Polynomial and rational matrices with the invariant rational functions and the four sequences of minimal indices prescribed

Froilán M. Dopico

joint work with **I. Baragaña**, **S. Marcaida** (U. del País Vasco, Spain) and **A. Roca** (U. Politècnica de València, Spain)

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The first time I met Volker in person was in ...

IV International Workshop on Accurate Solution of Eigenvalue Problems, Split, Croatia, June 24-27, 2002. Volker's talk was "Accurate solution of Quadratic Eigenvalue Problems with structure".

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PhD Defense of Andrii Dmytryshyn, December 2015

ICIAM, Valencia, 2019

Since then ...

- Volker has made several short visits to my department.
- Volker has had a profound influence and has motivated much of my research,
- through his very positive contagious attitude,
- and his pioneer contributions on (in addition to EMOSC, and port-Hamiltonian systems)
- Hamiltonian and symplectic matrices,
- Nonlinear eigenvalue problems, (Structured) Matrix Polynomials and their linearizations,
- Smith forms of structure matrix polynomials,
- Low rank perturbations of structured matrices, ...
- Thank you very much Volker and Congratulations for your great scientific contributions and for your leadership in our community through editorial work, presidencies of Mathematical Societies, and opening many new avenues of research.

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1. Preliminaries: Which are the data to be prescribed?
2. Goals of the talk
3. Reminders on prescribed complete eigenstructures
4. Prescribed data with minimal indices of row and column spaces
5. Conclusions

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Rational matrices and polynomial matrices

- A rational matrix $R(s)$ is a matrix whose entries are univariate rational functions with coefficients in an arbitrary field $\mathbb{F}$.
- A polynomial matrix $P(s)$ is a matrix whose entries are univariate polynomials with coefficients in $\mathbb{F}$.
- Polynomial are particular cases of rational matrices, but they are important on their own. The results in this talk were obtained first for polynomial matrices, but, for brevity, we focus on rational matrices.
- Any rational matrix $R(s)$ can be uniquely expressed as

$$R(s) = P(s) + R_{sp}(s), \quad \text{where}$$

  - $P(s)$ is a polynomial matrix (polynomial part), and
  - the rational matrix $R_{sp}(s)$ is the strictly proper part, whose entries have numerators with smaller degrees than the denominators.
- **Unimodular matrices** are square polynomial matrices with constant nonzero determinant.
- Polynomial and rational matrices arise in many applications.

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Definition

The **Smith-McMillan form** of a rational matrix $R(s) \in \mathbb{F}(s)^{m\times n}$ is the following **diagonal matrix** obtained under unimodular transformations $U(s)$ and $V(s)$:

$$U(s)R(s)V(s) = \left[\begin{array}{ccc|c} \frac{\epsilon_1(s)}{\psi_1(s)} & & & \\ & \ddots & & 0_{r\times(n-r)} \\ & & \frac{\epsilon_r(s)}{\psi_r(s)} & \\ \hline & 0_{(m-r)\times r} & & 0_{(m-r)\times(n-r)} \end{array}\right].$$

- $\epsilon_1(s) \mid \cdots \mid \epsilon_r(s)$ and $\psi_r(s) \mid \cdots \mid \psi_1(s)$ are scalar monic polynomials,
- the fractions $\frac{\epsilon_i(s)}{\psi_i(s)}$ are irreducible (invariant rational functions of $R(s)$),
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Definition (finite zeros and finite poles)

Given the **Smith-McMillan form** of a rational matrix $R(s) \in \mathbb{F}(s)^{m\times n}$:

$$\text{diag}\left(\frac{\epsilon_1(s)}{\psi_1(s)}, \ldots, \frac{\epsilon_r(s)}{\psi_r(s)}, 0_{(m-r)\times(n-r)}\right)$$

The **finite zeros** of $R(s)$ are the roots of the numerators and the **finite poles** are the roots of the denominators.

Remark

Given any $c \in \overline{\mathbb{F}}$, one can write for each $i = 1, \ldots, r$,

$$\frac{\epsilon_i(s)}{\psi_i(s)} = (s-c)^{\sigma_i(c)} \; \frac{\widetilde{\epsilon}_i(s)}{\widetilde{\psi}_i(s)}, \qquad \text{with } \widetilde{\epsilon}_i(c) \neq 0, \ \widetilde{\psi}_i(c) \neq 0.$$

Definition (Invariant orders at c)

The invariant orders at c of $R(s)$ are

$$\sigma_1(c) \leq \sigma_2(c) \leq \cdots \leq \sigma_r(c).$$

Finite zeros, finite poles, and invariant orders of a Rational Matrix

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The Smith form of a Polynomial Matrix

If $P(s)$ is a polynomial matrix, the denominators of its invariant rational functions are all 1 and the Smith-McMillan form reduces to the Smith form.

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Definition

The invariant orders of a rational matrix $R(s)$ at ∞ are the invariant orders of $R\left(\frac{1}{s}\right)$ at $s = 0$.

Proposition: The smallest invariant order at infinity

The smallest invariant order of $R(s)$ at infinity is

1. $-$degree (polynomial part of $R(s)$), if this polynomial part is nonzero,
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Definition (Reversal polynomial)

Let $P(s) = P_d s^d + P_{d-1} s^{d-1} + \cdots + P_0, \quad P_d \neq 0,$ be a **polynomial matrix of degree** d. The **reversal** of $P(s)$ is

$$\mathrm{rev}P(s) := s^d P\left(\frac{1}{s}\right) = P_d + P_{d-1}\, s + \cdots + P_0\, s^d\,.$$

Definition (Eigenvalue and partial mutiplicities at ∞)

The partial multiplicities of $P(s)$ at ∞ are those of $\mathrm{rev}P(s)$ at 0 and ∞ **is an eigenvalue of** $P(s)$ **if** 0 **is an eigenvalue of** $\mathrm{rev}P(s)$**.**

Proposition

Let $P(s) \in \mathbb{F}[s]^{m\times n}$ be a **polynomial matrix** of degree d and rank r.

1. $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_r$ are the invariant orders of $P(s)$ at ∞ if and only if $\sigma_1 + d \leq \sigma_2 + d \leq \cdots \leq \sigma_r + d$ are the partial multiplicities of $P(s)$ at ∞.
2. The smallest partial multiplicity of $P(s)$ at ∞ is zero.

Structure at infinity: partial multiplicities at infinity of a Polynomial Matrix

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Minimal bases of rational vector subspaces

- $\mathbb{F}[s]$ is the ring of univariate polynomials with coefficients in $\mathbb{F}$.
- $\mathbb{F}(s)$ is the field of univariate rational functions over $\mathbb{F}$ and
- $\mathbb{F}(s)^n$ is the vector space over $\mathbb{F}(s)$ of n-tuples with entries in $\mathbb{F}(s)$.
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There are many minimal bases of a rational subspace $\mathcal{V} \subseteq \mathbb{F}(s)^n$, but...

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The ordered list of degrees of the vector polynomials in any minimal basis of $\mathcal{V} \subseteq \mathbb{F}(s)^n$ is always the same.

Definition (Minimal indices)

These degrees are called the **minimal indices** of $\mathcal{V} \subseteq \mathbb{F}(s)^n$.

Remark: Minimal bases and indices of the **null spaces** of rational matrices (transfer functions) play a relevant role in several problems of Linear Systems and Control Theory that reduce to solving equations for rational matrices.

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An $m \times n$ rational matrix $R(s) \in \mathbb{F}(s)^{m\times n}$ whose rank r is smaller than m and/or n has non-trivial left and/or right rational null spaces (over the field $\mathbb{F}(s)$ of rational functions):

$$\begin{aligned}\mathcal{N}_\ell(R) &:= \left\{y(s) \in \mathbb{F}(s)^m \,:\, y(s)^T R(s) \equiv 0^T\right\} \subseteq \mathbb{F}(s)^m,\\ \mathcal{N}_r(R) &:= \{x(s) \in \mathbb{F}(s)^n \,:\, R(s)x(s) \equiv 0\} \subseteq \mathbb{F}(s)^n.\end{aligned}$$

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- The **left minimal bases and indices** of $R(s)$ are those of $\mathcal{N}_\ell(R)$.
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Rational null spaces of rational matrices and their minimal indices

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Given a rational matrix $R(s) \in \mathbb{F}(s)^{m \times n}$ with rank r, the complete eigenstructure of $R(s)$ consists of

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$$\frac{\epsilon_1(s)}{\psi_1(s)}, \cdots, \frac{\epsilon_r(s)}{\psi_r(s)} \quad \text{(finite pole/zero structure)},$$

(ii) the invariant orders at ∞ $\quad q_1 \leq \cdots \leq q_r \quad$ (infinite pole/zero structure),

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Remark 1: If the degree of $P(s)$ is one, i.e., $P(s)$ is a pencil, then its complete eigenstructure is revealed by the sizes of the blocks of the Kronecker Canonical Form under strict equivalence.

Remark 2: Such a form does not exist for rational and polynomial matrices (of degree larger than one), which makes it challenging the problems considered in this talk. No canonical form reveals the complete eigenstructure.

Complete eigenstructure of a polynomial matrix

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More structure: Row and column spaces

- The structural data described above have received a lot of attention in the literature on polynomial and rational matrices,
- but, as we teach in basic Linear Algebra courses, every matrix has four fundamental subspaces
- and, so far, we have only used two: the left and right null spaces.
- Given a rational matrix $R(s) \in \mathbb{F}(s)^{m \times n}$ the other two are

$$\mathcal{R}ow(R) = \{R(s)^T w(s) \, : \, w(s) \in \mathbb{F}(s)^m\} \subseteq \mathbb{F}(s)^n \, ,$$
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- which have minimal bases and indices as any other rational subspace.
- **Thus a rational matrix has four sequences of minimal indices.**
- As far as we know, minimal bases and indices of row and column spaces have not received much attention in the literature, but are fundamental for constructing rank revealing factorizations of polynomial matrices with good properties (Dmytryshyn, D, Van Dooren, LAA, 2025).

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Outline

Goals of the talk

In the last decade, **the following two problems have been solved:**

1. If a complete eigenstructure and a degree d are prescribed, to find necessary and sufficient conditions for the existence of a polynomial matrix with this complete eigenstructure and this degree.
2. If a complete eigenstructure is prescribed, to find necessary and sufficient conditions for the existence of a rational matrix with this complete eigenstructure.

In this talk, we present

1. necessary and sufficient conditions for the existence problems above when the minimal indices of the row and column spaces are prescribed
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Theorem

There exists a polynomial matrix $P(s) \in \mathbb{F}[s]^{m\times n}$ with degree d and normal rank $r \leq \min\{m,n\}$, and with

(i) *invariant factors* $\alpha_1(s) \mid \cdots \mid \alpha_r(s)$,

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Comments

- This result was proved by De Terán, D, Van Dooren, SIMAX 2015, for infinite fields $\mathbb{F}$
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The rational result

Theorem

There exists a rational matrix $R(s) \in \mathbb{F}(s)^{m\times n}$ *with normal rank* $r \leq \min\{m, n\}$, *with*

(i) *invariant rational functions*

$$\frac{\epsilon_1(s)}{\psi_1(s)}, \dots, \frac{\epsilon_r(s)}{\psi_r(s)},$$

(ii) *invariant orders at* ∞ $\quad q_1 \leq \cdots \leq q_r,$

(iii) *right minimal indices* $\quad d_1 \geq \cdots \geq d_{n-r},$

(iv) *left minimal indices* $\quad v_1 \geq \cdots \geq v_{m-r},$

if and only if

$$\sum_{i=1}^{r} \deg(\epsilon_i) - \sum_{i=1}^{r} \deg(\psi_i) + \sum_{i=1}^{r} q_i + \sum_{i=1}^{n-r} d_i + \sum_{i=1}^{m-r} v_i = 0.$$

- It was proved by Anguas, D, Hollister, Mackey, SIMAX 2019, for infinite fields $\mathbb{F}$ and extended to any field by Amparan, Baragaña, Marcaida, Roca, LAA 2025.

Outline

Theorem (consequence of Forney, SIAM J. Control 1975)

Let $R(s) \in \mathbb{F}(s)^{m\times n}$ *be a rational matrix of normal rank* r*, with*

- **(i)** *right minimal indices* $d_1 \geq \cdots \geq d_{n-r}$,
- **(ii)** *left minimal indices* $v_1 \geq \cdots \geq v_{m-r}$,
- **(iii)** *minimal indices of* $\mathcal{R}ow(R)$ $\ell_1 \geq \cdots \geq \ell_r$,
- **(iv)** *minimal indices of* $\mathcal{C}ol(R)$ $k_1 \geq \cdots \geq k_r$.

Then,

$$\sum_{i=1}^{m-r} v_i = \sum_{i=1}^{r} k_i \quad \text{and} \quad \sum_{i=1}^{n-r} d_i = \sum_{i=1}^{r} \ell_i .$$

Definition

Let

$$\mathbf{a} = (a_1 \geq \cdots \geq a_m) \quad \text{and} \quad \mathbf{b} = (b_1 \geq \cdots \geq b_m)$$

be two decreasingly ordered sequences of integers.

It is said that $\mathbf{a}$ is *majorized* by $\mathbf{b}$, denoted by

$$\mathbf{a} \prec \mathbf{b},$$

if

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i \quad \text{for } 1 \leq k \leq m-1 \text{ and}$$

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} b_i.$$

Theorem (Baragaña, D, Marcaida, Roca, submitted 2025)

Let $\mathbb{F}$ be algebraically closed. *There exists a rational matrix $R(s) \in \mathbb{F}(s)^{m\times n}$ with normal rank $r < \min\{m, n\}$, with*

- **(i)** *invariant rational functions* $\dfrac{\epsilon_1(s)}{\psi_1(s)}, \cdots, \dfrac{\epsilon_r(s)}{\psi_r(s)}$,
- **(ii)** *invariant orders at ∞* $q_1 \leq \cdots \leq q_r$,
- **(iii)** *minimal indices of $\mathcal{R}ow(P)$* $\ell_1 \geq \cdots \geq \ell_r$,
- **(iv)** *minimal indices of $\mathcal{C}ol(P)$* $k_1 \geq \cdots \geq k_r$

if and only if

$$(-g_r, \ldots, -g_1) \prec (\deg(\epsilon_r) - \deg(\psi_r) + q_r, \ldots, \deg(\epsilon_1) - \deg(\psi_1) + q_1),$$

where $g_1 \geq \cdots \geq g_r$ is the decreasing reordering of $k_r + \ell_1, \ldots, k_1 + \ell_r$.

- **The proof of the necessity is valid over arbitrary fields**. The proof of the sufficiency requires algebraically closed fields in an essential way and we have examples over $\mathbb{R}$ for which the sufficiency does not hold.
- The same comment applies to the result in the next slide.
- The majorization condition amounts in fact to r conditions. The last of such conditions is the Index Sum Theorem, i.e., the unique condition appearing when the complete eigenstructure is prescribed.
- If $r = \min\{m, n\}$, we have to add two trivial conditions:
  - $\ell_1 = \cdots = \ell_r = 0$ if $r = n$

    (coming from $\mathcal{R}ow(P) = \mathbb{F}(s)^n$ in this case),
  - $k_1 = \cdots = k_r = 0$ if $r = m$

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Remarks on previous theorems

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Theorem (Baragaña, D, Marcaida, Roca, submitted 2025)

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if and only if

(1) $(-g_r, \ldots, -g_1) \prec (\deg(\epsilon_r) - \deg(\psi_r) + q_r, \ldots, \deg(\epsilon_1) - \deg(\psi_1) + q_1)$, *where $g_1 \geq \cdots \geq g_r$ is the decreasing reordering of $k_r + \ell_1, \ldots, k_1 + \ell_r$,*

(2) *and* $\sum_{i=1}^{m-r} v_i = \sum_{i=1}^{r} k_i$ and $\sum_{i=1}^{n-r} d_i = \sum_{i=1}^{r} \ell_i$.

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Outline

- We have provided necessary and sufficient conditions for the existence of polynomial and rational matrices when **instead of or in addition to** their **classical complete eigenstructures** **the minimal indices of their row and column spaces** are prescribed.
- The obtained necessary and sufficient conditions are very simple and only require to check some equalities or inequalities of integer numbers related to the prescribed data.
- Extending these results to structured polynomial and rational matrices is essentially a completely open area where just very few particular results have been published so far.

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