}$ are irreducible (invariant rational functions of $R(s)$),
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Definition (finite zeros and finite poles)

Given the **Smith-McMillan form** of a rational matrix $R(s) \in \mathbb{F}(s)^{m \times n}$:

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The **finite zeros** of $R(s)$ are the roots of the numerators and the **finite poles** are the roots of the denominators.

Remark

Given any $c \in \overline{\mathbb{F}}$, one can write for each $i = 1, \ldots, r$,

$$\frac{\epsilon_i(s)}{\psi_i(s)} = (s-c)^{\sigma_i(c)} \frac{\widetilde{\epsilon}_i(s)}{\widetilde{\psi}_i(s)}, \qquad \text{with } \widetilde{\epsilon}_i(c) \neq 0,\ \widetilde{\psi}_i(c) \neq 0.$$

Definition (Invariant orders at c)

The invariant orders at c of $R(s)$ are

$$\sigma_1(c) \leq \sigma_2(c) \leq \cdots \leq \sigma_r(c).$$

Finite zeros, finite poles, and invariant orders of a Rational Matrix

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The invariant orders of a rational matrix $R(s)$ at ∞ are the invariant orders of $R\left(\frac{1}{s}\right)$ at $s = 0$.

Proposition: The smallest invariant order at infinity

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Minimal bases of rational vector subspaces

- $\mathbb{F}[s]$ is the ring of univariate polynomials with coefficients in $\mathbb{F}$.
- $\mathbb{F}(s)$ is the field of univariate rational functions over $\mathbb{F}$ and
- $\mathbb{F}(s)^n$ is the vector space over $\mathbb{F}(s)$ of n-tuples with entries in $\mathbb{F}(s)$.
- $\mathbb{F}(s)^n$ **is said to be a rational vector space and its subspaces are rational vector subspaces.** (Forney, SIAM J. Control 1975)
- Any rational subspace $\mathcal{V} \subseteq \mathbb{F}(s)^n$ has bases consisting entirely of vector polynomials $\longrightarrow$ polynomial bases of $\mathcal{V}$.

Definition (Minimal basis)

A **minimal basis** of a rational subspace $\mathcal{V} \in \mathbb{F}(s)^n$ is a basis

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Theorem (Forney, SIAM J. Control 1975)

The ordered list of degrees of the vector polynomials in any minimal basis of $\mathcal{V} \subseteq \mathbb{F}(s)^n$ is always the same.

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$$\begin{aligned} \mathcal{N}_\ell(R) &:= \left\{ y(s) \in \mathbb{F}(s)^m \,:\, y(s)^T R(s) \equiv 0^T \right\} \subseteq \mathbb{F}(s)^m, \\ \mathcal{N}_r(R) &:= \left\{ x(s) \in \mathbb{F}(s)^n \,:\, R(s)x(s) \equiv 0 \right\} \subseteq \mathbb{F}(s)^n. \end{aligned}$$

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Definition

Given a rational matrix $R(s) \in \mathbb{F}(s)^{m\times n}$ with rank r, the complete eigenstructure of $R(s)$ consists of

(i) the invariant rational functions

$$\frac{\epsilon_1(s)}{\psi_1(s)}, \cdots, \frac{\epsilon_r(s)}{\psi_r(s)} \quad \text{(finite pole/zero structure)},$$

(ii) the invariant orders at ∞ $\quad q_1 \leq \cdots \leq q_r \quad$ (infinite pole/zero structure),

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More structure: Row and column spaces

- The structural data described above have received a lot of attention in the literature on rational matrices,
- but, as we teach in basic Linear Algebra courses, every matrix has four fundamental subspaces
- and, so far, we have only used two: the left and right null spaces.
- Given a rational matrix $R(s) \in \mathbb{F}(s)^{m\times n}$ the other two are

$$\mathcal{R}ow(R) = \{R(s)^T w(s) \, : \, w(s) \in \mathbb{F}(s)^m\} \subseteq \mathbb{F}(s)^n \, ,$$
$$\mathcal{C}ol(R) = \{R(s) v(s) \, : \, v(s) \in \mathbb{F}(s)^n\} \subseteq \mathbb{F}(s)^m \, ,$$

- which have minimal bases and indices as any other rational subspace.
- **Thus a rational matrix has four sequences of minimal indices.**
- As far as we know, minimal bases and indices of row and column spaces have not received much attention in the literature, but are fundamental for constructing rank revealing factorizations of polynomial matrices with good properties (Dmytryshyn, D, Van Dooren, LAA, 2025).

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- As far as we know, minimal bases and indices of row and column spaces have not received much attention in the literature, but are fundamental for constructing rank revealing factorizations of polynomial matrices with good properties (Dmytryshyn, D, Van Dooren, LAA, 2025).

More structure: Row and column spaces

- The structural data described above have received a lot of attention in the literature on rational matrices,
- but, as we teach in basic Linear Algebra courses, every matrix has four fundamental subspaces
- and, so far, we have only used two: the left and right null spaces.
- Given a rational matrix $R(s) \in \mathbb{F}(s)^{m \times n}$ the other two are

$$\mathcal{R}ow(R) = \{R(s)^T w(s) \,:\, w(s) \in \mathbb{F}(s)^m\} \subseteq \mathbb{F}(s)^n \,,$$
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Outline

1. Preliminaries: Which are the data to be prescribed?
2. **Goals of the talk**
3. Reminders on prescribed complete eigenstructure
4. Prescribed data with minimal indices of row and column spaces
5. Conclusions

In the last decade, **the following problem has been solved:**

- If a complete eigenstructure is prescribed, to find necessary and sufficient conditions for the existence of a rational matrix with this complete eigenstructure.

In this talk, we present

- necessary and sufficient conditions for the existence problem above when the minimal indices of the row and column spaces are prescribed
  1. **instead of** the minimal indices of the right and left null spaces

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  2. **in addition** to the complete eigenstructure.

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Outline

Theorem

There exists a rational matrix $R(s) \in \mathbb{F}(s)^{m \times n}$ *with normal rank* $r \leq \min\{m, n\}$, *with*

(i) *invariant rational functions*

$$\frac{\epsilon_1(s)}{\psi_1(s)}, \dots, \frac{\epsilon_r(s)}{\psi_r(s)},$$

(ii) *invariant orders at* ∞ $\quad q_1 \leq \cdots \leq q_r,$

(iii) *right minimal indices* $\quad d_1 \geq \cdots \geq d_{n-r},$

(iv) *left minimal indices* $\quad v_1 \geq \cdots \geq v_{m-r},$

if and only if

$$\sum_{i=1}^{r} \deg(\epsilon_i) - \sum_{i=1}^{r} \deg(\psi_i) + \sum_{i=1}^{r} q_i + \sum_{i=1}^{n-r} d_i + \sum_{i=1}^{m-r} v_i = 0.$$

Comments

This result was

- proved for polynomial matrices by De Terán, D, Van Dooren, SIMAX 2015, for infinite fields $\mathbb{F}$,
- proved for rational matrices by Anguas, D, Hollister, Mackey, SIMAX 2019, for infinite fields $\mathbb{F}$,
- extended for polynomial matrices over any field recently by Amparan, Baragaña, Marcaida, Roca, SIMAX 2024, via a completely different proof,
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Outline

Theorem (consequence of Forney, SIAM J. Control 1975)

Let $R(s) \in \mathbb{F}(s)^{m\times n}$ *be a rational matrix of normal rank* r*, with*

(i) *right minimal indices* $d_1 \geq \cdots \geq d_{n-r}$,

(ii) *left minimal indices* $v_1 \geq \cdots \geq v_{m-r}$,

(iii) *minimal indices of* $\mathcal{R}ow(R)$ $\ell_1 \geq \cdots \geq \ell_r$,

(iv) *minimal indices of* $\mathcal{C}ol(R)$ $k_1 \geq \cdots \geq k_r$.

Then,

$$\sum_{i=1}^{m-r} v_i = \sum_{i=1}^{r} k_i \quad \text{and} \quad \sum_{i=1}^{n-r} d_i = \sum_{i=1}^{r} \ell_i.$$

An additional tool: majorization

Definition

Let

$$\mathbf{a} = (a_1 \geq \cdots \geq a_m) \quad \text{and} \quad \mathbf{b} = (b_1 \geq \cdots \geq b_m)$$

be two decreasingly ordered sequences of integers.

It is said that $\mathbf{a}$ is *majorized* by $\mathbf{b}$, denoted by

$$\mathbf{a} \prec \mathbf{b},$$

if

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i \quad \text{for } 1 \leq k \leq m-1 \text{ and}$$
$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} b_i.$$

Prescribed minimal indices of *Row/Col* instead of *right/left Nulls*

Theorem (Baragaña, D, Marcaida, Roca, submitted 2025)

Let $\mathbb{F}$ be algebraically closed. *There exists a rational matrix* $R(s) \in \mathbb{F}(s)^{m\times n}$ *with normal rank* $r < \min\{m,n\}$, *with*

- **(i)** *invariant rational functions* $\dfrac{\epsilon_1(s)}{\psi_1(s)}, \cdots, \dfrac{\epsilon_r(s)}{\psi_r(s)}$,
- **(ii)** *invariant orders at* ∞ $q_1 \leq \cdots \leq q_r$,
- **(iii)** *minimal indices of* $\mathcal{R}ow(R)$ $\ell_1 \geq \cdots \geq \ell_r$,
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if and only if

$$(-g_r, \ldots, -g_1) \prec (\deg(\epsilon_r) - \deg(\psi_r) + q_r, \ldots, \deg(\epsilon_1) - \deg(\psi_1) + q_1),$$

where

$g_1 \geq \cdots \geq g_r$ *is the decreasing reordering of* $k_r + \ell_1, \ldots, k_1 + \ell_r$.

Remarks on previous theorem

- **The proof of the necessity is valid over arbitrary fields**. The proof of the sufficiency requires algebraically closed fields in an essential way and we have examples over $\mathbb{R}$ for which the sufficiency does not hold.
- The same comment applies to the result in the next slide.
- The majorization condition amounts in fact to r conditions. The last of such conditions is the Index Sum Theorem, i.e., the unique condition appearing when the complete eigenstructure is prescribed.
- If $r = \min\{m, n\}$, we have to add two trivial conditions:
  - $\ell_1 = \cdots = \ell_r = 0$ if $r = n$

    (coming from $\mathcal{R}ow(R) = \mathbb{F}(s)^n$ in this case),
  - $k_1 = \cdots = k_r = 0$ if $r = m$

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Outline

Conclusions

- We have provided necessary and sufficient conditions for the existence of polynomial and rational matrices when **instead of the left and right minimal indices or in addition to** their **classical complete eigenstructures the minimal indices of their row and column spaces** are prescribed.

- The obtained necessary and sufficient conditions are very simple and only require to check some equalities or inequalities of integer numbers related to the prescribed data.

- These results can be easily extended to skew-symmetric polynomial and rational matrices, but

- extending these results to other structured polynomial and rational matrices is a completely open area.

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