Rosenbrock's Theorem on System Matrices over Elementary Divisor Domains and beyond

Froilán M. Dopico

joint work with **Vanni Noferini** (Aalto University, Finland)
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IWOTA 2025. Contributed Session: Matrix theory and linear algebra II, **in honour of Rien Kaashoek**
University of Twente. Enschede, the Netherlands. July 14-18, 2025

Preamble about Rien Kaashoek (I)

Preamble about Rien Kaashoek (II)

COLLOQUIUM

DEPARTAMENTO DE MATEMÁTICAS

UNIVERSIDAD CARLOS III DE MADRID

- ***Marinus A. Kaashoek***

  (Dept. of Mathematics, VU Univ. Amsterdam, Paises Bajos)

  Jueves, 24 de marzo de 2011

  The rational Corona-Bezout equation and Toeplitz operators

  Abstract:

  The interplay between complex function theory and operator theory is one of the great achievements of modern mathematics. In the later part of the previous century, another partner -- mathematical system theory -- entered in this successful cooperation. In this talk the connections between the three fields will be illustrated on some recent developments related to the Corona-Bezout equation with emphasis on rational matrix solutions.

Hora: 10:45

Lugar: Seminario del Departamento de Matemáticas

Aula 2.2.D08, Edificio Sabatini (2ª Planta)

Universidad Carlos III de Madrid

Avda. de la Universidad 30, Leganés (Madrid)

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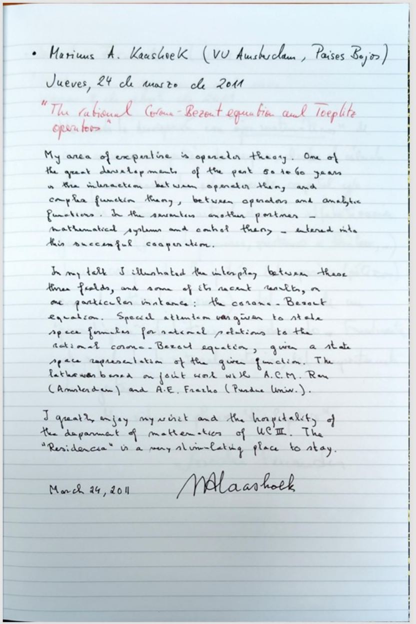
Más Información: Fernando Lledó (flledo (at) math.uc3m.es)



• Marinus A. Kaashoek (VU Amsterdam, Países Bajos)

Jueves, 24 de marzo de 2011

"The rational Corona-Bezout equation and Toeplitz operators"

My area of expertise is operator theory. One of the great developments of the past 50 to 60 years is the interaction between operator theory and complex function theory, between operators and analytic functions. In the seventies another partner – mathematical systems and control theory – entered into this successful cooperation.

In my talk I illustrated the interplay between these three fields, and some of its recent results, in one particular instance: the corona-Bezout equation. Special attention was given to state space formulas for rational solutions to the rational corona-Bezout equation, given a state space representation of the given function. The latter was based on joint work with A.C.M. Ran (Amsterdam) and A.E. Frazho (Purdue Univ.).

I greatly enjoy my visit and the hospitality of the department of mathematics of UC III. The "Residencia" is a very stimulating place to stay.

March 24, 2011 MAKaashoek

Preamble about Rien Kaashoek (IV)

State space formula for the least squares solution

THM 1 [Frazho-K-Ran, 2010]. *Equation $G(z)X(z) = I_m$ has a stab rational matrix solution if and only if the corresponding Riccati equatio (ARE) has a stabilizing solution Q, and $I_n - PQ$ is non-singular, where $P - APA^* = B$. In that case the least squares solution Φ is given by*

$$\Phi(z) = \left(I_p - zC_1(I_n - zA_0)^{-1}(I_n - PQ)^{-1}B\right)D_1, \text{ where}$$

$$A_0 = A - \Gamma(R_0 - \Gamma^* Q\Gamma)^{-1}(C - \Gamma^* QA),$$
$$C_1 = D^* C_0 + B^* QA_0,$$
$$\text{with } C_0 = (R_0 - \Gamma^* Q\Gamma)^{-1}(C - \Gamma^* QA),$$
$$D_1 = (D^* - B^* Q\Gamma)(R_0 - \Gamma^* Q\Gamma)^{-1} + C_1(I_n - PQ)^{-1}PC_0^*.$$

In particular, Φ is rational, and the McMillan degree of $\Phi \leq$ the McMil degree of G.

A few final remarks

- A state space version of Tolokonnikov's lemma. The corona problem viewed as a completion problem
- Continuous analogue [work in progress]. Role of Toeplitz operators is taken over by Wiener-Hopf integral operators.
- For a rational G we computed the optimal solution (when $m = 1$) and for the suboptimal case the maximum entropy solution (joint work in progress with Art Frazho and Sanne ter Horst).
- The corona-Bezout equation remains a source of inspiration. See the recent papers of Tavan Trent, Sergei Treil, and Sergei Treil and Brett Wick (several variables) and others.

Thank you for your attention!

Theorem (Rosenbrock, 1970)

Let $A(s) \in \mathbb{F}[s]^{n\times n}$, $B(s) \in \mathbb{F}[s]^{n\times m}$, $C(s) \in \mathbb{F}[s]^{p\times n}$ *and* $D(s) \in \mathbb{F}[s]^{p\times m}$ *with* $\det A(s) \neq 0$*. Let*

$$P(s) = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} \in \mathbb{F}[s]^{(n+p)\times(n+m)}, \quad G(s) = D(s) - C(s)A(s)^{-1}B(s) \in \mathbb{F}(s)^{p\times m}.$$

Assume that $P(s)$ is minimal. *If the Smith-McMillan form of* $G(s)$ *is*

$$S_G(s) = \operatorname{Diag}\left(\frac{\varepsilon_1(s)}{\psi_1(s)}, \dots, \frac{\varepsilon_r(s)}{\psi_r(s)}\right) \oplus 0_{(p-r)\times(m-r)} \in \mathbb{F}(s)^{p\times m},$$

and g *is the largest index in* $\{1, \dots, r\}$ *such that* $\psi_g(s) \neq 1$*, then the Smith forms of* P *and* A *are, respectively,*

$$S_P(s) = I_n \oplus \operatorname{Diag}(\varepsilon_1(s), \dots, \varepsilon_r(s)) \oplus 0_{(p-r)\times(m-r)} \in \mathbb{F}[s]^{(n+p)\times(n+m)},$$

and

$$S_A = I_{n-g} \oplus \operatorname{Diag}(\psi_g(s), \dots, \psi_1(s)) \in \mathbb{F}[s]^{n\times n}.$$

Comments, motivations, goals, ...

- This theorem is fundamental for numerical algorithms that compute the zeros and poles of rational matrices via linear polynomial system matrices $P(s)$. For instance, via (generalized) state-space representations.
- However, the proof in Rosenbrock's book is indirect since relies on results about the equivalence of polynomial system matrices.
- This also happens in other standard references, as Kailath's book (perhaps, because this theorem is not very relevant for people working in Linear Systems theory?).
- Personal comments: this fact always disturbed me and I chatted informally about it with my coauthors.
- In 1974, Coppel proved a more general version of this theorem for system matrices in any Principal Ideal Domain (PID) and transfer functions in its field of fractions via an elegant purely algebraic approach. Though more direct than previous proofs, still indirect since it was not the main goal of the author.
- We searched for more direct and general approaches.

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Outline

1. Elementary divisor domains and Rosenbrock's Theorem over EDDs

2. Beyond Rosenbrock's Theorem

Outline

Definition (elementary divisor ring)

A commutative ring $\mathfrak{R}$ with identity is an elementary divisor ring if for any $a, b, c \in \mathfrak{R}$, there exist $x, y, z, w \in \mathfrak{R}$ such that

$$\gcd(a, b, c) = (zx)\, a + (zy)\, b + (wy)\, c.$$

Theorem (Kaplansky, 1949)

$\mathfrak{R}$ is an elementary divisor ring

if and only if

for every $A \in \mathfrak{R}^{p\times m}$, there exist $U \in \mathfrak{R}^{p\times p}$, $S \in \mathfrak{R}^{p\times m}$, $V \in \mathfrak{R}^{m\times m}$ such that

1. $A = USV$,
2. *U, V are unimodular (invertible over $\mathfrak{R}$),*
3. *$S = \mathrm{Diag}(\alpha_1, \alpha_2, \ldots, \alpha_r) \oplus 0_{(p-r)\times(m-r)}$ is diagonal with $\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_r$.*

In words: $\mathfrak{R}$ is an elementary divisor ring if and only if every matrix $A \in \mathfrak{R}^{p\times m}$ has a Smith form.

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Definition (elementary divisor domain)

If $\mathfrak{R}$ is an elementary divisor ring and an integral domain (i.e., there are no nonzero zero divisors), it is called an elementary divisor domain (EDD).

Then, the smallest field $\mathbb{F}$ containing $\mathfrak{R}$ is called the field of fractions of $\mathfrak{R}$.

Corollary (EDD if and only if Smith-McMillan form)

$\mathfrak{R}$ is an elementary divisor domain with field of fractions $\mathbb{F}$

if and only if

for every $A \in \mathbb{F}^{p\times m}$, there exist $U \in \mathfrak{R}^{p\times p}$, $S \in \mathbb{F}^{p\times m}$, $V \in \mathfrak{R}^{m\times m}$ such that

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Examples of elementary divisor domains

- Any Principal Ideal Domain (PID), for instance,
  - $\mathbb{Z}$,
  - $\mathbb{F}[x]$, with $\mathbb{F}$ any field;
- Entire functions or, more generally, the ring of complex-valued functions that are holomorphic on an open connected subset $\Omega \subseteq \mathbb{C}$;
- Algebraic integers (roots of monic polynomials with integer coefficients).

Theorem

Let $\mathfrak{R}$ be an EDD, $G_1 \in \mathfrak{R}^{p\times m}$ and $G_2 \in \mathfrak{R}^{q\times m}$, $p+q \geq m$. The following are equivalent:

i) *G_1 and G_2 are right coprime in $\mathfrak{R}$, i.e., every common right divisor is unimodular.*

ii) *The Smith form over $\mathfrak{R}$ of $\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$ is $\begin{bmatrix} I_m \\ 0 \end{bmatrix}$.*

iii) *There exists a unimodular matrix $U \in \mathfrak{R}^{(p+q)\times(p+q)}$ such that $U \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$.*

iv) *There exist matrices $C \in \mathfrak{R}^{p\times(p+q-m)}$, $D \in \mathfrak{R}^{q\times(p+q-m)}$ such that $\begin{bmatrix} G_1 & C \\ G_2 & D \end{bmatrix}$ is unimodular.*

v) *There exist matrices $X \in \mathfrak{R}^{m\times p}$, $Y \in \mathfrak{R}^{m\times q}$ such that $XG_1 + YG_2 = I_m$.*

The polynomial matrices $G_1(s) \in \mathbb{C}[s]^{p\times m}, G_2(s) \in \mathbb{C}[s]^{q\times m}$ are right coprime if and only if

$$\operatorname{rank} \begin{bmatrix} G_1(z_0) \\ G_2(z_0) \end{bmatrix} = m, \quad \forall z_0 \in \mathbb{C}.$$

Coprimeness: Necessary and sufficient conditions

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iv) *There exist matrices $C \in \mathfrak{R}^{p\times(p+q-m)}$, $D \in \mathfrak{R}^{q\times(p+q-m)}$ such that $\left[\begin{smallmatrix} G_1 & C \\ G_2 & D \end{smallmatrix}\right]$ is unimodular.*

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Definition

Let $\mathfrak{R}$ be an EDD, $A \in \mathfrak{R}^{n\times n}$, $B \in \mathfrak{R}^{n\times m}$, $C \in \mathfrak{R}^{p\times n}$ and $D \in \mathfrak{R}^{p\times m}$ with $\det A \neq 0$. The matrix

$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{R}^{(n+p)\times(n+m)}$$

is minimal (or irreducible) if

- A and B are left coprime (i.e., their transposes are right coprime) and
- A and C are right coprime.

Definition

We say that

$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{R}^{(n+p)\times(n+m)}$$

is a **system matrix with transfer matrix** $G = D - CA^{-1}B \in \mathbb{F}^{p\times m}$, where $\mathbb{F}$ is the field of fractions of $\mathfrak{R}$.

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Theorem (D, Noferini, Zaballa, LAA 2025)

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Assume that P is minimal. *If the Smith-McMillan form of G is*

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and g is the largest index in $\{1, \dots, r\}$ such that $\psi_g \notin U(\mathfrak{R})$, then the Smith forms of P and A are, respectively,

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$$S_A \doteq I_{n-g} \oplus \operatorname{Diag}(\psi_g, \dots, \psi_1) \in \mathfrak{R}^{n\times n}.$$

- It involves the weakest possible conceivable assumptions on the underlying ring and field, since EDDs (an their fields of fractions) are the most general rings (fields) where the involved Smith-McMillan and Smith forms both exist and, so, where a Rosenbrock's like theorem makes sense.

- The proof is shorter and more direct than the proofs we have found in the literature (including the original by Rosenbrock and that by Coppel).

Outline

For the rest of the talk: $\mathfrak{R}$ is an EDD and $\mathbb{F}$ its field of fractions.

Theorem (Rosenbrock's Theorem over EDDs)

Let $A \in \mathfrak{R}^{n\times n}$, $B \in \mathfrak{R}^{n\times m}$, $C \in \mathfrak{R}^{p\times n}$ *and* $D \in \mathfrak{R}^{p\times m}$ *with* $\det A \neq 0$. *Let*

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$$S_P \doteq I_n \oplus \operatorname{Diag}(\varepsilon_1, \ldots, \varepsilon_r) \oplus 0_{(p-r)\times(m-r)} \in \mathfrak{R}^{(n+p)\times(n+m)},$$

and

$$S_A \doteq I_{n-g} \oplus \operatorname{Diag}(\psi_g, \ldots, \psi_1) \in \mathfrak{R}^{n\times n}.$$

Next goal

- **What happens when the system matrix is NOT minimal, i.e., when A and B are not left coprime or A and C are not right coprime?**
- Are there still any relations between the Smith-McMillan form of G and the Smith forms of A and P?
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Key auxiliary result: from non-minimal to minimal system matrices

Theorem (D, Noferini, Zaballa, LAA 2025)

Let $A \in \mathfrak{R}^{n\times n}$, $B \in \mathfrak{R}^{n\times m}$, $C \in \mathfrak{R}^{p\times n}$ and $D \in \mathfrak{R}^{p\times m}$ with $\det A \neq 0$. If A and B **are not left coprime** *or A and C* **are not right coprime**, *then there exist matrices $A_0 \in \mathfrak{R}^{n\times n}$, with $\det A_0 \neq 0$, $B_0 \in \mathfrak{R}^{n\times m}$, $C_0 \in \mathfrak{R}^{p\times n}$, $E \in \mathfrak{R}^{n\times n}$ and $F \in \mathfrak{R}^{n\times n}$ such that*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} E & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} A_0 & B_0 \\ C_0 & D \end{bmatrix} \begin{bmatrix} F & 0 \\ 0 & I_m \end{bmatrix}$$

and

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ii) $\det E \neq 0$, $\det F \neq 0$, *and at least one of these determinants is not a unit of $\mathfrak{R}$;*

iii) $D - CA^{-1}B = D - C_0 A_0^{-1} B_0$, **i.e., Schur complement does not change!!**

Essential idea

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This factorization can be combined with

1. The fact that Rosenbrock's Theorem holds for $\begin{bmatrix} A_0 & B_0 \\ C_0 & D \end{bmatrix}$.

2. **Proposition.** *Let* $A_1 \in \mathfrak{R}^{m\times n}$, $A_2 \in \mathfrak{R}^{n\times p}$ *and let* $A = A_1 A_2$. *Let* $\alpha_1^{(1)} \mid \cdots \mid \alpha_{r_1}^{(1)}$, $\alpha_1^{(2)} \mid \cdots \mid \alpha_{r_2}^{(2)}$ *and* $\alpha_1 \mid \cdots \mid \alpha_r$ *be the invariant factors of* A_1, A_2 *and* A, *respectively. Then* $\alpha_k^{(j)} \mid \alpha_k$ *for* $j = 1, 2$ *and* $k = 1, \ldots, r$.

   In words: Invariant factors of matrix factors divide the invariant factors of the product.

3. The classical expression of the minors of the Schur complement in terms of the minors of the whole matrix and $\det A$.

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$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{R}^{(n+p)\times(n+m)}, \quad \text{and} \quad G = D - CA^{-1}B \in \mathbb{F}^{p\times m},$$

and **assume that A and B are not left coprime or that A and C are not right coprime**. *Let*

$$S_G \doteq \operatorname{Diag}\left(\frac{\varepsilon_1}{\psi_1}, \dots, \frac{\varepsilon_r}{\psi_r}\right) \oplus 0_{(p-r)\times(m-r)} \in \mathbb{F}^{p\times m},$$

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be the Smith-McMillan form of G and the Smith forms of A and P, respectively. Let g be the largest index in $\{1, \dots, r\}$ such that $\psi_g \notin U(\mathfrak{R})$. Then

i) $n \geq g$ *and* $\psi_i \mid \widetilde{\psi}_i$, *for* $i = 1, \dots, g$;

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Main Theorem for non-minimal system matrices (II)

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In general, $\dfrac{\widetilde{\psi}_n\cdots\widetilde{\psi}_2\,\widetilde{\psi}_1}{\psi_g\cdots\psi_2\,\psi_1}$ can be any element in the ring!!!

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- One of the reasons why it is not easy to work on general EDDs is because they are not, in general, Unique Factorization Domains (UFD),
- i.e., we cannot assume that their elements have a unique factorization into prime elements.
- In particular, the invariant factors of the Smith forms of matrices over EDDs cannot be uniquely factorized into prime elements and “elementary divisors” cannot be defined.
- Thus, for matrices in general EDDs, we loose one of the fundamental concepts/tools of matrix polynomials: the elementary divisors.
- Moreover, not every UFD is an EDD,
- but if $\mathfrak{R}$ is a PID, then it is simultaneously an EDD and a UFD.
- PIDs include the ring of integers and rings of polynomials in one variable with coefficients in a field.

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Reminder: Elementary Divisors of Matrices over a PID $\mathfrak{R}$

Let $A \in \mathfrak{R}^{p\times m}$ with Smith form

$$S_A \doteq \mathrm{Diag}\,(\alpha_1, \ldots, \alpha_r) \oplus 0_{(p-r)\times(m-r)} \in \mathfrak{R}^{p\times m}.$$

We can write

$$\begin{aligned} \alpha_1 &= \beta_1^{e_{11}} \beta_2^{e_{12}} \cdots \beta_\ell^{e_{1\ell}}, \\ \alpha_2 &= \beta_1^{e_{21}} \beta_2^{e_{22}} \cdots \beta_\ell^{e_{2\ell}}, \\ &\vdots \qquad\qquad \vdots \\ \alpha_r &= \beta_1^{e_{r1}} \beta_2^{e_{r2}} \cdots \beta_\ell^{e_{r\ell}}, \end{aligned}$$

- where $\beta_1, \ldots, \beta_\ell$ are prime elements of $\mathfrak{R}$ and e_{ij} are nonnegative integers that satisfy $0 \leq e_{1j} \leq e_{2j} \leq \cdots \leq e_{rj}, \quad j = 1, \ldots, \ell$.
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Theorem (D, Noferini, Zaballa, LAA 2025)

Let $\mathfrak{R}$ be a PID and $\mathbb{F}$ its field of fractions. Let $A \in \mathfrak{R}^{n\times n}$, $B \in \mathfrak{R}^{n\times m}$, $C \in \mathfrak{R}^{p\times n}$ and $D \in \mathfrak{R}^{p\times m}$ with $\det A \neq 0$,

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be the Smith-McMillan form of G and g be the largest index in $\{1, \dots, r\}$ such that $\psi_g \notin U(\mathfrak{R})$. If $\pi \in \mathfrak{R}$ is prime and

$$\gcd\left(\pi, \frac{\det A}{\psi_g \cdots \psi_2 \psi_1}\right) \doteq 1,$$

then the sequence of the partial multiplicities of P at π is equal to the sequence of the partial multiplicities of $\operatorname{Diag}(\varepsilon_1, \dots, \varepsilon_r)$ at π.

Remarks

- This result holds under the more restrictive (but easier to verify) condition $\gcd(\pi\,,\,\det A) \doteq 1,$
- since this implies that $\gcd\left(\pi\,,\,\dfrac{\det A}{\psi_g \cdots \psi_2\,\psi_1}\right) \doteq 1.$
- Thus, if (1) we know the prime divisors of $\det A$ and (2) we are not interested in the possible elementary divisors of the Smith-McMillan numerators of G at that primes, then
- using non-minimal system matrices is safe.
- This was in fact the case in S. Güttel, R. Van Beeumen, K. Meerbergen, W. Michiels, "NLEIGS: a class of fully rational Krylov methods for nonlinear eigenvalue problems", *SIAM J. Sci. Comput.*, (2014).

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- Let $\mathfrak{R}$ be an EDD, $\mathbb{F}$ its field of fractions and

$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{F}^{(n+p)\times(n+m)}, \qquad G = D - CA^{-1}B \in \mathbb{F}^{p\times m}.$$

- We have also investigated the relations between the Smith-McMillan form of G and the Smith-McMillan forms of A and P.
- We have obtained results in the same spirit of Rosenbrock's Theorem, though they require some additional hypotheses, in addition to the coprimeness, and are more cumbersome.
- They may have applications for developing a unified approach to the study/computation of the structure at infinity of rational matrices.

Moving beyond Rosenbrock's Theorem in another direction

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