Uniqueness of solution of a generalized $\star$-Sylvester equation

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Generalized ★-Sylvester equation

Given \( A, B, C, D, E \in \mathbb{C}^{n \times n} \)

**Goal:** Find necessary and sufficient conditions for the equation

\[
AXB + CX^\star D = E
\]

to have a **unique solution**.

\((X \in \mathbb{C}^{n \times n}, \text{unknown})\)

\((\star = \top \text{ or } \ast)\)
Motivation

- **Natural extension of** $AX + X^*D = E$.
  - Numerical methods for palindromic eigenvalue problems
    [Byers-Kressner’06], [Kressner-Schröder-Watkins’09],
    [Dmytryshyn-Kågström’15]
  - Congruence orbits ($D = A, E = 0$) [D.-Dopico’11]

- Closely related to $AXB + CXD = E$ [Chu’87]

- Iterative algorithms for solving
  $\sum_{i=1}^{r} A_iXB_i + \sum_{j=1}^{s} C_jX^TD_j = E$
  [Wang-Cheng-Wei’07], [Xie-Ding-Ding’09], [Li-Wang-Zhou-Duan’10],
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Which kind of characterization are we looking for?

\[ \Lambda(A - \lambda B) = \text{Spectrum of } A - \lambda B \]

**Theorem** (Uniqueness of solution for generalized Sylvester) [Chu’87]

The equation \(AXB - CXD = E\) has a unique solution iff \(A - \lambda C\) and \(D - \lambda B\) are regular and \(\Lambda(A - \lambda C) \cap \Lambda(D - \lambda B) = \emptyset\).

\((A, C \in \mathbb{R}^{m \times m}; \quad B, D \in \mathbb{R}^{n \times n})\)

**Theorem** (Uniqueness of solution for \(\star\)-Sylvester) [Byers-Kressener’06, Kressner-Schröder-Watkins’09]

\(AX + X^*D = E\) has unique solution iff \(A - \lambda D^*\) is regular and:

- \(\star = \star\): If \(\lambda \in \Lambda(A - \lambda D^*)\), then \((1/\lambda) \not\in \Lambda(A - \lambda D^*)\).
- \(\star = \top\): If \(1 \neq \lambda \in \Lambda(A - \lambda D^\top)\), then \((1/\lambda) \not\in \Lambda(A - \lambda D^\top)\), and \(m_1(A - \lambda D^\top) \leq 1\).

\(m_{\mu}(A - \lambda B)\): algebraic multiplicity of \(\mu\) in \(A - \lambda B\)
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\( m_\mu(A - \lambda B) : \text{algebraic multiplicity of } \mu \text{ in } A - \lambda B \)
Which kind of characterization are we looking for? (cont.)

Know conditions for $AXB - CXD = E$ and $AX + X^*D = E$: in terms of **spectral properties** of **matrix pencils** constructed from the coefficient matrices.
Which kind of characterization are we looking for? (cont.)

Know conditions for $AXB - CXD = E$ and $AX + X^*D = E$: in terms of spectral properties of matrix pencils constructed from the coefficient matrices.

Q: Analogous characterization for $AXB + CX^*D = E$ ??
The vec approach

\[ \text{vec} (AXB + CX^* D) = \text{vec} (E) \] leads to

- \[ \star = \top : [B^\top \otimes A + \Pi (C \otimes D^\top)] \text{vec} (X) = \text{vec} (E) \]

- \[ \star = \star : (B^\top \otimes A) \text{vec} (X) + \Pi (C \otimes D^\top) \text{vec} (X) = \text{vec} (E) \]
The vec approach

\[ \text{vec}(AXB + CX^*D) = \text{vec}(E) \quad \text{leads to} \]

\[ \begin{array}{c}
\star = \top : [B^\top \otimes A + \Pi(C \otimes D^\top)] \text{vec}(X) = \text{vec}(E) \\
\star = \star : (B^\top \otimes A) \text{vec}(X) + \Pi(C \otimes D^\top) \text{vec}(\overline{X}) = \text{vec}(E)
\end{array} \]

Linear over \( \mathbb{C} \) \( \checkmark \)
The vec approach

\[ \text{vec} \left( AXB + CX^*D \right) = \text{vec} \left( E \right) \quad \text{leads to} \]

- \( \star = \top \): \[ [B^\top \otimes A + \Pi(C \otimes D^\top)] \text{vec} \left( X \right) = \text{vec} \left( E \right) \]
  Linear over \( \mathbb{C} \) ✓

- \( \star = \ast \): \( (B^\top \otimes A) \text{vec} \left( X \right) + \Pi(C \otimes D^\top) \text{vec} \left( X \right) = \text{vec} \left( E \right) \)
  Not linear over \( \mathbb{C} \)
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- \( \star = \top : \quad \left[ B^\top \otimes A + \Pi(C \otimes D^\top) \right] \text{vec}(X) = \text{vec}(E) \]
  \text{Linear over } \mathbb{C} \checkmark

- \( \star = \star : \quad (B^\top \otimes A) \text{vec}(X) + \Pi(C \otimes D^\top) \text{vec}(\overline{X}) = \text{vec}(E) \]
  \text{Not linear over } \mathbb{C} \leadsto \text{vec}(X) = [\text{vec}(\text{Re } X); \text{vec}(\text{Im } X)]
The vec approach

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leads to

- \( \star = \top : [B^\top \otimes A + \Pi(C \otimes D^\top)] \text{vec}(X) = \text{vec}(E) \)
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- \( \star = \star : (B^\top \otimes A) \text{vec}(X) + \Pi(C \otimes D^\top) \text{vec}(\bar{X}) = \text{vec}(E) \)
  \textbf{Linear over} \( \mathbb{R} \) \( \checkmark \sim \) \( \text{vec}(X) = [\text{vec}(\text{Re } X); \text{vec}(\text{Im } X)] \)
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  Linear over \( \mathbb{R} \) \( \checkmark \) \( \sim \) vec \( (X) = [\text{vec}(\text{Re } X); \text{vec}(\text{Im } X)] \)

\( AXB + CX^*D = E \) can be written as a linear system \( MY = b \):

\[ Y = \begin{cases} \text{vec}(X), & \text{if } \star = \top \\ [\text{vec}(\text{Re } X); \text{vec}(\text{Im } X)], & \text{if } \star = \ast \end{cases} \]
The vec approach (cont.)

\[ M \in \begin{cases} \mathbb{C}^{n^2 \times n^2}, & \text{if } \star = \top, \\ \mathbb{R}^{(2n^2) \times (2n^2)}, & \text{if } \star = \ast \end{cases} \]
The vec approach (cont.)

\[ M \in \begin{cases} 
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😊 Too large!
The vec approach (cont.)

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\[ \text{AXB} + \text{CX}^\ast D = E \text{ has a unique solution } \iff M \text{ is nonsingular} \]
The vec approach (cont.)

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\[ AXB + CX^\ast D = E \text{ has a unique solution } \iff M \text{ is nonsingular} \]

\[
\begin{align*}
AXB + CX^\ast D &= E \\
AXB + CX^\ast D &= 0
\end{align*}
\]

We only need to look at the homogeneous equation!
The vec approach (cont.)

\[ M \in \begin{cases} 
  \mathbb{C}^{n^2 \times n^2}, & \text{if } \star = \top, \\
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\end{cases} \]

 unloaded! Not easy to handle with

\[ AXB + CX^*D = E \text{ has a unique solution } \iff M \text{ is nonsingular} \]

\[ AXB + CX^*D = E \text{ has a unique solution} \]
\[ \iff \]
\[ AXB + CX^*D = 0 \text{ has a unique solution} \]

We only need to look at the homogeneous equation!
Two basic preparatory results

Lemma 1

If \( AXB + CX^*D = 0 \) has a unique solution, then

(a) At least one of \( A, C \) is invertible.

(b) At least one of \( B, D \) is invertible.
# Two basic preparatory results

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Proof. (a) If $A, C$ both singular, then $Au = 0 = Cv$, with $u, v \neq 0 \Rightarrow X = uv^*$ is a nonzero solution.
(b) If $B, D$ both singular, then $u^*D = v^*B = 0$ with $u, v \neq 0 \Rightarrow X = uv^*$ is a nonzero solution □
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If both $A, C$ or both $B, D$ are singular, then $AXB + CX^*D = 0$ has a rank-1 solution
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If both $A, C$ or both $B, D$ are singular, then $AXB + CX^*D = 0$ has a rank-1 solution.

We will see that also one of $A, D$, and one of $B, C$ must be invertible!
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If $A, B$ invertible: $X + A^{-1}CX^*DB^{-1} = 0 \iff \star$-Stein
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If \( A, D \) invertible: \( XBD^{-1} + A^{-1}CX^* = 0 \) \( \iff \star \)-Sylvester
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If \( A, D \) invertible: \( XBD^{-1} + A^{-1}CX^* = 0 \) ⇔ \( \ast \)-Sylvester
If \( C, B \) invertible: \( C^{-1}AX + X^*DB^{-1} = 0 \) ⇔ \( \ast \)-Sylvester
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If $C, B$ invertible: $C^{-1} AX + X^* DB^{-1} = 0 \iff \star$-Sylvester
If $C, D$ invertible: $C^{-1} AXBD^{-1} + X^* = 0 \iff \star$-Stein

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$AXB + X^* = 0$ has a unique solution $\iff AB^*Y + Y^* = 0$ has a unique solution
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Proof. \((\Leftarrow):\ AXB + X^* = 0 \ (X \neq 0) \ \Rightarrow (AB^*)(X^*A^*) + AX = 0, \) so \(Y = (AX)^* \neq 0\) is solution of \(AB^*Y + Y^* = 0.\)
\((\Rightarrow):\ AB^*Y + Y^* = 0 \ (Y \neq 0) \ \Rightarrow X = B^*Y \neq 0\) is a solution of \(AXB + X^* = 0.\) \(\Box\)
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Corollary
$AXB + CX^*D = 0$ has a unique solution if and only if

(a) $A$ is invertible and $D^*A^{-1}CY + Y^*B = 0$ has a unique solution, or
(b) $C$ is invertible and $B^*C^{-1}AY + Y^*D = 0$ has a unique solution.
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Characterization for $\star$-Sylvester (again)

**Theorem** (Uniqueness of solution for $\star$-Sylvester) [Byers-Kressner’06, Kressner-Schröder-Watkins’09]

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**S** $\subseteq C \cup \{\infty\}$ is reciprocal free if $\lambda \neq \mu - 1$ for all $\lambda, \mu \in S$.

**$\star$-reciprocal free** if $\lambda \neq (\mu)^{-1}$ for all $\lambda, \mu \in S$. 

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Two different proofs:

- **[BK’06]** ($\star = \top$): Relies on some continuity arguments of operators.
- **[KSW’09]** ($\star = \star$)

- **[D-Dopico-Guillery-Montealegre-Reyes’11]**: Using The **Kronecker canonical form** of $A + \lambda B^*$. 

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$AX + X^D = E$ has unique solution if and only if $A - \lambda D^\star$ is regular and:

- $\star = \star$: If $\lambda \in \Lambda(A - \lambda D^\star)$, then $(1/\lambda) \not\in \Lambda(A - \lambda D^\star)$.
- $\star = \top$: If $1 \neq \lambda \in \Lambda(A - \lambda D^\top)$, then $(1/\lambda) \not\in \Lambda(A - \lambda D^\top)$, and $m_1(A - \lambda D^\top) \leq 1$.

$S \subseteq \mathbb{C} \cup \{\infty\}$ is

- reciprocal free if $\lambda \neq \mu^{-1}$ for all $\lambda, \mu \in S$
- $\star$-reciprocal free if $\lambda \neq (\mu)^{-1}$ for all $\lambda, \mu \in S$
Theorem (Uniqueness of solution for $\star$-Sylvester) [Byers-Kressner’06, Kressner-Schröder-Watkins’09]

$AX + X^* D = E$ has unique solution if and only if $A - \lambda D^*$ is regular and:
- $\star = \star$: $\Lambda(A - \lambda D^*)$ is $\star$-reciprocal free.
- $\star = \top$: $\Lambda(A - \lambda D^\top) \setminus \{1\}$ is reciprocal free, and $m_1(A - \lambda D^\top) \leq 1$.

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Characterization of uniqueness of solution

**Theorem** *(Uniqueness for generalized \( \star \)-Sylvester)*

\[ AXB + CX^*D = E \] has a **unique solution** if and only if the pencil

\[
P(\lambda) := \begin{bmatrix} \lambda D^* & B^* \\ A & \lambda C \end{bmatrix}
\]

is **regular** and:

- \( \star = \star \): \( \Lambda(P) \) is \( \star \)-reciprocal free.
- \( \star = \top \): \( \Lambda(P) \setminus \{\pm 1\} \) is reciprocal free and \( m_1(P) = m_{-1}(P) \leq 1 \).

**Remark:** \( m_{\lambda}(P) = m_{-\lambda}(P) \)
The main result

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Proof of the main result

\[ AXB + CX^*D = E \]

has unique sol. \iff \[ P(\lambda) := \begin{bmatrix} \lambda D^* & B^* \\ A & \lambda C \end{bmatrix} \text{ regular and } \begin{cases} \star = \star : \Lambda(P) \text{-rec, free} \\
\star = \top : \Lambda(P) \setminus \{\pm 1\} \text{ rec. free, } m_{\pm 1}(P) \leq 1 \end{cases} \]

Proof:

- **A** invertible: \[ \det P(\lambda) = \pm \det(A) \det(B^* - \lambda^2 D^* A^{-1} C) \]
  \[
  \begin{bmatrix} 0 & I \\ I & -\lambda D^* A^{-1} \end{bmatrix} \begin{bmatrix} \lambda D^* & B^* \\ A & \lambda C \end{bmatrix} = \begin{bmatrix} A & \lambda C \\ 0 & B^* - \lambda^2 D^* A^{-1} C \end{bmatrix}.
  \]

- **C** invertible: \[ \det P(\lambda) = \pm \det(C) \det(B^* C^{-1} A - \lambda^2 D^*) \]
Proof of the main result

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has unique sol. \iff \[ P(\lambda) := \begin{bmatrix} \lambda D^* & B^* \\ A & \lambda C \end{bmatrix} \] is regular and

\[ \Lambda(P) \text{ regular, free, } \star = * \] \quad \text{and} \quad \Lambda(P) \setminus \{\pm 1\} \text{ regular, free, } m_{\pm 1}(P) \leq 1

Proof:

- A invertible: \( \det P(\lambda) = \pm \det(A) \det(B^* - \lambda^2 D^* A^{-1} C) \)
- C invertible: \( \det P(\lambda) = \pm \det(C) \det(B^* C^{-1} A - \lambda^2 D^*) \)

\[
\begin{bmatrix}
\lambda I & -\lambda B^* C^{-1} \\
0 & I
\end{bmatrix} \cdot \begin{bmatrix}
\lambda D^* & B^* \\
A & \lambda C
\end{bmatrix} = \begin{bmatrix}
\lambda^2 D^* - B^* C^{-1} A & 0 \\
A & \lambda C
\end{bmatrix}.
\]

\[ \Box \]
Proof of the main result

\( AXB + CX^*D = E \) has unique sol. \( \iff \) \( P(\lambda) := \begin{bmatrix} \lambda D^* & B^* \\ A & \lambda C \end{bmatrix} \) regular and \( \Lambda(P) \) *-rec, free and \( \Lambda(P) \) \{±1\} rec. free, \( m_{±1}(P) \leq 1 \)

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- **A** invertible: \( \det P(\lambda) = \pm \det(A) \det(B^* - \lambda^2 D^* A^{-1} C) \)
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**Proof:**

- **A invertible:** \( \det P(\lambda) = \pm \det(A) \det(B^* - \lambda^2 D^* A^{-1} C) \)
- **C invertible:** \( \det P(\lambda) = \pm \det(C) \det(B^* C^{-1} A - \lambda^2 D^*) \)

Recall:

**AXB + CX^*D = 0** has a unique solution iff

(a) **A is invertible** and \( D^* A^{-1} CY + Y^* B = 0 \) has a unique solution, or

(b) **C is invertible** and \( B^* C^{-1} AY + Y^* D = 0 \) has a unique solution.

**AX + X^*D = E** has unique solution iff **A** is regular and:

- \( \star = \star \): \( \Lambda(A - \lambda D^*) \) is \(*\)-reciprocal free.
- \( \star = \top \): \( \Lambda(A - \lambda D^\top) \setminus \{1\} \) is reciprocal free, and \( m_{1}(A - \lambda D^\top) \leq 1 \).
The periodic Schur decomposition

**Theorem [Bojanczyk-Golub-Van Dooren’92]**

There are $U_1, U_2, V_1, V_2$ unitary such that

\[
U_1 AV_1 = T_A, \quad U_1 CV_2 = T_C, \\
U_2 B^* V_1 = T_B^*, \quad U_2 D^* V_2 = T_D^*,
\]

with $T_A, T_B^*, T_C, T_D^*$ upper triangular.
The periodic Schur decomposition

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with $T_A, T_B^*, T_C, T_D^*$ upper triangular.

Connection with the pencil $P(\lambda)$:

$$\begin{bmatrix} U_2 & U_1 \end{bmatrix} \begin{bmatrix} \lambda D^* & B^* \\ A & \lambda C \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \lambda T_D^* & T_B^* \\ T_A & \lambda T_C \end{bmatrix}$$
An $O(n^3)$ algorithm

(Based on the algorithm in [D-Dopico’11] for $AX + X^\top D = E$, outlined in [Chiang-Chu-Lin’12])

\[ T_A \cdot X \cdot T_B + T_C \cdot X^\top \cdot T_D = E \]
An \( O(n^3) \) algorithm

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\[ T_A \cdot X \cdot T_B + T_C \cdot X^\top \cdot T_D = E \]

\[
\begin{array}{cccc}
X_{11} & \ldots & X_{1,k-1} & X_{1k} \\
& \ddots & \ddots & \vdots \\
X_{k-1,1} & \ldots & X_{k-1,k-1} & X_{k-1,k} \\
X_{k1} & \ldots & X_{k,k-1} & X_{kk} \\
\end{array}
\]
An $O(n^3)$ algorithm

(Based on the algorithm in [D-Dopico’11] for $AX + X^T D = E$, outlined in [Chiang-Chu-Lin’12])
An $O(n^3)$ algorithm

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(Based on the algorithm in [D-Dopico’11] for $AX + X^T D = E$, outlined in [Chiang-Chu-Lin’12])

$$TA \cdot X \cdot TB + TC \cdot X^T \cdot TD = E$$
An $O(n^3)$ algorithm

(Based on the algorithm in [D-Dopico’11] for $AX + X^T D = E$, outlined in [Chiang-Chu-Lin’12])
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Goal 1:
Obtain necessary and sufficient conditions for uniqueness of solution of systems of equations of the form $AXB + CX^*D = E$ (with both $X = Y$ or $X \neq Y$) and $\star = 1, \top, \ast$.

Goal 2:
Write an algorithm to compute the unique solution.
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(Ongoing work with B. Iannazzo, F. Poloni, and L. Robol)
Goal 1:

Obtain **necessary and sufficient conditions** for **uniqueness of solution** of systems of equations of the form $AXB + CX^*D = E$ (with both $X = Y$ or $X \neq Y$) and $\star = 1, \top, \ast$.

Goal 2:

Write an **algorithm** to compute the unique solution.

(Ongoing work with B. Iannazzo, F. Poloni, and L. Robol)

More on this at the forthcoming **ILAS2016 Conference in Leuven**


C.-Y. CHIANG, K.-W. E. CHU, W.-W. LIN, *On the *\(-\)Sylvester equation* $AX \pm X^*B = C$, AMC 218 (2012)*

F. DE TÉRÁN, F. M. DOPICO, *Consistency and efficient solution of the Sylvester equation for *\(-\)-congruence*, ELA 22 (2011)


D. KRESSNER, C. SCHRODER, D. S. WATKINS, *Implicit QR algorithms for palindromic and even eigenvalue problems*, NA 51(2) (2009)
<table>
<thead>
<tr>
<th>Author(s)</th>
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<td>F. De Terán, B. Iannazzo</td>
<td>Uniqueness of solution of a generalized $\star$-Sylvester matrix equation</td>
<td>LAA 493 (2016)</td>
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<tr>
<td>R. Byers, D. Kressner</td>
<td>Structured condition numbers for invariant subspaces</td>
<td>SIMAX 28 (2) (2006)</td>
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<td>C.-Y. Chiang, K.-W. E. Chu, W.-W. Lin</td>
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<td>F. De Terán, F. M. Dopico, N. Guillery, D. Montealegre, N. Z. Reyes</td>
<td>The solution of the equation $AX + X^\star B = 0$</td>
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THANKS FOR YOUR ATTENTION !!!!!