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Uniqueness of solution of generalized Sylvester equations with rectangular coefficients

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Federico Poloni
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What is this talk about?

MS1: Matrix equations: analysis and algorithms.

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What is this talk about?

MS1: Matrix equations: analysis and algorithms.

In this talk: No algorithms at all!

☞ Theoretical characterization for the uniqueness of solution of generalized Sylvester equations **explicitly in terms of their coefficients**.

👍 Just basic linear algebra techniques.

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Generalized Sylvester equations

(GS) $AXB + CXD = E$ $\rightsquigarrow$ Generalized Sylvester equation.

(GS$\star$) $AXB + CX^{\star}D = E$ $\rightsquigarrow$ Generalized $\star$-Sylvester equation $\quad (\star = \top, *)$.

$X \in \mathbb{C}^{m \times n}$ (unknown) $\quad$ A, B, C, D, E complex matrices with appropriate size.

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$X \in \mathbb{C}^{m\times n}$ (unknown) A, B, C, D, E complex matrices with appropriate size.

(GS) is linear over $\mathbb{C}$.

(GS$\top$) is linear over $\mathbb{C}$.

(GS$*$) is linear over $\mathbb{R}$.

The vec approach

You can use (for $\star = \top$):

$$\mathrm{vec}(AXB - CX^\top D) = \mathrm{vec}(E) \Leftrightarrow M\,\mathrm{vec}(X) = \mathrm{vec}(E)$$

with

$$M = B^\top \otimes A + (D^\top \otimes C)\Pi$$

(Π is a permutation matrix associated with the transposition).

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☟ You'll need to keep track of the entries of A, B, C, D in M.

We will not follow this approach.

Existence and uniqueness of solution

(Eq) $\quad AXB + CX^{\sigma}D = E \quad (\sigma = 1, \top, *)$

Solvability (**S**)	(Eq) has a solution, for some given A, B, C, D, E.
Unique solvability (**US**)	(Eq) has a **unique** solution, for given A, B, C, D, E.
Solvability for any right-hand side (**SR**)	(Eq) has a solution for **any** E, and given A, B, C, D
At most one solution, for any right-hand side (**OR**)	(Eq) has **at most one** solution, for **any** E, and given A, B, C, D
Exactly one solution, for any right-hand side (**UR**)	(Eq) has **unique** solution, for **any** E, and given A, B, C, D

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$\Leftrightarrow$ The operator $X \mapsto AXB + CX^{\star}D$ is **invertible**.

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Some history

Characterization for **S**, **US**, **SR**, **OR**, **UR**, in terms of A, B, C, D, E:

	$AXB + CXD = E$ square coefficients	$AXB + CXD = E$ general coefficients	$AXB + CX^\star D = E$ square coefficients	$AXB + CX^\star D = E$ general coefficients
S	[DK, 2016]	[DK, 2016], [Košir, 1992]	[DK, 2016]	[DK, 2016]
US	[Chu, 1987]	[Košir, 1992]	[DI, 2016]	open
SR	same as **US**	[DIPR, 2018] (after [Košir, 1992])	same as **US**	open
OR	same as **US**	[Košir, 1996]	same as **US**	open
UR	same as **US**	[DIPR, 2018] (after [Košir, 1992])	same as **US**	[DIPR, 2018]

[DI, 2016]=[D-Iannazzo, 2016]

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[DK, 2016]=[Dmytryshyn-Kågström, 2016]

[Byers-Kressner, 2006]: **US**, **UR** $\rightsquigarrow AX + X^\top D = E$ $(A, D, X \in \mathbb{C}^{n\times n})$.

[Kressner-Schröder-Watkins, 2009]: **US**, **UR** $\rightsquigarrow AX + X^* D = E$ $(A, D, X \in \mathbb{C}^{n\times n})$.

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Some basic notions

A matrix pencil $X + \lambda Y$ ($X, Y \in \mathbb{C}^{m \times n}$) is regular if $m = n$ and $\det(X + \lambda Y) \not\equiv 0$.

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A matrix pencil $X + \lambda Y$ ($X, Y \in \mathbb{C}^{m\times n}$) is regular if $m = n$ and $\det(X + \lambda Y) \not\equiv 0$.

Definition: If $X + \lambda Y$ is regular:

(1) $\Lambda(X + \lambda Y) := \{\mu \in \mathbb{C} : \det(X + \mu Y) = 0\} \cup \{\infty\}$ (Spectrum of $X + \lambda Y$)

($\infty \in \Lambda(X + \lambda Y) \Leftrightarrow \operatorname{rank} Y < n$).

(2) If $\mu \in \mathbb{C}$, then $m_\mu(X + \lambda Y) :=$ algebraic multiplicity of μ
(as a root of $\det(X + \lambda Y)$).

(3) $m_\infty(X + \lambda Y) := m_0(Y + \lambda X)$.

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(as a root of $\det(X+\lambda Y)$).

(3) $m_\infty(X+\lambda Y):=m_0(Y+\lambda X)$.

Definition: $\mathscr{S}\subseteq\mathbb{C}\cup\{\infty\}$. Then $\mathscr{S}$ is

(a) reciprocal free if $\lambda\neq\mu^{-1}$, for all $\lambda,\mu\in\mathscr{S}$;

(b) $*$-reciprocal free if $\lambda\neq(\overline{\mu})^{-1}$, for all $\lambda,\mu\in\mathscr{S}$.

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Previous results: Sylvester equations

$X \in \mathbb{C}^{m\times n}$

Characterization for **UR**:

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Characterization for **UR**:

Equation	Conditions	Sizes	Ref.
$AX+XD=E$	$\Lambda(A)\cap\Lambda(-D)=\emptyset$	$A\in\mathbb{C}^{m\times m}$ $D\in\mathbb{C}^{n\times n}$ $M\in\mathbb{C}^{mn\times mn}$	[Sylvester'1884]

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$AX + X^* D = E$	$A - \lambda D^*$ is regular $\Lambda(A - \lambda D^*)$ is $*$-reciprocal free	$A \in \mathbb{C}^{m\times n}$ $D \in \mathbb{C}^{n\times m}$ $M \in \mathbb{C}^{n^2\times mn}$	[Kressner-Schröder-Watkins'09]

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$AX + X^*D = E$	$A-\lambda D^*$ is regular $\Lambda(A-\lambda D^*)$ is $*$-reciprocal free	$A\in\mathbb{C}^{m\times n}$ $D\in\mathbb{C}^{n\times m}$ $M\in\mathbb{C}^{n^2\times mn}$	[Kressner-Schröder-Watkins'09]
$AX + X^\top D = E$	$A-\lambda D^\top$ is regular $\Lambda(A-\lambda D^\top)\setminus\{1\}$ is reciprocal free, $m_1(A-\lambda D^\top)\leq 1$	$A\in\mathbb{C}^{m\times n}$ $D\in\mathbb{C}^{n\times m}$ $M\in\mathbb{C}^{n^2\times mn}$	[Byers-Kressner'06]

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$AX + X^* D = E$	$A - \lambda D^*$ is regular $\Lambda(A - \lambda D^*)$ is $*$-reciprocal free	$A \in \mathbb{C}^{m\times n}$ $D \in \mathbb{C}^{n\times m}$ $M \in \mathbb{C}^{n^2\times mn}$	[Kressner-Schröder-Watkins'09]
$AX + X^\top D = E$	$A - \lambda D^\top$ is regular $\Lambda(A - \lambda D^\top) \setminus \{1\}$ is reciprocal free, $m_1(A - \lambda D^\top) \leq 1$	$A \in \mathbb{C}^{m\times n}$ $D \in \mathbb{C}^{n\times m}$ $M \in \mathbb{C}^{n^2\times mn}$	[Byers-Kressner'06]

$\Rightarrow m = n$.

Previous results: generalized Sylvester equations

$X \in \mathbb{C}^{m\times n}$

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Equation	Conditions	Sizes	Ref.
$AXB + CXD = E$	$A - \lambda C$, $B - \lambda D$ are regular, $\Lambda(A-\lambda C) \cap \Lambda(B - \lambda D) = \emptyset$	$A, C \in \mathbb{C}^{m\times m}$ $B, D \in \mathbb{C}^{n\times n}$ $M \in \mathbb{C}^{mn\times mn}$	[Chu'87]

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$AXB + CXD = E$	$A - \lambda C$, $B - \lambda D$ are regular, $\Lambda(A-\lambda C)\cap\Lambda(B-\lambda D)=\emptyset$	$A, C \in \mathbb{C}^{m\times m}$ $B, D \in \mathbb{C}^{n\times n}$ $M \in \mathbb{C}^{mn\times mn}$	[Chu'87]
$AXB + CX^* D = E$	$\begin{bmatrix} \lambda D^* & B^* \\ A & \lambda C \end{bmatrix}$ is regular, $\Lambda\left(\begin{bmatrix} \lambda D^* & B^* \\ A & \lambda C \end{bmatrix}\right)$ is $*$-reciprocal free	$A \in \mathbb{C}^{n\times n}$ $D \in \mathbb{C}^{n\times n}$ $M \in \mathbb{C}^{n^2\times n^2}$ $(m = n)$	[D-Iannazzo'16]

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$AXB + CXD = E$	$A-\lambda C$, $B-\lambda D$ are regular, $\Lambda(A-\lambda C)\cap\Lambda(B-\lambda D)=\emptyset$	$A,C\in\mathbb{C}^{m\times m}$ $B,D\in\mathbb{C}^{n\times n}$ $M\in\mathbb{C}^{mn\times mn}$	[Chu'87]
$AXB + CX^*D = E$	$\begin{bmatrix} \lambda D^* & B^* \\ A & \lambda C \end{bmatrix}$ is regular, $\Lambda\left(\begin{bmatrix} \lambda D^* & B^* \\ A & \lambda C \end{bmatrix}\right)$ is $*$-reciprocal free	$A\in\mathbb{C}^{n\times n}$ $D\in\mathbb{C}^{n\times n}$ $M\in\mathbb{C}^{n^2\times n^2}$ $(m=n)$	[D-Iannazzo'16]
$AXB + CX^\top D = E$	$\begin{bmatrix} \lambda D^\top & B^\top \\ A & \lambda C \end{bmatrix}$ is regular, $\Lambda\left(\begin{bmatrix} \lambda D^\top & B^\top \\ A & \lambda C \end{bmatrix}\right)\setminus\{\pm1\}$ is reciprocal free, $m_{\pm1}(A-\lambda D^\top)\leq 1$	$A\in\mathbb{C}^{n\times n}$ $D\in\mathbb{C}^{n\times n}$ $M\in\mathbb{C}^{n^2\times n^2}$ $(m=n)$	[D-Iannazzo'16]

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Previous results: generalized Sylvester equations

$X \in \mathbb{C}^{m\times n}$

Characterization for **UR**:

Equation	Conditions	Sizes	Ref.
$AXB+CXD=E$	$A-\lambda C$, $B-\lambda D$ are regular, $\Lambda(A-\lambda C)\cap\Lambda(B-\lambda D)=\emptyset$	$A,C\in\mathbb{C}^{m\times m}$ $B,D\in\mathbb{C}^{n\times n}$ $M\in\mathbb{C}^{mn\times mn}$	[Chu'87]
$AXB+CX^*D=E$	$\begin{bmatrix}\lambda D^* & B^*\\ A & \lambda C\end{bmatrix}$ is regular, $\Lambda\left(\begin{bmatrix}\lambda D^* & B^*\\ A & \lambda C\end{bmatrix}\right)$ is $*$-reciprocal free	$A\in\mathbb{C}^{n\times n}$ $D\in\mathbb{C}^{n\times n}$ $M\in\mathbb{C}^{n^2\times n^2}$ $(m=n)$	[D-Iannazzo'16]
$AXB+CX^\top D=E$	$\begin{bmatrix}\lambda D^\top & B^\top\\ A & \lambda C\end{bmatrix}$ is regular, $\Lambda\left(\begin{bmatrix}\lambda D^\top & B^\top\\ A & \lambda C\end{bmatrix}\right)\setminus\{\pm1\}$ is reciprocal free, $m_{\pm1}(A-\lambda D^\top)\leq 1$	$A\in\mathbb{C}^{n\times n}$ $D\in\mathbb{C}^{n\times n}$ $M\in\mathbb{C}^{n^2\times n^2}$ $(m=n)$	[D-Iannazzo'16]

What happens for A,B,C,D,E rectangular?

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Conditions on the eigenvalues are not enough

The characterization for **UR** in the “square" case depends on the eigenvalues of $\begin{bmatrix} \lambda D^\top & B^\top \\ A & \lambda C \end{bmatrix}$ (provided it's regular).

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☞ However, for "rectangular" coefficients this is not enough:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \Leftrightarrow x = 0 \qquad \text{Not } \textbf{US} \qquad (1)$$

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$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} [0] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \Leftrightarrow x = 0 \qquad \text{Not } \mathbf{US} \tag{1}$$

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The associated pencils are:

$$\mathscr{Q}_1(\lambda) = \left[\begin{array}{cc|c} \lambda & 0 & 0 \\ \hline 1 & 0 & \lambda \\ 0 & 1 & 0 \end{array}\right], \qquad \mathscr{Q}_2(\lambda) = \left[\begin{array}{cc|c} \lambda & 0 & 1 \\ \hline 0 & 0 & \lambda \\ 0 & 1 & 0 \end{array}\right].$$

which are **regular** and with the same eigenstructure.

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which are **regular** and with the same eigenstructure.

The main result: previous considerations

$A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}, D \in \mathbb{C}^{m\times q}$.

Set $\mathscr{Q}(\lambda) := \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix} \in \mathbb{C}^{(q+p)\times(m+n)}$

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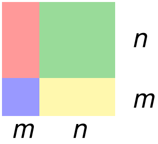
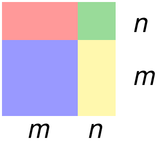
Set $\mathscr{Q}(\lambda) := \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix} \in \mathbb{C}^{(q+p)\times(m+n)}$

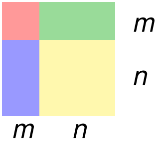
- If $p = m, q = n$, then $m_\infty(\mathscr{Q}) \geq |m-n|$:



or



- If $p = n, q = m$, then $m_0(\mathscr{Q}) \geq |m-n|$:



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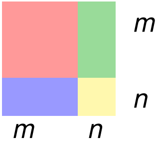

Removing the "dimension induced" $0/\infty$ e-vals

If $p=m, q=n$, set:

$$\widehat{\Lambda}(\mathscr{Q}) := \begin{cases} \Lambda(\mathscr{Q}), & \text{if } m_\infty(\mathscr{Q}) > |m-n|, \\ \Lambda(\mathscr{Q}) \setminus \{\infty\}, & \text{if } m_\infty(\mathscr{Q}) = |m-n|. \end{cases}$$

If $p=n, q=m$, set:

$$\widetilde{\Lambda}(\mathscr{Q}) := \begin{cases} \Lambda(\mathscr{Q}), & \text{if } m_0(\mathscr{Q}) > |m-n|, \\ \Lambda(\mathscr{Q}) \setminus \{0\}, & \text{if } m_0(\mathscr{Q}) = |m-n|. \end{cases}$$

Size constraints

$$\overbrace{A}^{p\times m}\overbrace{X}^{m\times n}\overbrace{B}^{n\times q}+\overbrace{C}^{p\times n}\overbrace{X^\star}^{n\times m}\overbrace{D}^{m\times q}=\overbrace{E}^{p\times q}$$

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Size constraints

$$\overbrace{A}^{p\times m}\overbrace{X}^{m\times n}\overbrace{B}^{n\times q}+\overbrace{C}^{p\times n}\overbrace{X^{\star}}^{n\times m}\overbrace{D}^{m\times q}=\overbrace{E}^{p\times q}\Rightarrow\begin{cases} pq \text{ equations}\\ mn \text{ unknowns}\end{cases}$$

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UR $\Rightarrow \boxed{pq = mn}$

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The main result: statement

$A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}$, and $D \in \mathbb{C}^{m\times q}$, $\mathscr{Q}(\lambda) := \left[\begin{smallmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{smallmatrix}\right]$.

Theorem (**UR** for $AXB + CX^\star D = E$) [D-Iannazzo-Poloni-Robol'18]

$AXB + CX^\star D = E$ has a unique solution, for any E, iff $\mathscr{Q}(\lambda)$ is regular and one of the following holds:

(i) $p = m \neq n = q$, either $m < n$ and A is invertible or $m > n$ and B is invertible, and

- If $\star = \top$, $\widehat{\Lambda}(\mathscr{Q}) \setminus \{\pm 1\}$ is reciprocal free and $m_1(\mathscr{Q}) = m_{-1}(\mathscr{Q}) \leq 1$.
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(ii) $p = n \neq m = q$, either $m > n$ and C is invertible or $m < n$ and D is invertible, and

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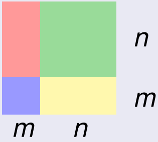

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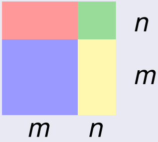

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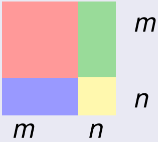

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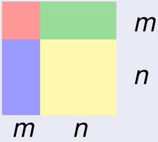

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Proof: some ideas

1. $p < \min\{m,n\}$. $\exists u, v \neq 0$ such that $Au = 0 = Cv$ (because of the dimensions of A, C). Then $X = uv^\star$ is a nonzero solution of $AXB + CX^\star D = 0$.
2. If $p > \max\{m,n\}$: $mn = pq \Rightarrow q < \min\{m,n\} \Rightarrow \exists u, v \neq 0$ such that $v^\star B = 0 = u^\star D$, and $X = uv^\star$ is a nonzero solution of $AXB + CX^\star D = 0$.
3. $m < p < n$ and $mn = pq \Rightarrow m < q < n \Rightarrow m < \min\{p,q\} \Rightarrow \exists u, v \neq 0$ such that $u^\top A = v^\top D^\top = 0$.
   For $\star = \top$:
   $$AXB + CX^\top D = 0 \Leftrightarrow M\,\mathrm{vec}(X) = 0, \qquad M = B^\top \otimes A + (D^\top \otimes C)\Pi.$$
   Then, $(v^\top \otimes u^\top)M = 0$, so M is singular and $AXB + CX^\top D = 0$ has a nonzero solution.
4. $n < p < m$. By setting $Y = X^\top$, $AXB + CX^\top D = 0 \Leftrightarrow CYD + AY^\top B = 0$, so we use the previous result.
5. The case $mn = pq$ and $p \in \{m,n\}$, with $m \neq n$ is more involved.

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The equation $AXB - CXD = E$

Theorem [D-Iannazzo-Poloni-Robol'18]

$AXB - CXD = E$ has **exactly** one solution, for **all** E, iff:

- $A - \lambda C$ and $D^\top - \lambda B^\top$ are regular and $\Lambda(A - \lambda C) \cap \Lambda(D^\top - \lambda B^\top) = \emptyset$, or
- there is some $s \in \mathbb{Z}^+$ such that $\mathrm{KCF}(A - \lambda C) = \bigoplus L_s$ and $\mathrm{KCF}(B^\top - \lambda D^\top) = \bigoplus L_s^\top$ or viceversa.

(KCF: Kronecker canonical form, $L_s = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \end{bmatrix}_{s \times (s+1)}$).

Some observation on the $\star = *$ case

Lemma

$AXB + CX^*D = 0$ has a unique solution iff

$$\begin{aligned} AXB + CYD &= 0, \\ D^*XC^* + B^*YA^* &= 0, \end{aligned}$$

has a unique solution.

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Summary

- We have provided necessary and sufficient conditions for $AXB + CX^\star D = E$ (with $\star = *, \top$) to have a unique solution, for all E, and allowing A, B, C, D, E to be rectangular $\rightsquigarrow$ In terms of properties of $\begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix}$.
- Interesting differences with the case of A, B, C, D, E being square:
  - Spectral information is not enough.
  - Some invertibility conditions on A, B, C, D arise.
- We have also provided conditions for $AXB - CXD = E$ to have a unique solution, for all E $\rightsquigarrow$ Depend on the **KCF** of $A - \lambda C$ and $B^\top - \lambda D^\top$.

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