

LOW RANK PERTURBATION OF KRONECKER STRUCTURES WITHOUT FULL RANK*

FERNANDO DE TERÁN[†] AND FROILÁN M. DOPICO[†]

Abstract. Let $P(\lambda) = A_0 + \lambda A_1$ be a singular $m \times n$ matrix pencil without full rank whose Kronecker canonical form (KCF) is given. Let ρ be a positive integer such that $\rho \leq \min\{m, n\} - \text{rank}(P)$ and $\rho \leq \text{rank}(P)$. We study the change of the KCF of $P(\lambda)$ due to perturbation pencils $Q(\lambda)$ with $\text{rank}(Q) = \rho$. We focus on the generic behavior of the KCF of $(P + Q)(\lambda)$, i.e., the behavior appearing for perturbations $Q(\lambda)$ in a dense open subset of the pencils with rank ρ . The most remarkable generic properties of the KCF of the perturbed pencil $(P + Q)(\lambda)$ are (i) if λ_0 is an eigenvalue of $P(\lambda)$, finite or infinite, then λ_0 is an eigenvalue of $(P + Q)(\lambda)$; (ii) if λ_0 is an eigenvalue of $P(\lambda)$, then the number of Jordan blocks associated with λ_0 in the KCF of $(P + Q)(\lambda)$ is equal to or greater than the number of Jordan blocks associated with λ_0 in the KCF of $P(\lambda)$; (iii) if λ_0 is an eigenvalue of $P(\lambda)$, then the dimensions of the Jordan blocks associated with λ_0 in $(P + Q)(\lambda)$ are equal to or greater than the dimensions of the Jordan blocks associated with λ_0 in $P(\lambda)$; (iv) the row (column) minimal indices of $(P + Q)(\lambda)$ are equal to or greater than the largest row (column) minimal indices of $P(\lambda)$. Moreover, if the sum of the row (column) minimal indices of the perturbations $Q(\lambda)$ is known, apart from their rank, then the whole set of the row (column) minimal indices of $(P + Q)(\lambda)$ is generically obtained, and in the case $\rho < \min\{m, n\} - \text{rank}(P)$ the whole KCF of $(P + Q)(\lambda)$ is generically determined.

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1. Introduction. Matrix spectral canonical forms are very important both in theory and in applications like the behavior of dynamical systems near bifurcations. Spectral canonical forms are mathematical structures that are very fragile under perturbations. For instance, it is well known that, although the *Jordan* canonical form of a matrix A has blocks of dimension larger than one, all the blocks in the Jordan form of the perturbed matrix $A + E$ have dimension one and correspond to eigenvalues different from those of A , for almost all perturbations E . The same can be said on the behavior of the *Weierstrass* canonical form of a regular matrix pencil $A_0 + \lambda A_1$, and on the *Kronecker* canonical form (KCF) of singular matrix pencils. In this latter case, in fact, the perturbed pencil has full rank for almost all perturbations. However, there are perturbations that allow us to guarantee that some part of the spectral canonical form of the original pencil is also a part of the spectral canonical form of the perturbed pencil. One example of perturbations of this kind is *low rank perturbations*, i.e., perturbations with a fixed rank that is small in some way specified by a property of the unperturbed matrix or pencil.

Low rank perturbations of spectral canonical forms have received attention since the 1980s. At least two kinds of contributions can be considered in this area. Given an

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[†]Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. Universidad 30, 28911 Leganés, Spain (fteran@math.uc3m.es, dopico@math.uc3m.es).

$m \times n$ pencil (or matrix) $P(\lambda)$ and perturbations $Q(\lambda)$ with fixed rank, the first class of works tries to classify all the spectral canonical forms of $(P + Q)(\lambda)$ compatible with the canonical form of $P(\lambda)$ and the rank of the perturbations $Q(\lambda)$. As far as we know, this has only been done for rank one perturbations; see [1] and [18] in this context. A second class of papers in this area characterizes generic properties of the spectral canonical form of $(P + Q)(\lambda)$, i.e., properties that hold for perturbations in a dense open subset of the matrices or pencils with a fixed rank; for this problem, see the references [4, 10, 13, 16, 17]. Generic properties have been considered only for the Jordan canonical form of matrices and for the Weierstrass canonical form of regular matrix pencils, and the study of explicit necessary and sufficient conditions for the generic behaviors to hold has been performed only in [4, 13]. The purpose of this paper is to determine generic properties of the KCF of singular matrix pencils without full rank under certain low rank perturbations, and to provide sufficient conditions for these properties to hold.

Throughout this work the term *generic* will frequently be used. This word appears in many mathematical works, but it is not a well-defined technical term, and its precise meaning is not always the same in the literature. In this paper, we use *generic* in the following sense: *a property is said to be generic in a set \mathcal{C} if it holds in a dense open subset of \mathcal{C} .* In our context, \mathcal{C} will be the set of allowable perturbations, and we identify the set of $m \times n$ complex matrix pencils, $A_0 + \lambda A_1$, with \mathbb{C}^{2mn} , where the usual topology is considered. Therefore every subset \mathcal{C} of pencils can be seen as a subset of \mathbb{C}^{2mn} . In this setting, we have that a set $\mathcal{G} \subset \mathcal{C}$ is dense in \mathcal{C} if and only if every element in \mathcal{C} is the limit of a sequence of elements in \mathcal{G} , and we will say that \mathcal{G} is open in \mathcal{C} if \mathcal{G} is the intersection of \mathcal{C} with an open subset of \mathbb{C}^{2mn} ; i.e., we consider in \mathcal{C} the *subspace topology* induced by the usual topology of \mathbb{C}^{2mn} . To finish these comments on the term *generic*, let us remark that it will not be used in the statement of most theorems, where precise assumptions will be included. Discussions on the genericity of these assumptions will be separately addressed.

We will consider as unperturbed pencil a singular $m \times n$ matrix pencil $P(\lambda)$ without full rank, i.e., $\text{rank}(P) < \min\{m, n\}$. Given an integer number ρ such that

$$(1) \quad 0 < \rho \leq \min\{m, n\} - \text{rank}(P),$$

and $\rho \leq \text{rank}(P)$, the set of perturbations is restricted to pencils $Q(\lambda)$ with $\text{rank}(Q) = \rho$. Notice that (1) and $\rho \leq \text{rank}(P)$ are both low rank conditions imposed on the perturbations. Here, the rank has to be understood as the rank of matrix polynomials, which is also known as the *normal rank* of a pencil.

For the set of perturbations defined in the previous paragraph the first problem we deal with is to get information on the generic regular part of the perturbed pencil $(P + Q)(\lambda)$. This is addressed in section 4, where it is proved that, generically, if λ_0 is an eigenvalue of $P(\lambda)$, finite or infinite, then λ_0 is also an eigenvalue of $(P + Q)(\lambda)$ with partial multiplicities greater than or equal to the corresponding partial multiplicities of λ_0 relative to $P(\lambda)$. These results are consequences of Theorem 4.4, *which is our first major contribution*. The second problem we deal with is to get information on the generic minimal indices of $(P + Q)(\lambda)$. For the sake of brevity, let us summarize the results only for the column or right minimal indices. Similar results hold for the row minimal indices. It is known that the number of column minimal indices of $P(\lambda)$ is $n - \text{rank}(P)$. The initial result we present is that, generically, $P + Q$ has $n - \text{rank}(P) - \rho$ column minimal indices. This implies, in particular, that if $\rho = n - \text{rank}(P)$, then $P + Q$ has no column minimal indices; i.e., it has full column

rank. These results follow from Theorem 3.1 and its direct consequence, Corollary 3.2. The case $\rho < n - \text{rank}(P)$ is much more difficult, and it is addressed in Theorem 5.8, where all the column minimal indices of $P + Q$ are generically determined if, apart from the rank, the sum of the column minimal indices of the perturbations $Q(\lambda)$ is known. As a corollary, Theorem 5.10 presents generic partial information on the column minimal indices of $P + Q$ when $\rho = \text{rank}(Q)$ is the only property known on the perturbations. Loosely speaking, one can say that the generic column minimal indices of $P + Q$ are equal to or greater than the $n - \text{rank}(P) - \rho$ largest column minimal indices of P . Theorems 5.8 and 5.10 constitute *our second major contribution*. All the results previously described remain valid in the limit case

$$\rho = \min\{m, n\} - \text{rank}(P).$$

If the strict inequality is assumed in (1), i.e., $\text{rank}(P) + \text{rank}(Q) < \min\{m, n\}$, it is possible to fully determine the generic KCF of $(P + Q)(\lambda)$ in terms of the sums of the column and row minimal indices and of the regular part of the KCF of $Q(\lambda)$. In the case that $\text{rank}(Q)$ is the only information available on the perturbations, the generic KCF of $(P + Q)(\lambda)$ can only be partially determined. These results appear in Theorems 6.2 and 6.3, which are *our last major contribution*. It should be stressed that all the generic results on the KCF of $(P + Q)(\lambda)$ that we present are very easy to describe, although to prove that they occur under certain generic sufficient conditions is a hard task that requires techniques very different from those used in [4, 13].

The class of low rank perturbations considered in this work includes very interesting problems. To cite one of them: the study of the generic variation of the minimal indices of a square pencil ($m = n$) under low rank perturbations requires necessarily the assumptions $\text{rank}(P) < n$ (because otherwise $P(\lambda)$ has no minimal indices) and $\text{rank}(P) + \text{rank}(Q) < n$ (because otherwise generically $\text{rank}(P + Q) = n$ and $P + Q$ has no minimal indices). However, this class of perturbations does not cover all the relevant situations. There are still open problems in the area of generic low rank perturbations of spectral canonical forms. Some of them will be discussed in section 7, where we will explain why the results obtained in this paper are, apart from being relevant by themselves, an essential step towards the solution of new open problems.

The perturbations considered in this work are not of small norm. The change of KCF of matrix pencils under small normwise perturbations was studied in [14], where the set of Kronecker structures nearby to a given one was characterized in terms of some majorization conditions on the sequences of column and row minimal indices and on the regular structure. Further results of this kind were obtained in [2] and [5].

Low rank perturbations of spectral properties have appeared in several applied problems. For instance, in the area of structural modifications of dynamical systems, it is of particular relevance to study how a system must be modified in order to fix certain eigenvalues in the new system. This is known generically as the “pole-zero assignment” problem [15]. In [7], low rank perturbations of the damping matrices of vibrating systems are considered in order to obtain defective systems.

The paper is organized as follows. In section 2 the notation and some preliminary results are introduced. In section 3 the meaning and genericity of the low rank assumptions used in different sections of this work are discussed, and, as a consequence, the generic number of row and column minimal indices of the perturbed pencil is determined. In section 4 generic properties of the regular structure of the perturbed pencil $(P + Q)(\lambda)$ are established. Section 5 deals with the minimal indices of $(P + Q)(\lambda)$. Section 6 describes the whole generic KCF of $(P + Q)(\lambda)$, assuming

that the strict inequality $\rho < \min\{m, n\} - \text{rank}(P)$ holds. Finally, in section 7 the conclusions and some open problems are presented.

2. Notation, definitions, and preliminary results. Several basic definitions and results are presented in this section. Some of them are well known and are stated just to establish the notation used throughout the paper. In addition, some other definitions and elementary results are presented.

2.1. Kronecker canonical form and rank of a pencil. We begin by introducing the concepts of singular pencil, rank or normal rank of a pencil, and eigenvalue of a pencil.

DEFINITION 2.1 (see [8, Chapter XII]). *Let $A_0, A_1 \in \mathbb{C}^{m \times n}$ be two complex $m \times n$ matrices. The matrix pencil*

$$(2) \quad P(\lambda) = A_0 + \lambda A_1$$

is called singular if one of the following conditions hold: $m \neq n$, or $m = n$ and $\det(P(\lambda))$ is the zero polynomial in the variable λ . Otherwise the pencil is called regular.

DEFINITION 2.2. *The rank of the pencil $P(\lambda)$ is the dimension of its largest minor that is not equal to the zero polynomial in λ . For the sake of simplicity, we will simply denote the rank of $P(\lambda)$ by $\text{rank}(P)$, omitting the variable λ .*

The rank of a pencil is also called its normal rank [2, 5]. However, we prefer the classical name *rank*, because this concept corresponds to the usual rank of matrices whose entries are rational functions of λ .

DEFINITION 2.3. *A complex number μ is a finite eigenvalue of the pencil $P(\lambda)$ if the rank of the constant matrix $P(\mu)$ is less than $\text{rank}(P)$. The pencil $P(\lambda) = A_0 + \lambda A_1$ has an infinite eigenvalue if zero is an eigenvalue of the dual pencil $A_1 + \lambda A_0$.*

For every pencil $P(\lambda)$ there exist two nonsingular matrices $R \in \mathbb{C}^{m \times m}$ and $S \in \mathbb{C}^{n \times n}$ such that $RP(\lambda)S = \mathcal{K}_P(\lambda)$ is the KCF of $P(\lambda)$ (see [8, Chapter XII]). The KCF is a block diagonal matrix and is unique up to permutations of the diagonal blocks. To be more precise,

$$(3) \quad \mathcal{K}_P(\lambda) = \text{diag}(L_{\varepsilon_1}, \dots, L_{\varepsilon_p}, L_{\eta_1}^T, \dots, L_{\eta_q}^T, \mathcal{J}_P),$$

where L_{ε_i} is the $\varepsilon_i \times (\varepsilon_i + 1)$ matrix pencil

$$L_{\varepsilon_i} = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \end{bmatrix},$$

the superscript T means transposition, and \mathcal{J}_P is a square pencil that constitutes the regular structure of the KCF of $P(\lambda)$. The matrix pencil \mathcal{J}_P contains the spectral information on the eigenvalues of $P(\lambda)$. This means that \mathcal{J}_P is a direct sum of Jordan blocks

$$J_k(\lambda_i) = \begin{bmatrix} \lambda - \lambda_i & 1 & & & \\ & \lambda - \lambda_i & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & \\ & & & & \lambda - \lambda_i \end{bmatrix}_{k \times k},$$

associated with certain finite eigenvalues $\lambda_i \in \mathbb{C}$ of $P(\lambda)$, and, eventually, of Jordan blocks associated with the infinite eigenvalue

$$J_k(\infty) = \begin{bmatrix} 1 & \lambda & & & \\ & 1 & \ddots & & \\ & & \ddots & \lambda & \\ & & & & 1 \end{bmatrix}_{k \times k}.$$

The numbers $\varepsilon_1, \dots, \varepsilon_p$ are called the *column (or right) minimal indices of $P(\lambda)$* , and η_1, \dots, η_q are called the *row (or left) minimal indices of $P(\lambda)$* [8, Chapter XII]. Notice that the row minimal indices of $P(\lambda)$ are the column minimal indices of $P(\lambda)^T$ and vice versa. We will assume that they are indexed in nondecreasing order, i.e.,

$$0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \dots \leq \varepsilon_p \quad \text{and} \quad 0 \leq \eta_1 \leq \eta_2 \leq \dots \leq \eta_q.$$

Analogously the matrix pencils L_{ε_i} ($L_{\eta_j}^T$) are called the *column or right (row or left) singular blocks of the KCF of $P(\lambda)$* . These blocks reveal the *singular structure of $P(\lambda)$* .

Observe that, if the KCF of $P(\lambda)$ is given by (3), then

$$(4) \quad \text{rank}(P) = n - p = m - q;$$

i.e., the rank of a pencil is related to the number of column and row singular blocks in its KCF. Notice also that if \mathcal{J}_P is a $j \times j$ pencil, then

$$(5) \quad \text{rank}(P) = j + \varepsilon_1 + \dots + \varepsilon_p + \eta_1 + \dots + \eta_q.$$

2.2. The vector space of n -tuples of rational functions. Minimal bases.

The entries of an $m \times n$ pencil $P(\lambda) = A_0 + \lambda A_1$ are polynomials of degree one over \mathbb{C} . Moreover, it is well known that the column (row) minimal indices of $P(\lambda)$ are related to the degrees of certain polynomial solutions of $(A_0 + \lambda A_1)x(\lambda) = 0$ ($(A_0 + \lambda A_1)^T y(\lambda) = 0$) [8, Chapter XII], where $x(\lambda)$ ($y(\lambda)$) is an n -tuple (m -tuple) whose entries are polynomials. The vector $x(\lambda)$ will be called a *vector polynomial*. Previous comments make clear that vector polynomials can naturally arise in dealing with singular pencils. The set of polynomials with complex coefficients is a *ring* but not a *field*. This means that to extend many elementary ideas of linear algebra to vector polynomials one has to consider the field of all rational functions with complex coefficients. For instance, let $v_1 = [1 + \lambda, 1 + \lambda]^T$ and $v_2 = [1 + \lambda^2, 1 + \lambda^2]^T$ be two vector polynomials. The determinant of the matrix $[v_1 | v_2]$ is obviously zero, but a rational function has to be necessarily used as a coefficient to express v_2 as a linear combination of v_1 : $v_2 = \frac{1+\lambda^2}{1+\lambda} v_1$. The *field of rational functions* with complex coefficients will be denoted by $\mathbb{C}(\lambda)$, and the *vector space over $\mathbb{C}(\lambda)$ of the n -tuples of rational functions* will be denoted by $\mathbb{C}^n(\lambda)$.

The following definitions are taken from [6] (see also [11]). The *degree*, $\text{deg}(x)$, of a *vector polynomial* $x(\lambda)$ is the greatest degree of its components. Every vector subspace \mathcal{V} of $\mathbb{C}^n(\lambda)$ always has a basis consisting of vector polynomials. It can be obtained from a general basis simply by multiplying each vector by the denominators of its entries. The *order* of such a polynomial basis is defined as the sum of the degrees of its vectors. A *minimal basis* of \mathcal{V} is a polynomial basis of \mathcal{V} that has least order among all polynomial bases of \mathcal{V} .

Let us introduce some additional concepts that we will use very often. Given an $m \times n$ matrix pencil $P(\lambda) = A_0 + \lambda A_1$, the *right (left) null space* of $P(\lambda)$ is the subspace of $\mathbb{C}^n(\lambda)$ ($\mathbb{C}^m(\lambda)$), $\mathcal{N}(P) = \{x(\lambda) \in \mathbb{C}^n(\lambda) : P(\lambda)x(\lambda) = 0\}$ ($\mathcal{N}(P^T) = \{y(\lambda) \in \mathbb{C}^m(\lambda) : P^T(\lambda)y(\lambda) = 0\}$). A *right (left) null space vector* of $P(\lambda)$ is a *vector polynomial* contained in $\mathcal{N}(P)$ ($\mathcal{N}(P^T)$). A *right ordered minimal basis* of $P(\lambda)$ (*ROMB*) is a minimal basis, $\{x_1(\lambda), \dots, x_p(\lambda)\}$, of $\mathcal{N}(P)$ with $\deg(x_1) \leq \deg(x_2) \leq \dots \leq \deg(x_p)$. A *left ordered minimal basis* of $P(\lambda)$ (*LOMB*) is a minimal basis, $\{y_1(\lambda), \dots, y_q(\lambda)\}$, of $\mathcal{N}(P^T)$ with $\deg(y_1) \leq \deg(y_2) \leq \dots \leq \deg(y_q)$.

Lemma 2.4 shows that the degrees of the vectors in an ROMB (LOMB) of $P(\lambda)$ are equal to the column (row) minimal indices of $P(\lambda)$.

LEMMA 2.4. *Let $\varepsilon_1 \leq \dots \leq \varepsilon_p$ and $\eta_1 \leq \dots \leq \eta_q$ be, respectively, the column and row minimal indices of the pencil $P(\lambda)$. Let $\{x_1(\lambda), \dots, x_p(\lambda)\}$ and $\{y_1(\lambda), \dots, y_q(\lambda)\}$ be, respectively, an ROMB and an LOMB of $P(\lambda)$. Then $\deg(x_i) = \varepsilon_i$, for $i = 1, \dots, p$, and $\deg(y_j) = \eta_j$, for $j = 1, \dots, q$.*

Proof. We prove the result for the column minimal indices. For the row minimal indices simply use $P^T(\lambda)$ and invoke the result for column minimal indices. Let us recall [8, Chapter XII, p. 38] the relationship between the column minimal indices of $P(\lambda)$ and the polynomial solutions of $P(\lambda)x(\lambda) = 0$. Among all the polynomial solutions of this system of equations we choose a nonzero solution $z_1(\lambda)$ of least degree. This degree is ε_1 . Among all the polynomial solutions that are linearly independent of $z_1(\lambda)$ we take a solution $z_2(\lambda)$ of least degree. This degree is ε_2 . We continue this process until we get a *fundamental series of solutions* $\{z_1(\lambda), \dots, z_p(\lambda)\}$, i.e., $p = \dim \mathcal{N}(P)$ linearly independent polynomial solutions of $P(\lambda)x(\lambda) = 0$ of degrees $\varepsilon_1 \leq \dots \leq \varepsilon_p$. A fundamental series of solutions is not uniquely determined, but the degrees of its vectors are, and, as we prove in the next paragraph, every fundamental series of solutions is an ROMB and vice versa.

Let us assume that there exists some index j such that $\deg(x_j) < \varepsilon_j$. Let j_0 be the least of these indices, i.e., $\deg(x_{j_0}) < \varepsilon_{j_0}$ and $\deg(x_k) \geq \varepsilon_k$ for $k = 1, \dots, j_0 - 1$. Obviously $j_0 > 1$. Therefore, $\varepsilon_{j_0-1} \leq \deg(x_{j_0-1}) \leq \deg(x_{j_0}) < \varepsilon_{j_0}$. The definition of the minimal indices implies that the linearly independent vectors $\{x_1(\lambda), \dots, x_{j_0}(\lambda)\}$ are linear combinations of $\{z_1(\lambda), \dots, z_{j_0-1}(\lambda)\}$. This is impossible. Then $\deg(x_j) \geq \varepsilon_j$ for all $j = 1, \dots, p$, and, in fact, $\deg(x_j) = \varepsilon_j$ for all j , because $\{x_1(\lambda), \dots, x_p(\lambda)\}$ is an ROMB. \square

We will also use the following related lemma.

LEMMA 2.5. *Let $P(\lambda)$ be a pencil with KCF given by (3), and $\{x_1(\lambda), \dots, x_p(\lambda)\}$ be an ROMB of $P(\lambda)$. Then every right null space vector of $P(\lambda)$ of degree at most ε_i is a linear combination of $\{x_1(\lambda), \dots, x_j(\lambda)\}$ with polynomial coefficients, where j is the largest index such that $\deg(x_j) \leq \varepsilon_i$. In particular, every right null space vector of $P(\lambda)$ is a linear combination of $\{x_1(\lambda), \dots, x_p(\lambda)\}$ with polynomial coefficients. A similar result holds for left null space vectors and an LOMB.*

Proof. The fact that every right null space vector is a linear combination of the mentioned vectors is a straightforward consequence of the definition of minimal indices. The fact that the coefficients are polynomials follows from [6, Main Theorem, p. 495]. \square

We will need to ascertain the linear independence of some sets of vector polynomials of $\mathbb{C}^n(\lambda)$. In some situations, this problem can be solved through a standard linear independence problem in \mathbb{C}^n . This is shown by Lemma 2.6.

LEMMA 2.6. *Let $\{v_1(\lambda), \dots, v_r(\lambda)\}$ be a set of vector polynomials of $\mathbb{C}^n(\lambda)$. Let*

us express these vectors as

$$v_i(\lambda) = v_{i0} + \lambda v_{i1} + \dots + \lambda^{d_i} v_{id_i} \quad \text{for } 1 \leq i \leq r,$$

where $v_{ij} \in \mathbb{C}^n$ for all i, j , and $d_i = \deg(v_i(\lambda))$.

1. If $\{v_{10}, \dots, v_{r0}\}$ is a linearly independent set in \mathbb{C}^n , then $\{v_1(\lambda), \dots, v_r(\lambda)\}$ is a linearly independent set in $\mathbb{C}^n(\lambda)$.

2. If $\{v_{1d_1}, \dots, v_{rd_r}\}$ is a linearly independent set in \mathbb{C}^n , then $\{v_1(\lambda), \dots, v_r(\lambda)\}$ is a linearly independent set in $\mathbb{C}^n(\lambda)$.

Proof. To prove the first item, the linear combination

$$(6) \quad \alpha_1(\lambda) v_1(\lambda) + \dots + \alpha_r(\lambda) v_r(\lambda) = 0$$

is considered, where $\alpha_i(\lambda)$, $1 \leq i \leq r$, can be chosen to be polynomials, because if they were rational functions, one could multiply by their denominators. Let us express these polynomials as

$$\alpha_i(\lambda) = \alpha_{i0} + \lambda \alpha_{i1} + \dots + \lambda^{t_i} \alpha_{it_i} \quad \text{for } 1 \leq i \leq r,$$

where $\alpha_{ij} \in \mathbb{C}$ for all i, j . Therefore, the coefficient vector of the term of degree zero in (6) is

$$\sum_{i=1}^r \alpha_{i0} v_{i0} = 0.$$

If $\{v_{10}, \dots, v_{r0}\}$ is a linearly independent set in \mathbb{C}^n , then $\alpha_{10} = \alpha_{20} = \dots = \alpha_{r0} = 0$. Thus, the coefficient vector of the term of degree one in (6) is $\sum_{i=1}^r \alpha_{i1} v_{i0} = 0$; this implies $\alpha_{11} = \alpha_{21} = \dots = \alpha_{r1} = 0$. A simple inductive argument completes the proof of the first item. To prove the second item one simply begins with the coefficient of the term with greatest degree, and performs downward the inductive step. \square

We finish this section with another technical result on the linear independence of vector polynomials.

LEMMA 2.7. *Let $\{z_1(\lambda), \dots, z_k(\lambda)\}$, $k < n$, be a linearly independent set of vector polynomials in $\mathbb{C}^n(\lambda)$ and $\{z'_1(\lambda), \dots, z'_l(\lambda)\}$ be another set of vector polynomials in $\mathbb{C}^n(\lambda)$ such that $k + l \leq n$ and $\text{rank}[z_1(\lambda) | \dots | z_k(\lambda) | z'_1(\lambda) | \dots | z'_l(\lambda)] = k$. Let us denote by $\{u_1, \dots, u_n\}$ the canonical basis of \mathbb{C}^n ; i.e., the entries of these vectors are $(u_i)_j = \delta_{ij}$. Then there exist l vectors of the canonical basis, u_{j_1}, \dots, u_{j_l} , such that $\{z_1(\lambda), \dots, z_k(\lambda), z'_1(\lambda) + \alpha_1 u_{j_1}, \dots, z'_l(\lambda) + \alpha_l u_{j_l}\}$ is a linearly independent set in $\mathbb{C}^n(\lambda)$ for all nonzero complex numbers $\alpha_1, \dots, \alpha_l$.*

Proof. There exists at least one u_{j_1} such that $\{z_1(\lambda), \dots, z_k(\lambda), u_{j_1}\}$ is linearly independent because, otherwise, all the vectors in $\{u_1, \dots, u_n\}$ would be linear combinations of $\{z_1(\lambda), \dots, z_k(\lambda)\}$. This is impossible because $\{u_1, \dots, u_n\}$ is also a basis of $\mathbb{C}^n(\lambda)$ and $k < n$. This argument can be successively applied to prove that there exist u_{j_1}, \dots, u_{j_l} vectors of the canonical basis such that $\{z_1(\lambda), \dots, z_k(\lambda), u_{j_1}, \dots, u_{j_l}\}$ is linearly independent. Thus $\{z_1(\lambda), \dots, z_k(\lambda), \alpha_1 u_{j_1}, \dots, \alpha_l u_{j_l}\}$ is linearly independent for all nonzero complex numbers $\alpha_1, \dots, \alpha_l$. Notice that the assumption $\text{rank}[z_1(\lambda) | \dots | z_k(\lambda) | z'_1(\lambda) | \dots | z'_l(\lambda)] = k$ implies that the vectors $z'_i(\lambda)$ are linear combinations of $\{z_1(\lambda), \dots, z_k(\lambda)\}$ with coefficients in $\mathbb{C}(\lambda)$. Therefore, elementary column operations can be used to transform the matrix $[z_1(\lambda) | \dots | z_k(\lambda) | \alpha_1 u_{j_1} | \dots | \alpha_l u_{j_l}]$ into $[z_1(\lambda) | \dots | z_k(\lambda) | z'_1(\lambda) + \alpha_1 u_{j_1} | \dots | z'_l(\lambda) + \alpha_l u_{j_l}]$. This does not change the rank of the matrix, which proves the result. \square

2.3. Expansion of a pencil as sum of rank-one pencils. The expansion presented in Lemma 2.8 will play a key role in this paper.

LEMMA 2.8. *Let $Q(\lambda)$ be an $m \times n$ matrix pencil with rank ρ , and let $\tilde{\varepsilon}$ be the sum of its column (or right) minimal indices. Then $Q(\lambda)$ can be expressed in the form*

$$(7) \quad Q(\lambda) = v_1(\lambda)w_1(\lambda)^T + \cdots + v_\rho(\lambda)w_\rho(\lambda)^T,$$

where

- (i) $\{v_1(\lambda), \dots, v_\rho(\lambda)\}$ is a linearly independent set of vector polynomials in $\mathbb{C}^m(\lambda)$ with degrees at most one;
- (ii) $\{w_1(\lambda), \dots, w_\rho(\lambda)\}$ is a linearly independent set of vector polynomials in $\mathbb{C}^n(\lambda)$ with degrees at most one;
- (iii) each summand $v_i(\lambda)w_i(\lambda)^T$, $1 \leq i \leq \rho$, is an $m \times n$ matrix pencil with rank equal to one, i.e., $v_i(\lambda)$ and $w_i(\lambda)$ have both degree zero, or if one has degree one, the other has degree zero;
- (iv) there are $\tilde{\varepsilon}$ vectors among $w_1(\lambda), \dots, w_\rho(\lambda)$ with degree exactly one, and the remaining vectors $w_i(\lambda)$ are of degree zero.

A decomposition (7) satisfying these conditions will be called a right decomposition of $Q(\lambda)$. Any other decomposition of $Q(\lambda)$ as a sum of ρ rank-one matrix pencils contains at least $\tilde{\varepsilon}$ vectors among $w_1(\lambda), \dots, w_\rho(\lambda)$ with degree exactly one.

Proof. The result is a direct consequence of the KCF. Let the KCF of Q be

$$\mathcal{K}_Q(\lambda) = \text{diag}(L_{\tilde{\varepsilon}_1}, \dots, L_{\tilde{\varepsilon}_h}, L_{\tilde{\eta}_1}^T, \dots, L_{\tilde{\eta}_l}^T, \mathcal{J}_Q),$$

where \mathcal{J}_Q is the regular structure of $Q(\lambda)$ and there exist two nonsingular matrices, X and Y , such that $Q(\lambda) = X\mathcal{K}_Q(\lambda)Y$. Now, notice that a block $L_{\tilde{\varepsilon}_i}$ can be expanded as a sum of $\tilde{\varepsilon}_i$ rank-one pencils,

$$\begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} [\lambda \quad 1 \quad \dots \quad 0] + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} [0 \quad \dots \quad \lambda \quad 1],$$

where the row (column) vectors have degree equal to one (zero). An expansion for a block $L_{\tilde{\eta}_j}^T$ is obtained by transposition, but now the column (row) vectors have degree equal to one (zero). For the Jordan blocks in \mathcal{J}_Q , corresponding to finite or infinite eigenvalues, similar expansions with row vectors of degree zero are possible. All these expansions can be combined with $Q(\lambda) = X\mathcal{K}_Q(\lambda)Y$ to prove straightforwardly the four items of the lemma.

Let us prove now the fact that any other decomposition of $Q(\lambda)$ as a sum of ρ rank-one matrix pencils contains at least $\tilde{\varepsilon}$ vectors among $w_1(\lambda), \dots, w_\rho(\lambda)$ with degree exactly one. Notice that the set of solutions of $Q(\lambda)x(\lambda) = 0$ is equal to the set of solutions of $[w_1(\lambda), \dots, w_\rho(\lambda)]^T x(\lambda) = 0$, and therefore the column minimal indices of the pencils $Q(\lambda)$ and $D_0 + \lambda D_1 \equiv [w_1(\lambda), \dots, w_\rho(\lambda)]^T$ are equal. If there were less than $\tilde{\varepsilon}$ vectors among $w_1(\lambda), \dots, w_\rho(\lambda)$ with degree exactly one, then $\text{rank}(D_1) < \tilde{\varepsilon}$. This implies that the matrix coefficient of λ in the KCF of $[w_1(\lambda), \dots, w_\rho(\lambda)]^T$ has also rank smaller than $\tilde{\varepsilon}$. This is in contradiction with $\tilde{\varepsilon}$ being the sum of its column minimal indices. \square

Remark 1. A result similar to that in Lemma 2.8 can be obtained by considering the sum of the row (or left) minimal indices, $\tilde{\eta}$, of $Q(\lambda)$ and choosing the column

vectors of the expansions of the Jordan blocks in \mathcal{J}_Q to be of degree zero. In this case, we will consider a *left decomposition* of $Q(\lambda)$:

$$(8) \quad Q(\lambda) = \widehat{v}_1(\lambda)\widehat{w}_1(\lambda)^T + \cdots + \widehat{v}_\rho(\lambda)\widehat{w}_\rho(\lambda)^T,$$

where the vectors $\{\widehat{v}_1(\lambda), \dots, \widehat{v}_\rho(\lambda)\}$ and $\{\widehat{w}_1(\lambda), \dots, \widehat{w}_\rho(\lambda)\}$ have the properties appearing in items (i), (ii), and (iii) of Lemma 2.8, but (iv) is replaced by “there are $\tilde{\eta}$ vectors among $\{\widehat{v}_i(\lambda), \dots, \widehat{v}_\rho(\lambda)\}$ with degree exactly one, and the remaining vectors are of degree zero.” Notice that left and right decompositions are not unique. Besides, a left decomposition of $Q(\lambda)$ may not be simultaneously a right decomposition.

Example 1. Let us show right and left decompositions of a pencil with $\tilde{\varepsilon} = 1$, $\tilde{\eta} = 0$, and $\rho = 2$:

$$\begin{aligned} \begin{bmatrix} \lambda & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \lambda & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \lambda \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \\ \begin{bmatrix} \lambda & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \lambda & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \lambda \end{bmatrix}. \end{aligned}$$

2.4. Jordan blocks, invariant polynomials, elementary divisors, and dual pencils. Given an arbitrary $m \times n$ complex matrix pencil $P(\lambda)$ with rank r , there exist two matrix polynomials $U(\lambda)$ and $V(\lambda)$ with dimensions $m \times m$ and $n \times n$, respectively, and nonzero constant determinants, such that

$$(9) \quad U(\lambda)P(\lambda)V(\lambda) = \text{diag}(h_1(P), \dots, h_r(P), 0, \dots, 0),$$

where $h_i(P)$ are nonzero monic polynomials in λ satisfying $h_i(P)|h_{i+1}(P)$; i.e., $h_i(P)$ divides $h_{i+1}(P)$ for $i = 1, \dots, r - 1$ [8, Chapter VI]. These polynomials are called the *invariant polynomials (or factors)* of $P(\lambda)$, and the diagonal matrix in the right-hand side of (9) is called the *Smith canonical form* of $P(\lambda)$. This form is unique and, in fact, *exists for general matrix polynomials* and not only for pencils. If each

$$(10) \quad h_i(P) = (\lambda - \lambda_1)^{\nu_{i1}} \cdots (\lambda - \lambda_d)^{\nu_{id}} \quad \text{for } i = 1, \dots, r$$

is decomposed in powers of different irreducible factors, then those factors among $(\lambda - \lambda_1)^{\nu_{i1}}, \dots, (\lambda - \lambda_d)^{\nu_{id}}, \dots, (\lambda - \lambda_1)^{\nu_{r1}}, \dots, (\lambda - \lambda_d)^{\nu_{rd}}$ with $\nu_{ij} > 0$ are called the *elementary divisors of $P(\lambda)$* . There exists a close relationship between the elementary divisors and the dimensions of the Jordan blocks associated with the finite eigenvalues in the regular structure of the KCF of the pencil $P(\lambda)$. This is revealed by the following result. It is a simple consequence of the theory developed in [8, Chapter VI].

LEMMA 2.9. *Let $P(\lambda)$ be an $m \times n$ complex matrix pencil. For each elementary divisor $(\lambda - \lambda_j)^{\nu_{ij}}$ of $P(\lambda)$ there exists a Jordan block of dimension ν_{ij} associated with a finite eigenvalue λ_j in the regular structure of the KCF of $P(\lambda)$. Conversely, each Jordan block of dimension ν_{ij} associated with a finite eigenvalue λ_j in the KCF form of $P(\lambda)$ gives an elementary divisor $(\lambda - \lambda_j)^{\nu_{ij}}$.*

The reader should notice that Lemma 2.9 gives no information for the infinite eigenvalue of the pencil $P(\lambda)$. This information can be obtained from the zero eigenvalue of the *dual pencil* through Lemma 2.10, whose trivial proof is omitted.

LEMMA 2.10. *Let A and B be two complex $m \times n$ matrices. The pencils $A + \lambda B$ and $B + \lambda A$ have the same column and row minimal indices. Besides, the number and dimensions of the Jordan blocks corresponding to the infinite eigenvalue in the KCF of $A + \lambda B$ are equal to the number and dimensions of the Jordan blocks corresponding to the zero eigenvalue in the KCF of $B + \lambda A$, and vice versa.*

Given an eigenvalue λ_j of the pencil $P(\lambda)$, the exponents $0 \leq \nu_{1j} \leq \nu_{2j} \leq \dots \leq \nu_{rj}$ in (10) are called the *partial multiplicities* of λ_j relative to P , and if a number μ is not an eigenvalue of $P(\lambda)$, then all its partial multiplicities relative to P are defined as zero [9, p. 331]. Anyway, for any number λ_0 its partial multiplicities relative to P coincide with the dimensions of the Jordan blocks associated with λ_0 in the regular structure of the KCF of $P(\lambda)$, whenever Jordan blocks of zero dimension are admitted as nonexisting blocks. This also holds for the infinite eigenvalue. The partial multiplicities of an eigenvalue λ_0 , *finite or infinite*, of $P(\lambda)$ with g associated Jordan blocks in the KCF are usually arranged in an infinite sequence called *Segre characteristic* of λ_0 relative to $P(\lambda)$. This sequence is

$$\mathcal{S}_P(\lambda_0) = (n_g(\lambda_0), n_{g-1}(\lambda_0), \dots, n_1(\lambda_0), 0, \dots),$$

where $n_g(\lambda_0) \geq n_{g-1}(\lambda_0) \geq \dots \geq n_1(\lambda_0)$ are the dimensions of the Jordan blocks associated with λ_0 in the KCF of $P(\lambda)$. Notice that in the case when λ_0 is not an eigenvalue of $P(\lambda)$, all the terms in $\mathcal{S}_P(\lambda_0)$ are equal to zero.

The concepts of partial multiplicities and Segre characteristics are also valid for *general matrix polynomials*. The eigenvalues of a matrix polynomial can be defined as the roots of its invariant polynomials. Given two matrix polynomials $P(\lambda)$ and $Q(\lambda)$, we write

$$\mathcal{S}_P(\lambda_0) \geq \mathcal{S}_Q(\lambda_0) \quad \text{if} \quad (\mathcal{S}_P(\lambda_0))_i \geq (\mathcal{S}_Q(\lambda_0))_i \quad \text{for all } i > 0;$$

i.e., the inequality holds for each entry in the Segre characteristics.

The Smith canonical form (9) allows us to express every matrix polynomial $P(\lambda)$ of rank r as

$$(11) \quad P(\lambda) = h_1(P) a_1(\lambda) z_1^T(\lambda) + \dots + h_r(P) a_r(\lambda) z_r^T(\lambda),$$

where $h_1(P), \dots, h_r(P)$ are its invariant polynomials and $a_i(\lambda)$ and $z_i(\lambda)$ are vector polynomials. Besides, according to (9), the vectors $a_i(\lambda)$ and $z_i(\lambda)$ are, respectively, the columns of $U^{-1}(\lambda)$ and $V^{-T}(\lambda)$. This implies that neither $a_i(\lambda)$ nor $z_i(\lambda)$ can be written as the product of a scalar polynomial of degree greater than zero times a vector polynomial, because the matrices $U^{-1}(\lambda)$ and $V^{-T}(\lambda)$ are matrix polynomials with constant nonzero determinants. Notice that in the case when $P(\lambda)$ is a pencil the expansion (11) is not an expansion of the type (7), in general, because the summands $h_i(P) a_i(\lambda) z_i^T(\lambda)$ have, in general, degree larger than one.

The KCF of the direct sum, $P(\lambda) \oplus Q(\lambda)$, of two pencils $P(\lambda)$ and $Q(\lambda)$ is the direct sum of the KCFs of $P(\lambda)$ and $Q(\lambda)$, up to some permutations of the diagonal blocks. Therefore, the Segre characteristic of λ_0 relative to $P(\lambda) \oplus Q(\lambda)$ is obtained simply by putting together the Segre characteristics of λ_0 relative to $P(\lambda)$ and to $Q(\lambda)$, and then reordering the resulting sequence. The same holds in the case when $P(\lambda)$ and $Q(\lambda)$ are general matrix polynomials. This follows from [8, Chapter VI, Theorem 5].

3. Low rank assumptions: Meaning and genericity. Throughout this work we will deal with three $m \times n$ complex pencils: the fixed *unperturbed* pencil $P(\lambda)$, the *perturbation* pencil $Q(\lambda)$, and the *perturbed* pencil $(P + Q)(\lambda)$. The pencil $P(\lambda)$ does not have full rank, and its KCF will always be assumed to be known; it will be denoted by (3). We will frequently omit the variable λ when there is no risk of confusion.

As announced in the Introduction, the set of perturbations we considered is the set of pencils

$$(12) \quad \mathcal{C} = \{Q(\lambda) : \text{rank}(Q) = \rho\},$$

where $\rho > 0$ is an integer such that

$$(13) \quad \text{rank}(P) + \rho \leq \min\{m, n\}$$

and $\rho \leq \text{rank}(P)$. These are the two low rank conditions imposed on the set of perturbations. Notice also that $\rho > 0$, and (13) implies that $\text{rank}(P) < \min\{m, n\}$, i.e., that $P(\lambda)$ does not have full rank.

A key result in this work is that the property

$$(14) \quad \text{rank}(P + Q) = \text{rank}(P) + \text{rank}(Q)$$

is generic in the set \mathcal{C} . This is rigorously proved in Theorem 3.1 below. By combining this result with the identity (4), one can say that, for perturbations in the set \mathcal{C} , the perturbed pencils $(P + Q)(\lambda)$ have generically $n - \text{rank}(P) - \rho$ column minimal indices and $m - \text{rank}(P) - \rho$ row minimal indices. See Corollary 3.2 below on this point.

Notice that, taking into account that $P + Q$ is an $m \times n$ pencil, the condition (14) implies $\text{rank}(P) + \text{rank}(Q) \leq \min\{m, n\}$, i.e., the assumption (13), and that $P(\lambda)$ does not have full rank for $Q(\lambda) \neq 0$, i.e., for nontrivial perturbations. These facts and the genericity of (14) in \mathcal{C} lead us to impose $\text{rank}(P + Q) = \text{rank}(P) + \text{rank}(Q)$ in most of the lemmas and theorems we prove, without explicitly mentioning the initial low rank condition (13).

Section 5 is devoted to studying the generic column minimal indices of $(P + Q)(\lambda)$. In section 5

$$(15) \quad \text{rank}(P + Q) = \text{rank}(P) + \text{rank}(Q) < n$$

is assumed as a hypothesis in most of the results. Notice that (15) is implied by (14) only if $\min\{m, n\} = m < n$; otherwise (15) is an additional assumption. The reason for assuming (15) in section 5 is that, according to (4), the number of column minimal indices of $P + Q$ is $n - \text{rank}(P + Q)$, which is zero if $\text{rank}(P + Q) = n$. Therefore the study of the generic column minimal indices of $P + Q$ makes sense only if (15) holds. In the case of the row minimal indices, m instead of n has to be used in (15).

Finally, let us comment on the additional low rank assumption,

$$\rho = \text{rank}(Q) \leq \text{rank}(P).$$

This assumption is very natural for considering $Q(\lambda)$ as a low rank perturbation of $P(\lambda)$, and it is essential to guarantee that other hypotheses used in the study of the minimal indices of $P + Q$ are really *generic*. This will be discussed in subsection 5.5.

3.1. Genericity of the assumption $\text{rank}(P + Q) = \text{rank}(P) + \text{rank}(Q)$. Number of minimal indices of $P + Q$. The purpose of this section is to present a rigorous proof of the genericity of the most pervasive and crucial assumption in this work. This assumption determines the generic number of row and column minimal indices of the perturbed pencil $(P + Q)(\lambda)$.

THEOREM 3.1. *Let $P(\lambda)$ be an $m \times n$ complex matrix pencil and ρ be a positive integer such that $\text{rank}(P) + \rho \leq \min\{m, n\}$. Then the set of $m \times n$ complex matrix pencils*

$$\mathcal{G} = \{Q(\lambda) \text{ } m \times n \text{ pencil} : \text{rank}(Q) = \rho \text{ and } \text{rank}(P + Q) = \text{rank}(P) + \text{rank}(Q)\}$$

is dense and open in the set of $m \times n$ complex matrix pencils with rank ρ .

Proof. First, let us prove that \mathcal{G} is dense in the set of pencils with rank ρ . Notice that $\text{rank}(P + E) \leq \text{rank}(P) + \text{rank}(E)$ for every pencil $E(\lambda)$. Therefore, we have to prove that for every pencil $E(\lambda)$ with rank ρ and $\text{rank}(P + E) < \text{rank}(P) + \text{rank}(E)$ there exists a sequence $\{Q^{(t)}(\lambda)\}_{t=1}^\infty \subset \mathcal{G}$ whose limit is $E(\lambda)$. Let $r \equiv \text{rank}(P)$. According to (7), we can write

$$\begin{aligned} P(\lambda) &= v_1(\lambda)w_1(\lambda)^T + \dots + v_r(\lambda)w_r(\lambda)^T, \\ E(\lambda) &= a_1(\lambda)b_1(\lambda)^T + \dots + a_\rho(\lambda)b_\rho(\lambda)^T, \end{aligned}$$

and

$$(P + E)(\lambda) = [v_1 | \dots | v_r | a_1 | \dots | a_\rho] [w_1 | \dots | w_r | b_1 | \dots | b_\rho]^T,$$

where we have omitted some λ 's for simplicity. Elementary arguments show that $\text{rank}(P + E) < \text{rank}(P) + \text{rank}(E) = r + \rho$ if and only if $\text{rank}[v_1 | \dots | v_r | a_1 | \dots | a_\rho] < r + \rho$ or $\text{rank}[w_1 | \dots | w_r | b_1 | \dots | b_\rho] < r + \rho$. Suppose that $\text{rank}[v_1 | \dots | v_r | a_1 | \dots | a_\rho] < r + \rho$. This implies, due to the fact that the set $\{v_1, \dots, v_r\}$ is linearly independent, that

- (i) $\text{rank}[v_1 | \dots | v_r | a_1 | \dots | a_\rho] = r + \widehat{\rho}$ with $0 \leq \widehat{\rho} < \rho$; and
- (ii) the vectors a_1, \dots, a_ρ can be reordered as $a_{i_1}, \dots, a_{i_{\widehat{\rho}}}, a_{k_1}, \dots, a_{k_{\rho - \widehat{\rho}}}$, where $\{v_1, \dots, v_r, a_{i_1}, \dots, a_{i_{\widehat{\rho}}}\}$ is a linearly independent set.

Now, Lemma 2.7 is used to show that there exist $\rho - \widehat{\rho}$ vectors, $u_{j_1}, \dots, u_{j_{\rho - \widehat{\rho}}}$, of the canonical bases of \mathbb{C}^m such that, for every $t = 1, 2, \dots$,

$$\text{rank} \left[v_1 | \dots | v_r | a_{i_1} | \dots | a_{i_{\widehat{\rho}}} | a_{k_1} + \frac{1}{t}u_{j_1} | \dots | a_{k_{\rho - \widehat{\rho}}} + \frac{1}{t}u_{j_{\rho - \widehat{\rho}}} \right] = r + \rho.$$

Let $\{a_1^{(t)}, \dots, a_\rho^{(t)}\}$ be the set of vectors that is obtained from $\{a_1, \dots, a_\rho\}$ by replacing $a_{k_1}, \dots, a_{k_{\rho - \widehat{\rho}}}$ by $a_{k_1} + \frac{1}{t}u_{j_1}, \dots, a_{k_{\rho - \widehat{\rho}}} + \frac{1}{t}u_{j_{\rho - \widehat{\rho}}}$. If $\text{rank}[w_1 | \dots | w_r | b_1 | \dots | b_\rho] < r + \rho$, we proceed in a similar way to produce a set of vectors $\{b_1^{(t)}, \dots, b_\rho^{(t)}\}$. Finally, let us define the sequence of pencils

$$Q^{(t)}(\lambda) = a_1^{(t)}(\lambda)(b_1^{(t)}(\lambda))^T + \dots + a_\rho^{(t)}(\lambda)(b_\rho^{(t)}(\lambda))^T, \quad t = 1, 2, \dots$$

It is trivial to check that (i) $\lim_{t \rightarrow \infty} Q^{(t)}(\lambda) = E(\lambda)$; (ii) $\text{rank}(Q^{(t)}) = \rho$ for all t ; and (iii) $\text{rank}(P + Q^{(t)}) = \text{rank}(P) + \text{rank}(Q^{(t)})$ for all t . This proves that \mathcal{G} is dense.

Now, we will prove that \mathcal{G} is open in the set of matrix pencils with rank ρ . To this purpose, let us proceed as follows: as explained in the Introduction, the set of $m \times n$ complex matrix pencils is identified with \mathbb{C}^{2mn} , and the set of matrix pencils with

rank ρ is a subset \mathcal{C} of \mathbb{C}^{2mn} . Thus $\mathcal{G} \subset \mathcal{C} \subset \mathbb{C}^{2mn}$. We consider in \mathcal{C} the *subspace topology* induced by the usual topology of \mathbb{C}^{2mn} , as we explained in the Introduction. Therefore, for proving that \mathcal{G} is open in \mathcal{C} , it is sufficient to prove that every $Q(\lambda) \in \mathcal{G}$ is included in an open subset \mathcal{X}_Q of \mathbb{C}^{2mn} such that

$$\text{rank}(P + E) \geq \text{rank}(P + Q) = \text{rank}(P) + \text{rank}(Q) \quad \text{for all } E \in \mathcal{X}_Q.$$

The reason is that in this case the following hold:

1. $\mathcal{X}_Q \cap \mathcal{C}$ is open in \mathcal{C} ; and
2. the fact that $\text{rank}(P) + \text{rank}(Q) \leq \text{rank}(P + E) \leq \text{rank}(P) + \text{rank}(E)$ for all $E \in \mathcal{X}_Q$ implies that $\text{rank}(P + E) = \text{rank}(P) + \text{rank}(E)$ for all $E \in \mathcal{X}_Q \cap \mathcal{C}$. This means that $\mathcal{X}_Q \cap \mathcal{C} \subset \mathcal{G}$ and that Q is an interior point of \mathcal{G} .

Let us see how $\mathcal{X}_Q \subset \mathbb{C}^{2mn}$ is constructed. Given $Q \in \mathcal{G}$, the equation $\text{rank}(P + Q) = \text{rank}(P) + \rho \equiv r + \rho$ implies that the pencil $(P + Q)(\lambda)$ has a $(r + \rho) \times (r + \rho)$ minor that is a polynomial in λ with at least one nonzero coefficient. Let $\det(P + Q)(\alpha, \beta)$ be this minor, where the sets $\alpha \subseteq \{1, \dots, m\}$ and $\beta \subseteq \{1, \dots, n\}$ denote, respectively, the rows and columns that define the minor. By identifying every pencil $E(\lambda) = E_0 + \lambda E_1$ with an element of \mathbb{C}^{2mn} , the coefficients of $\det(P + E)(\alpha, \beta)$ define a continuous function $f(E)$, $f : \mathbb{C}^{2mn} \rightarrow \mathbb{C}^{r+\rho+1}$, because these coefficients are polynomials in the entries of the complex matrices E_0 and E_1 . Taking into account that $f(Q) \neq 0$, there exists an open ball, \mathcal{B} , in $\mathbb{C}^{r+\rho+1}$ whose center is $f(Q)$ and such that $0 \notin \mathcal{B}$. Then we can take $\mathcal{X}_Q = f^{-1}(\mathcal{B})$, because it is open, $f(E) \neq 0$ for all $E \in \mathcal{X}_Q$, and, therefore, $\text{rank}(P + E) \geq r + \rho$ for all $E \in \mathcal{X}_Q$. \square

As a consequence of Theorem 3.1 and (4) the generic number of row and column minimal indices of $P + Q$ is determined.

COROLLARY 3.2. *Let $P(\lambda)$ be an $m \times n$ complex matrix pencil with p column minimal indices and q row minimal indices, and ρ be a positive integer such that $\text{rank}(P) + \rho \leq \min\{m, n\}$. Then the set of perturbations $Q(\lambda)$ with $\text{rank}(Q) = \rho$ and such that $P + Q$ has $p - \rho$ column minimal indices and $q - \rho$ row minimal indices is dense and open in the set of $m \times n$ complex matrix pencils with rank ρ .*

4. The regular structure of the perturbed pencil. In this section we get information on the regular structure of the KCF of the perturbed pencil $(P + Q)(\lambda)$ in terms of the regular structures of $P(\lambda)$ and $Q(\lambda)$, i.e., \mathcal{J}_P and \mathcal{J}_Q . With only the hypothesis $\text{rank}(P + Q) = \text{rank}(P) + \text{rank}(Q)$, we prove that for every eigenvalue of P or Q , the regular structure of $P + Q$ has as least as many blocks as $\mathcal{J}_P \oplus \mathcal{J}_Q$, with dimensions larger than or equal to the dimensions of the blocks in $\mathcal{J}_P \oplus \mathcal{J}_Q$. Besides, other blocks may be present. This is presented in Theorem 4.4, and it is our first major contribution. In section 6, we will see that the generic regular structure of $P + Q$ is precisely $\mathcal{J}_P \oplus \mathcal{J}_Q$ if $\text{rank}(P + Q) = \text{rank}(P) + \text{rank}(Q) < \min\{m, n\}$.

In this section, the auxiliary lemma, Lemma 4.1, will be used. It appears without proof in [12]. The proof is elementary.

LEMMA 4.1 (see [12, p. 799]). *If $D = \text{diag}(z_1, \dots, z_k)$ and G is an arbitrary $k \times k$ matrix, then*

$$\det(D + G) = \det G + \sum z_{\nu_1} \dots z_{\nu_j} \cdot \det \check{G}(\nu_1, \dots, \nu_j),$$

where the sum runs over all $j \in \{1, \dots, k\}$ and all ν_1, \dots, ν_j such that $1 \leq \nu_1 < \dots < \nu_j \leq k$, and $\check{G}(\nu_1, \dots, \nu_j)$ denotes the matrix obtained from G by deleting the rows and columns with indices ν_1, \dots, ν_j , with $\det \check{G}(1, \dots, k) \equiv 1$.

Lemma 4.2 extends [18, Theorem 1] under more stringent assumptions.

LEMMA 4.2. Let $\mathcal{L}(\lambda)$ be an $m \times n$ matrix polynomial with rank equal to r , and $e_1 \leq \dots \leq e_r$ be the partial multiplicities of λ_0 relative to $\mathcal{L}(\lambda)$. Let $\mathcal{M}(\lambda)$ be a rank-one matrix polynomial, and e be the partial multiplicity of λ_0 relative to $\mathcal{M}(\lambda)$. Let us assume that $\text{rank}(\mathcal{L} + \mathcal{M}) = \text{rank}(\mathcal{L}) + \text{rank}(\mathcal{M})$, and that $e_i < e \leq e_{i+1}$, for some $i \in \{0, 1, \dots, r\}$, where we define $e_0 \equiv -1$ and $e_{r+1} \equiv \infty$. Then, the partial multiplicities $f_1 \leq \dots \leq f_{r+1}$ of λ_0 relative to $(\mathcal{L} + \mathcal{M})(\lambda)$ satisfy

$$f_1 = e_1, \dots, f_i = e_i, \quad e \leq f_{i+1}, \quad e_{i+1} \leq f_{i+2}, \dots, e_r \leq f_{r+1}.$$

Remark 2. Notice that in Lemma 4.2 it is possible that $e_1 = \dots = e_r = 0$ or that $e = 0$; i.e., λ_0 may not be an eigenvalue of $\mathcal{L}(\lambda)$ or of $\mathcal{M}(\lambda)$.

Proof of Lemma 4.2. Theorem 1 in [18] implies that

$$e_1 \leq f_2, e_2 \leq f_3, \dots, e_r \leq f_{r+1}.$$

So, we only need to prove that $f_1 = e_1, \dots, f_i = e_i, e \leq f_{i+1}$. Let $U(\lambda)$ and $V(\lambda)$ be the matrix polynomials, with nonzero constant determinants, that transform \mathcal{L} into its Smith normal form, i.e., $U(\lambda)\mathcal{L}(\lambda)V(\lambda) = \text{diag}((\lambda - \lambda_0)^{e_1}p_1(\lambda), \dots, (\lambda - \lambda_0)^{e_r}p_r(\lambda), 0, \dots, 0)$, with the polynomials $p_1(\lambda), \dots, p_r(\lambda)$ such that $p_j(\lambda_0) \neq 0$, for $j = 1, \dots, r$. Invariant polynomials and partial multiplicities remain unchanged under multiplication by $U(\lambda)$ and $V(\lambda)$; therefore we can focus on the partial multiplicities of the matrix polynomial:

$$(16) \quad U(\lambda)(\mathcal{L} + \mathcal{M})(\lambda)V(\lambda) = \text{diag}((\lambda - \lambda_0)^{e_1}p_1(\lambda), \dots, (\lambda - \lambda_0)^{e_r}p_r(\lambda), 0, \dots, 0) + (\lambda - \lambda_0)^e x(\lambda)y(\lambda)^T,$$

where the second term of the right-hand side is $U(\lambda)\mathcal{M}(\lambda)V(\lambda)$ (see (11)).

In the case $e_0 < e \leq e_1$, i.e., $i = 0$, the exponent of the factor $(\lambda - \lambda_0)$ of the greatest common divisor of all 1×1 minors in (16) is greater than or equal to e ; thus $e \leq f_1$ by the definition of invariant polynomials [8, Chapter VI, section 3], and the result is proven. Let us assume from now on that $i \geq 1$. In the rest of the proof, we will prove that if $c_k, k = 1, \dots, r + 1$, denotes the exponent of the factor $(\lambda - \lambda_0)$ of the greatest common divisor of all $k \times k$ minors in (16), then

$$(17) \quad c_1 = e_1, \quad c_2 = e_1 + e_2, \dots, \quad c_i = e_1 + \dots + e_i, \quad c_{i+1} \geq e_1 + \dots + e_i + e.$$

This and the definition of invariant polynomials imply $f_1 = e_1, \dots, f_i = e_i, e \leq f_{i+1}$.

The lowest power of $(\lambda - \lambda_0)$ in a 1×1 minor of (16) is easily seen to be e_1 , so $c_1 = e_1$. For $k \geq 2$, let us notice that all the nonzero $k \times k$ minors of (16) must contain at least $k - 1$ of the $(1, 1), \dots, (r, r)$ diagonal entries. Then, a nonzero $k \times k$ minor of (16) must be of one of these two types:

(i)

$$\det(\text{diag}((\lambda - \lambda_0)^{e_{i_1}}p_{i_1}(\lambda), \dots, 0, \dots, (\lambda - \lambda_0)^{e_{i_{k-1}}}p_{i_{k-1}}(\lambda)) + (\lambda - \lambda_0)^e [x(\lambda)y(\lambda)^T]_k),$$

(ii)

$$(18) \quad \det(\text{diag}((\lambda - \lambda_0)^{e_{i_1}}p_{i_1}(\lambda), \dots, (\lambda - \lambda_0)^{e_{i_k}}p_{i_k}(\lambda)) + (\lambda - \lambda_0)^e [x(\lambda)y(\lambda)^T]_k),$$

where $[x(\lambda)y(\lambda)^T]_k$ is some $k \times k$ submatrix of $x(\lambda)y(\lambda)^T$. If we apply Lemma 4.1 to these minors, we see that

(i) every minor of type (i) may be written as

$$(19) \quad (\lambda - \lambda_0)^{e_1 + \dots + e_{k-1} + e} q(\lambda),$$

where $q(\lambda)$ is a polynomial;

(ii) every minor of type (ii) may be written as

$$(20) \quad (\lambda - \lambda_0)^{e_1 + \dots + e_{k-1} + \min\{e, e_k\}} t(\lambda),$$

where $t(\lambda)$ is a polynomial.

In the case $k = i + 1$, these results directly imply that $c_{i+1} \geq e_1 + \dots + e_i + e$. In the case $k \leq i$, (19) and (20) imply $c_k \geq e_1 + \dots + e_k$. Moreover, the equality follows by taking $i_1 = 1, i_2 = 2, \dots, i_k = k$, in (18) and applying Lemma 4.1. \square

Next we prove a corollary of Lemma 4.2.

COROLLARY 4.3. *Let $\mathcal{L}(\lambda)$ and $\mathcal{M}(\lambda)$ be two $m \times n$ matrix polynomials such that $\text{rank}(\mathcal{M}) = 1$ and $\text{rank}(\mathcal{L} + \mathcal{M}) = \text{rank}(\mathcal{L}) + \text{rank}(\mathcal{M})$. Let $h(\mathcal{M})$ be the unique invariant polynomial of $\mathcal{M}(\lambda)$. Then*

$$\mathcal{S}_{\mathcal{L} + \mathcal{M}}(\lambda_0) \geq \mathcal{S}_{\mathcal{L} \oplus h(\mathcal{M})}(\lambda_0) = \mathcal{S}_{\mathcal{L} \oplus \mathcal{M}}(\lambda_0) \quad \text{for any complex number } \lambda_0.$$

Proof. Let us use the notation in Lemma 4.2 for the partial multiplicities of λ_0 . The partial multiplicities of λ_0 relative to $\mathcal{L} \oplus h(\mathcal{M})$ are $e_1 \leq \dots \leq e_i < e \leq e_{i+1} \leq \dots \leq e_r$, by Theorem 5 in [8, Chapter VI, p. 142]. These are also the partial multiplicities of λ_0 relative to $\mathcal{L} \oplus \mathcal{M}$, by the same argument. Lemma 4.2 implies the inequality $\mathcal{S}_{\mathcal{L} + \mathcal{M}}(\lambda_0) \geq \mathcal{S}_{\mathcal{L} \oplus h(\mathcal{M})}(\lambda_0)$. \square

Now we prove the main theorem in this section.

THEOREM 4.4. *Let $P(\lambda)$ and $Q(\lambda)$ be two $m \times n$ complex matrix pencils such that $\text{rank}(P + Q) = \text{rank}(P) + \text{rank}(Q)$. Then, for every complex number λ_0 , including the infinite, $\mathcal{S}_{P+Q}(\lambda_0) \geq \mathcal{S}_{P \oplus Q}(\lambda_0)$. This means, in particular, that if λ_0 is an eigenvalue of $P(\lambda)$ or if λ_0 is an eigenvalue of $Q(\lambda)$, then λ_0 is an eigenvalue of $(P + Q)(\lambda)$.*

Proof. We consider only finite numbers λ_0 . The result for the infinite eigenvalue follows from considering the zero eigenvalue in the dual pencils of $P(\lambda)$ and $Q(\lambda)$. According to (11), $Q(\lambda)$ can be expressed as

$$Q(\lambda) = h_1(Q) b_1(\lambda) c_1^T(\lambda) + \dots + h_\rho(Q) b_\rho(\lambda) c_\rho^T(\lambda),$$

where $\rho \equiv \text{rank}(Q)$ and where $h_1(Q), \dots, h_\rho(Q)$ are the invariant polynomials of $Q(\lambda)$. The property $\text{rank}(P + Q) = \text{rank}(P) + \text{rank}(Q)$ implies that

$$(21) \quad \begin{aligned} & \text{rank}(P + h_1(Q) b_1 c_1^T + \dots + h_k(Q) b_k c_k^T) \\ &= \text{rank}(P + h_1(Q) b_1 c_1^T + \dots + h_{k-1}(Q) b_{k-1} c_{k-1}^T) + \text{rank}(h_k(Q) b_k c_k^T), \end{aligned}$$

for $k = 1, \dots, \rho$. We have omitted the variable λ for the sake of simplicity. Therefore, Corollary 4.3 can be applied ρ times to prove

$$\begin{aligned} \mathcal{S}_{P+Q}(\lambda_0) &\geq \mathcal{S}_{(P+h_1(Q)b_1c_1^T+\dots+h_{\rho-1}(Q)b_{\rho-1}c_{\rho-1}^T)\oplus h_\rho(Q)}(\lambda_0) \geq \dots \\ &\geq \mathcal{S}_{P \oplus h_1(Q) \oplus \dots \oplus h_\rho(Q)}(\lambda_0), \end{aligned}$$

where we have used $(A + B) \oplus C = (A \oplus C) + (B \oplus 0)$. Finally, Theorem 5 in [8, Chapter VI, p. 142] implies $\mathcal{S}_{P \oplus h_1(Q) \oplus \dots \oplus h_\rho(Q)}(\lambda_0) = \mathcal{S}_{P \oplus Q}(\lambda_0)$. \square

5. The minimal indices of the perturbed pencil. The purpose of this section is to determine the minimal indices of the perturbed pencil $(P+Q)(\lambda)$ in terms of data of $P(\lambda)$ and $Q(\lambda)$. For the sake of brevity, we will develop the results only for the column minimal indices. A set of counterpart results for the row minimal indices can be obtained just by considering the column minimal indices of the transpose pencil $(P+Q)^T(\lambda)$.

The main result in this section is Theorem 5.8, where the whole set of column minimal indices of $P+Q$ is found for most perturbations Q having a given rank and a given sum of its column minimal indices. The genericity of the hypotheses of Theorem 5.8 is discussed in subsection 5.5. Theorem 5.10 presents some generic information on the column minimal indices of $P+Q$ when only the rank of the perturbation is available.

According to Lemma 2.4, determining the column minimal indices of $P+Q$ is equivalent to finding the degrees of an ROMB of $P+Q$. This ROMB will not be explicitly constructed, but the degrees of its vectors will be precisely determined. Lemma 5.1 is the key result for this task. It allows us to delimit the search for this basis.

LEMMA 5.1. *Let $P(\lambda)$ and $Q(\lambda)$ be two complex matrix pencils such that $\text{rank}(P+Q) = \text{rank}(P) + \text{rank}(Q)$. Then $x(\lambda)$ is a right null space vector of $P+Q$ if and only if $x(\lambda)$ is, simultaneously, a right null space vector of $P(\lambda)$ and $Q(\lambda)$, i.e., $\mathcal{N}(P+Q) = \mathcal{N}(P) \cap \mathcal{N}(Q)$.*

Proof. Let $\text{col}(P)$ be the column space of $P(\lambda)$ in $\mathbb{C}^m(\lambda)$. Then,

$$\dim(\text{col}(P+Q)) \leq \dim(\text{col}[P \ Q]) = \dim(\text{col}(P)) + \dim(\text{col}(Q)) - \dim(\text{col}(P) \cap \text{col}(Q)).$$

The assumption $\text{rank}(P+Q) = \text{rank}(P) + \text{rank}(Q)$ implies that $\text{col}(P) \cap \text{col}(Q) = \{0\}$. If $x(\lambda)$ is a right null space vector of $P+Q$, then $P(\lambda)x(\lambda) = -Q(\lambda)x(\lambda)$. Notice that the vector $z(\lambda) \equiv P(\lambda)x(\lambda) = -Q(\lambda)x(\lambda)$ is a vector in $\text{col}(P) \cap \text{col}(Q)$, and thus $z(\lambda) = 0$ and $P(\lambda)x(\lambda) = Q(\lambda)x(\lambda) = 0$. The converse is trivial. \square

We have already remarked in section 3 that if $\text{rank}(P+Q) = \text{rank}(P) + \text{rank}(Q) = n$, then the pencil $(P+Q)(\lambda)$ does not have column minimal indices. Therefore, in the rest of this section, it will be assumed $\text{rank}(P+Q) = \text{rank}(P) + \text{rank}(Q) < n$. This implies that $\text{rank}(Q) < p$, where p is the number of column minimal indices of P .

5.1. Connection polynomials and associated mosaic Toeplitz matrices.

From Lemma 5.1, it is possible to obtain a more specific characterization of the right null space vectors of $P+Q$.

LEMMA 5.2. *Let $P(\lambda)$ and $Q(\lambda)$ be two complex $m \times n$ matrix pencils such that $\text{rank}(P+Q) = \text{rank}(P) + \text{rank}(Q) < n$, $\tilde{\varepsilon}$ be the sum of the column minimal indices of $Q(\lambda)$, and $\{x_1(\lambda), \dots, x_p(\lambda)\}$ be an ROMB of $P(\lambda)$. Let us consider a right decomposition of $Q(\lambda)$ given by (7), where the first $\tilde{\varepsilon}$ vectors in $\{w_1(\lambda), \dots, w_\rho(\lambda)\}$ are assumed to be of degree one without loss of generality. Then, every right null space vector of $(P+Q)(\lambda)$ is a linear combination of the vectors $\{x_1(\lambda), \dots, x_p(\lambda)\}$ with polynomial coefficients. Moreover,*

$$(22) \quad x(\lambda) = \alpha_1(\lambda)x_1(\lambda) + \dots + \alpha_p(\lambda)x_p(\lambda)$$

is a right null space vector of $(P+Q)(\lambda)$ if and only if the polynomials $\alpha_1(\lambda), \dots, \alpha_p(\lambda)$ satisfy the system of linear equations in $\mathbb{C}^p(\lambda)$,

$$(23) \quad \begin{array}{rcl} a_{11}(\lambda)\alpha_1(\lambda) + \dots + a_{1p}(\lambda)\alpha_p(\lambda) & = & 0 \\ & \vdots & \vdots \\ a_{\rho 1}(\lambda)\alpha_1(\lambda) + \dots + a_{\rho p}(\lambda)\alpha_p(\lambda) & = & 0, \end{array}$$

where

$$(24) \quad a_{ij}(\lambda) = w_i(\lambda)^T x_j(\lambda), \quad i = 1, \dots, \rho, \quad j = 1, \dots, p.$$

Proof. A vector polynomial, $x(\lambda)$, is a right null space vector of $P+Q$ if and only if it is a right null space vector of P and Q , by Lemma 5.1. $P(\lambda)x(\lambda) = 0$ is equivalent to the fact that $x(\lambda)$ is a linear combination of $\{x_1(\lambda), \dots, x_p(\lambda)\}$ with polynomial coefficients, by Lemma 2.5. $Q(\lambda)x(\lambda) = 0$ is equivalent, taking into account (7), to $w_1(\lambda)^T x(\lambda) = \dots = w_\rho(\lambda)^T x(\lambda) = 0$, and this is the system of equations (23). \square

The system of equations (23) is of capital importance in this work, because the set of its solutions allows us to obtain the right null space of $P+Q$ through (22), and we are looking for the degrees of an ROMB of $\mathcal{N}(P+Q)$. Thus, the coefficients $a_{ij}(\lambda)$ of the system (23) play an essential role. They are polynomials in λ and link the pencils P and Q . They are used so often that we introduce the following definition.

DEFINITION 5.3. *A set of polynomials $\{a_{ij}(\lambda) : i = 1, \dots, \rho, j = 1, \dots, p\}$ like those appearing in (24) will be called a complete set of right connection polynomials of $P(\lambda)$ and $Q(\lambda)$.*

Since neither an ROMB of P nor a right decomposition (7) of Q is unique, a complete set of right connection polynomials of P and Q is not necessarily unique.

Remark 3. A left decomposition (8) of $Q(\lambda)$ and an LOMB $\{y_1(\lambda), \dots, y_q(\lambda)\}$ of $P(\lambda)$ can be considered to define a complete set of left connection polynomials of $P(\lambda)$ and $Q(\lambda)$. These are the polynomials

$$b_{ij}(\lambda) = \hat{v}_i(\lambda)^T y_j(\lambda), \quad i = 1, \dots, \rho, \quad j = 1, \dots, q.$$

These polynomials are needed to obtain the row minimal indices of $(P+Q)(\lambda)$.

Let us denote by $\varepsilon_1 \leq \dots \leq \varepsilon_p$ the column minimal indices of the unperturbed pencil $P(\lambda)$, and by $\tilde{\varepsilon}$ the sum of the column minimal indices of the perturbation pencil $Q(\lambda)$, as in section 2. Then the degrees of the right connection polynomials of P and Q are bounded as follows:

$$(25) \quad \deg(a_{ij}(\lambda)) \leq \begin{cases} \varepsilon_j + 1, & i = 1, \dots, \tilde{\varepsilon}, \\ \varepsilon_j, & i = \tilde{\varepsilon} + 1, \dots, \rho. \end{cases}$$

For most perturbations $Q(\lambda)$ these inequalities are, in fact, equalities, but this will not be assumed in the subsequent developments. Nevertheless, the generic behavior for the minimal indices of the perturbed pencil $P+Q$ holds under certain conditions that limit the number of right connection polynomials with degrees strictly less than the right-hand side of (25). These generic conditions involve some of the mosaic Toeplitz matrices appearing in the following definition.

DEFINITION 5.4. *Let $\varepsilon_1 \leq \dots \leq \varepsilon_p$ be the column minimal indices of the pencil $P(\lambda)$, $\tilde{\varepsilon}$ be the sum of the column minimal indices of the pencil $Q(\lambda)$, and $\{a_{ij}(\lambda) : i = 1, \dots, \rho, j = 1, \dots, p\}$ be a complete set of right connection polynomials of P and*

Q. Let us express these polynomials as follows:¹

$$a_{ij}(\lambda) = a_{ij}^0 + \lambda a_{ij}^1 + \cdots + \lambda^{\epsilon_{ij}} a_{ij}^{\epsilon_{ij}}, \quad \text{where } \epsilon_{ij} = \begin{cases} \epsilon_j + 1, & i = 1, \dots, \tilde{\epsilon}, \\ \epsilon_j, & i = \tilde{\epsilon} + 1, \dots, \rho. \end{cases}$$

Let k and d be nonnegative integer numbers such that $1 \leq k \leq p$ and $d \geq \epsilon_k - 1$. The k th mosaic Toeplitz matrix of degree d associated with the connection polynomials $a_{ij}(\lambda)$ is denoted by $A_k(d)$ and is a matrix partitioned into ρ rows and k columns of blocks whose (s, t) -block, $s = 1, \dots, \rho$, $t = 1, \dots, k$, is the Toeplitz matrix

$$(A_k(d))_{st} = \underbrace{\begin{bmatrix} a_{st}^0 & & & & \\ \vdots & \ddots & & & \\ a_{st}^{\epsilon_{st}} & & \ddots & & \\ & \ddots & & a_{st}^0 & \\ & & \ddots & \vdots & \\ & & & a_{st}^{\epsilon_{st}} & \end{bmatrix}}_{d - \epsilon_t + 1},$$

with $d - \epsilon_t + 1$ columns and therefore a number of rows equal to

$$\epsilon_{st} + 1 + d - \epsilon_t = \begin{cases} d + 2, & s = 1, \dots, \tilde{\epsilon}, \\ d + 1, & s = \tilde{\epsilon} + 1, \dots, \rho. \end{cases}$$

Remark 4. Notice that in the case $d = \epsilon_k - 1$ the k th column of blocks of $A_k(d)$ is formed by matrices having a “number of columns equal to zero,” i.e., by empty matrices. This also happens for those columns whose index j satisfies $d = \epsilon_j - 1$. We understand in this case that $A_k(d)$ has less than k columns of blocks. This convention will simplify the notation and the statements of our results.

The importance of the family of mosaic Toeplitz matrices $A_k(d)$ is made clear by the next lemma, Lemma 5.5. This result extends and complements Lemma 5.2, characterizing right null space vectors of $P + Q$ of a given degree through systems of constant linear equations, i.e., systems of equations in \mathbb{C}^n and not in $\mathbb{C}^n(\lambda)$ as (23). Degrees are the fundamental quantities in this section, because our goal is to get the degrees of the vectors of an ROMB of $P + Q$, i.e., the column minimal indices of $P + Q$. According to Lemma 5.2 all the right null space vectors, $x(\lambda)$, of $P + Q$ are of the form (22), and they have $\deg(x) = \max_{1 \leq i \leq p} \{\epsilon_i + \deg(\alpha_i) : \alpha_i(\lambda) \neq 0\}$ [6, Main Theorem, p. 495]. This implies that if j is the largest index such that $\alpha_j(\lambda) \neq 0$, then $\deg(x) \geq \epsilon_j$. To look for smaller degrees, one has to consider necessarily linear combinations $x(\lambda) = \alpha_1(\lambda)x_1(\lambda) + \cdots + \alpha_k(\lambda)x_k(\lambda)$, with $k < j$. See also Lemma 2.5.

LEMMA 5.5. *Let $P(\lambda)$ and $Q(\lambda)$ be two complex $m \times n$ matrix pencils such that $\text{rank}(P + Q) = \text{rank}(P) + \text{rank}(Q) < n$, $\{x_1(\lambda), \dots, x_p(\lambda)\}$ be an ROMB of P , and $\epsilon_1 \leq \cdots \leq \epsilon_p$ be the column minimal indices of P , i.e., $\deg(x_i) = \epsilon_i$. Then the following hold:*

1. Every right null space vector, $x(\lambda)$, of $(P + Q)(\lambda)$ of degree d can be expressed in the form $x(\lambda) = \alpha_1(\lambda)x_1(\lambda) + \cdots + \alpha_k(\lambda)x_k(\lambda)$ for the largest number k such that $1 \leq k \leq p$ and $d \geq \epsilon_k$, and a unique set of polynomials $\{\alpha_1(\lambda), \dots, \alpha_k(\lambda)\}$.

¹The reader should notice that the superscript notation a_{ij}^k does not mean a_{ij} to the k th power.

2. Moreover,

$$(26) \quad x(\lambda) = \alpha_1(\lambda)x_1(\lambda) + \cdots + \alpha_k(\lambda)x_k(\lambda)$$

is a right null space vector of $(P+Q)(\lambda)$ of degree $d \geq \varepsilon_k$ if and only if the polynomials $\alpha_1(\lambda), \dots, \alpha_k(\lambda)$ satisfy the following two conditions:

(i) The polynomials $\alpha_i(\lambda)$ can be expressed as

$$(27) \quad \alpha_i(\lambda) = \alpha_{i0} + \lambda \alpha_{i1} + \cdots + \lambda^{d-\varepsilon_i} \alpha_{i,d-\varepsilon_i}$$

for all $i = 1, \dots, k$, with $\alpha_{j,d-\varepsilon_j} \neq 0$ for at least one index j .

(ii) If $A_k(d)$ is the k th mosaic Toeplitz matrix of degree d associated with a complete set of right connection polynomials of P and Q defined with $\{x_1(\lambda), \dots, x_p(\lambda)\}$, the coefficients α_{il} satisfy the system of constant linear equations

$$(28) \quad A_k(d) \begin{bmatrix} \alpha_{10} \\ \vdots \\ \alpha_{1,d-\varepsilon_1} \\ \vdots \\ \vdots \\ \alpha_{k0} \\ \vdots \\ \alpha_{k,d-\varepsilon_k} \end{bmatrix} = 0.$$

Notice that, if all the solutions of the system (28) are such that $\alpha_{1,d-\varepsilon_1} = \cdots = \alpha_{k,d-\varepsilon_k} = 0$, then all nonzero right null space vectors of $P + Q$ of the form (26) are of degree less than d .

Proof. The proof of the first item is a direct consequence of Lemma 5.2 and the fact that in (22) $\deg(x) = \max_{1 \leq i \leq p} \{\varepsilon_i + \deg(\alpha_i) : \alpha_i(\lambda) \neq 0\}$ according to [6, Main Theorem, p. 495]. Once the index k is chosen, the uniqueness of $\{\alpha_1(\lambda), \dots, \alpha_k(\lambda)\}$ follows from the linear independence of $\{x_1(\lambda), \dots, x_k(\lambda)\}$.

The second item follows from (22) and (23) by setting $\alpha_{k+1}(\lambda) = \cdots = \alpha_p(\lambda) = 0$. Notice that (27) simply states that there are no indices i , $1 \leq i \leq k$, such that $\deg(\alpha_i) > d - \varepsilon_i$, because this would imply $\deg(x) > d$. The condition $\alpha_{j,d-\varepsilon_j} \neq 0$ for at least one index j guarantees that $\deg(x) = d$. On the other hand, the linear system (28) is the system obtained from (23) by expanding products and sums of polynomials and equating the coefficients to zero. With these remarks in mind, the proof is trivial. \square

5.2. Properties of mosaic Toeplitz matrices. Lemma 5.6 gathers the properties of mosaic Toeplitz matrices that we will use to deduce the generic minimal indices of the pencil $P + Q$.

LEMMA 5.6. Let $\mathcal{T} = \{A_k(d) : 1 \leq k \leq p, d \geq \varepsilon_k - 1\}$ be the set of mosaic Toeplitz matrices defined in Definition 5.4. Then the following hold:

1. The number of rows of $A_k(d)$ is equal to $\rho(d + 1) + \tilde{\varepsilon}$.
2. The number of columns of $A_k(d)$ is equal to $k(d + 1) - \sum_{j=1}^k \varepsilon_j$.
3. $A_k(d)$ has more columns than rows if and only if

$$k > \rho \quad \text{and} \quad d > \frac{\sum_{i=1}^k \varepsilon_i + \tilde{\varepsilon}}{k - \rho} - 1.$$

- 4. If the columns of $A_k(d)$ are linearly independent, i.e., $A_k(d)$ has full column rank, then every matrix $A_{k'}(d')$ in \mathcal{T} with $k' \leq k$ and $d' \leq d$ has full column rank.
- 5. If the rows of $A_k(d)$ are linearly independent, i.e., $A_k(d)$ has full row rank, then every matrix $A_{k'}(d')$ in \mathcal{T} with $k' \geq k$ and $d' \geq d$ has full row rank.
- 6. If the rows of $A_k(d)$ are linearly independent, then

$$(29) \quad \text{rank} \begin{bmatrix} a_{11}(\lambda) & a_{12}(\lambda) & \dots & a_{1j}(\lambda) \\ \vdots & \vdots & & \vdots \\ a_{\rho 1}(\lambda) & a_{\rho 2}(\lambda) & \dots & a_{\rho j}(\lambda) \end{bmatrix} = \rho \quad \text{for } j \geq k.$$

Proof. The first three items are direct consequences of the number of rows and columns of the blocks appearing in Definition 5.4.

Item 4. Notice that $A_{k-1}(d)$ is obtained from $A_k(d)$ just by erasing the last column of blocks. As a consequence, the columns of $A_{k-1}(d)$ are a subset of the columns of $A_k(d)$. Then, $A_{k-1}(d)$ has full column rank if $A_k(d)$ has full column rank, and, by induction, $A_{k'}(d)$ has full column rank whenever $k' \leq k$.

If $d - 1 \geq \varepsilon_k - 1$, then $A_k(d - 1)$ is an element of \mathcal{T} , and it is obtained from $A_k(d)$ by erasing the last column of each block of $A_k(d)$ to get a certain matrix $A'_k(d)$ and, after that, erasing the last row of each block of $A'_k(d)$ to get $A_k(d - 1)$. However, notice that $A'_k(d)$ is of full column rank, and that the last rows of the blocks of $A'_k(d)$ are zero rows; then $A_k(d - 1)$ also has full column rank.

If $d - 1 < \varepsilon_k - 1$, then $A_k(d - 1)$ is not in \mathcal{T} . Let $k' < k$ be the largest index such that $d - 1 \geq \varepsilon_{k'} - 1$. Then $A_{k'}(d - 1)$ is an element of \mathcal{T} , and $A_{k'}(d)$ has full column rank. The argument in the paragraph above is applied to prove that $A_{k'}(d - 1)$ has full column rank.

Finally, the results above can be combined inductively to prove item 4.

Item 5. Let $k' \geq k$. Then the submatrix of $A_{k'}(d)$ that lies in the first $k(d + 1) - \sum_{j=1}^k \varepsilon_j$ columns is precisely $A_k(d)$. Therefore, if $A_k(d)$ has full row rank, then $A_{k'}(d)$ also has full row rank. To complete the proof, let us prove that $A_k(d + t)$ has full row rank for any integer $t > 0$ whenever $A_k(d)$ has full row rank. It is enough to prove this statement for $t = 1$ and then to apply an inductive argument. Notice that the submatrix of $A_k(d)$ that contains the last row of each of the row blocks of $A_k(d)$ has linearly independent rows. This means that for the matrix

$$B \equiv \begin{bmatrix} a_{11}^{\varepsilon_{11}} & \dots & a_{1k}^{\varepsilon_{1k}} \\ \vdots & & \vdots \\ a_{\rho 1}^{\varepsilon_{\rho 1}} & \dots & a_{\rho k}^{\varepsilon_{\rho k}} \end{bmatrix},$$

$\text{rank}(B) = \rho$. Observe that the matrix B is also the $\rho \times k$ submatrix of $A_k(d + 1)$ that lies in the last rows and columns of the blocks of $A_k(d + 1)$. If the last rows and columns of the blocks of $A_k(d + 1)$ are moved down and back by permutations to the last positions, the rank does not change, and the matrix we get has the structure

$$\begin{bmatrix} A_k(d) & * \\ 0 & B \end{bmatrix}.$$

The rank of this matrix is clearly $\text{rank}(A_k(d)) + \rho = \rho(d + 2) + \tilde{\varepsilon}$, i.e., the number of rows of $A_k(d + 1)$. Therefore $A_k(d + 1)$ has full row rank.

Item 6. It is enough to prove this property for $j = k$. If the rows of $A_k(d)$ are linearly independent, then the submatrix of $A_k(d)$ that contains the first row of each

of the row blocks of $A_k(d)$ has linearly independent rows. This means that

$$\text{rank} \begin{bmatrix} a_{11}^0 & \cdots & a_{1k}^0 \\ \vdots & & \vdots \\ a_{\rho 1}^0 & \cdots & a_{\rho k}^0 \end{bmatrix} = \rho.$$

The result follows by applying Lemma 2.6.1 to the rows of the matrix (29) for $j = k$. \square

According to [8, Chapter XII, p. 38] (see also the proof of Lemma 2.4 in this paper), the smallest column minimal index of $P + Q$ is the least degree among the degrees of nonzero right null space vectors of $P + Q$. Taking into account Lemma 5.5.2, this smallest minimal index corresponds to the smallest d for which a linear system of the family (28) ($1 \leq k \leq p$) has nonzero solutions with $\alpha_{j,d-\varepsilon_j} \neq 0$ for at least one index j . Our intuition here is that solutions of this kind do not exist, generically, if $A_k(d)$ has a number of rows larger than or equal to the number of columns, and they do exist, generically, in the opposite case. This intuition is based on the idea that if the coefficients of the connection polynomials are random for random perturbations pencils Q , then the columns of $A_k(d)$ should be linearly independent if $A_k(d)$ has more rows than columns or the same number of rows and columns. Based on this intuition, one can tentatively think that the most likely value of the smallest minimal index of $P + Q$ for random perturbations Q is the smallest d such that some of the $A_k(d)$ has more columns than rows. Of course, these naive arguments have to be supported with rigorous assumptions, but they, together with Lemma 5.6.3, make it natural to consider the following sequence of integer numbers:

$$d_k = \left\lfloor \frac{\sum_{i=1}^k \varepsilon_i + \tilde{\varepsilon}}{k - \rho} \right\rfloor \quad \text{for } k = \rho + 1, \dots, p,$$

where $\lfloor x \rfloor$ denotes the floor function of x , i.e., the largest integer that is less than or equal to x . Notice that $A_k(d_k)$ exists only if $d_k \geq \varepsilon_k - 1$; in this case Lemma 5.6.3 guarantees that $A_k(d_k)$ has more columns than rows. However, it is not difficult to devise examples for which $d_k < \varepsilon_k - 1$ for some k . The natural candidate for the smallest column minimal index of $P + Q$ is $\min_{\rho+1 \leq k \leq p} d_k$. To prove that this is the case under certain generic assumptions, and also to find the rest of the column minimal indices, it is necessary to study the properties of the sequence $\{d_k\}$.

LEMMA 5.7. *Let $0 \leq \varepsilon_1 \leq \dots \leq \varepsilon_p$ be p integer numbers, and ρ and $\tilde{\varepsilon}$ be integer numbers such that $0 < \rho < p$ and $0 \leq \tilde{\varepsilon} \leq \rho$. Let us consider the sequence of integer numbers*

$$(30) \quad d_k = \left\lfloor \frac{\sum_{i=1}^k \varepsilon_i + \tilde{\varepsilon}}{k - \rho} \right\rfloor \quad \text{for } k = \rho + 1, \dots, p.$$

Then we have the following:

1. $d_{\rho+1} \geq \varepsilon_{\rho+1} \geq \dots \geq \varepsilon_1$.
2. If $d_k < d_{k-1}$, then $d_k \geq \varepsilon_k \geq \dots \geq \varepsilon_1$.
3. If $d_k < d_{k+1}$, then $d_k < d_{k+1} \leq d_{k+2} \leq \dots \leq d_p$.
4. If $d_k < d_{k+1}$, then $d_i < \varepsilon_i$ for all $i \geq (k + 1)$.
5. Let

$$d_{\min} \equiv \min_{\rho+1 \leq k \leq p} d_k.$$

Then all the indices j such that $d_j = d_{\min}$ are consecutive; i.e., if $d_j = d_{\min}$ holds for more than one index, then there exist two indices $j_1 < j_2$ such that

$$d_j = d_{\min} \quad \text{if } j_1 \leq j \leq j_2 \quad \text{and} \quad d_j > d_{\min} \quad \text{if } j < j_1 \quad \text{or} \quad j_2 < j.$$

In addition, $d_{j_1} \geq \varepsilon_{j_1}$.

6. Let s be the largest index such that $d_s = d_{\min}$ and $d_s \geq \varepsilon_s$. Then

$$\varepsilon_k > d_k \geq d_s \quad \text{for all } k > s.$$

7. Let s be the index defined in the previous item, and $A_s(d_{\min})$ and $A_s(d_{\min} - 1)$ be mosaic Toeplitz matrices introduced in Definition 5.4. Then

(i) $A_s(d_{\min})$ has more columns than rows and has s columns of blocks;

(ii) $A_s(d_{\min} - 1)$ has a number of rows larger than or equal to the number of columns, or it is the empty matrix;

(iii) for any $k > s$, $A_k(d_{\min})$ is not defined or $A_k(d_{\min}) = A_s(d_{\min})$.

Before proving this lemma, we would like to point out that the index s appearing in item 6 will play an essential role in determining the generic column minimal indices of $P + Q$.

Proof of Lemma 5.7. The first item is trivial.

Item 2. Let us consider the integer divisions

$$(31) \quad \sum_{i=1}^k \varepsilon_i + \tilde{\varepsilon} = (k - \rho)d_k + r_k, \quad \text{where } 0 \leq r_k < k - \rho,$$

$$(32) \quad \sum_{i=1}^{k-1} \varepsilon_i + \tilde{\varepsilon} = (k - 1 - \rho)d_{k-1} + r_{k-1}, \quad \text{where } 0 \leq r_{k-1} < k - 1 - \rho.$$

Let us subtract (32) from (31) to get

$$\varepsilon_k = (k - \rho - 1)(d_k - d_{k-1}) + d_k + r_k - r_{k-1} \leq (k - \rho - 1)(d_k - d_{k-1}) + d_k + k - \rho - 1.$$

Thus, $\varepsilon_k \leq (k - \rho - 1)(d_k - d_{k-1} + 1) + d_k \leq d_k$. The last step is a consequence of $(d_k - d_{k-1} + 1) \leq 0$ and $(k - 1) > \rho$.

Item 3. Let us consider the integer division

$$(33) \quad \sum_{i=1}^{k+1} \varepsilon_i + \tilde{\varepsilon} = (k + 1 - \rho)d_{k+1} + r_{k+1}, \quad \text{where } 0 \leq r_{k+1} < k + 1 - \rho.$$

Let us subtract (31) from (33) to get

$$\varepsilon_{k+1} = (k - \rho)(d_{k+1} - d_k) + d_{k+1} + r_{k+1} - r_k \geq (k - \rho) + d_{k+1} + r_{k+1} - r_k > d_{k+1},$$

where we have used that $r_{k+1} - r_k > -(k - \rho)$. Therefore, we have proved that

$$(34) \quad d_k < d_{k+1} \quad \text{implies} \quad \varepsilon_{k+1} > d_{k+1}.$$

Let us consider now an index l such that $l \geq (k + 2)$, and the integer division

$$(35) \quad \sum_{i=1}^l \varepsilon_i + \tilde{\varepsilon} = (l - \rho)d_l + r_l, \quad \text{where } 0 \leq r_l < l - \rho.$$

Let us subtract (33) from (35) to get

$$\varepsilon_l + \varepsilon_{l-1} + \cdots + \varepsilon_{k+2} = (d_l - d_{k+1})(l - \rho) + d_{k+1}(l - (k + 1)) + r_l - r_{k+1},$$

and then,

$$(\varepsilon_l - d_{k+1}) + (\varepsilon_{l-1} - d_{k+1}) + \cdots + (\varepsilon_{k+2} - d_{k+1}) = (d_l - d_{k+1})(l - \rho) + r_l - r_{k+1}.$$

The inequality (34) implies $(d_l - d_{k+1})(l - \rho) + r_l - r_{k+1} > 0$, and therefore $(d_l - d_{k+1} + 1)(l - \rho) > 0$. Thus, we have proven that

$$(36) \quad d_k < d_{k+1} \quad \text{implies} \quad d_l \geq d_{k+1} \quad \text{for all } l \geq (k + 2).$$

This result allows us to prove the more general result appearing in item 3. Let us proceed by contradiction. Assume that $d_{k+1} \leq d_{k+2} \leq \cdots \leq d_p$ is false. This means that there exists an index $l \geq (k + 2)$ such that $d_{k+1} \leq d_{k+2} \leq \cdots \leq d_{l-1} > d_l$. Let j be the smallest integer such that $(k + 1) \leq j \leq (l - 1)$ and $d_j = d_{j+1} = \cdots = d_{l-1}$. Notice that this integer is at least $k + 1$, because $d_k < d_{l-1}$ by (36). Then $d_{j-1} < d_j$, and (36) can be applied with $k = j - 1$ to see that $d_j \leq d_l$; on the other hand, $d_j = d_{l-1} > d_l$. This is absurd.

Item 4. Let us prove the result by induction. In (34), we have already proven the base case of the induction: $d_{k+1} < \varepsilon_{k+1}$. Let us assume that $d_i < \varepsilon_i$ for some $i \geq (k + 1)$. On the other hand, $d_i \leq d_{i+1}$ due to the result in item 3. If $d_i < d_{i+1}$, one can apply (34) with $k = i$ to see that $d_{i+1} < \varepsilon_{i+1}$. Otherwise, $d_i = d_{i+1}$ and $d_{i+1} < \varepsilon_i \leq \varepsilon_{i+1}$.

Item 5. The fact that the indices are consecutive is a direct consequence of item 3. The fact that $d_{j_1} \geq \varepsilon_{j_1}$ is a consequence of items 1 and 2.

Item 6. If there is only one index s such that $d_s = d_{\min}$, the result is a simple consequence of items 4 and 5. Otherwise, let j_1 and j_2 be the two indices appearing in item 5. If $s = j_2 \leq p$, the result follows again from item 4. If $s < j_2$, then, by definition, $d_s = d_{s+1} < \varepsilon_{s+1} \leq \cdots \leq \varepsilon_{j_2}$. Therefore, $d_k < \varepsilon_k$ for $s + 1 \leq k \leq j_2$. Also, by definition, $d_{j_2} < d_{j_2+1}$, and item 4 implies $d_k < \varepsilon_k$ for $k \geq (j_2 + 1)$.

Item 7. The assertions on the number of rows and columns of $A_s(d_{\min})$ and $A_s(d_{\min} - 1)$ follow from Lemma 5.6.3. Let us remember that the j th column of blocks of $A_s(d_{\min})$ has $d_{\min} - \varepsilon_j + 1$ columns; therefore, $d_{\min} \geq \varepsilon_s$ guarantees that all the blocks of $A_s(d_{\min})$ have at least one column. Notice that $d_{\min} \geq \varepsilon_s$ also implies that $d_{\min} - 1 \geq \varepsilon_s - 1$; thus $A_s(d_{\min} - 1)$ is defined, but some (or all) of its blocks may be empty. Finally, for $j > s$ we know that $\varepsilon_j > d_{\min}$, i.e., $\varepsilon_j - 1 \geq d_{\min}$. This means that $A_k(d_{\min})$, with $k > s$, is not defined unless $\varepsilon_j - 1 = d_{\min}$ for $s + 1 \leq j \leq k$, but in this case the j th blocks $(A_k(d_{\min}))_{ij}$ are empty matrices. \square

5.3. Generic column minimal indices of $P + Q$. Now we are in position to find out which are the generic column minimal indices of the perturbed pencil $(P + Q)(\lambda)$, assuming that, apart from the rank, the sum of the column minimal indices of the perturbation is known. This is done in Theorem 5.8, our second major contribution.

THEOREM 5.8. *Let $P(\lambda)$ and $Q(\lambda)$ be two $m \times n$ complex matrix pencils such that $\text{rank}(P + Q) = \text{rank}(P) + \text{rank}(Q) < n$ and $\rho \equiv \text{rank}(Q)$. Let $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ be the column minimal indices of P , $\tilde{\varepsilon}$ be the sum of the column minimal indices of Q , $\{d_{\rho+1}, \dots, d_p\}$ be the sequence of numbers defined in (30), d_{\min} be the minimum of this sequence, and s be the largest index such that $d_s = d_{\min}$ and $d_s \geq \varepsilon_s$. Finally, let $A_s(d_{\min} - 1)$ and $A_s(d_{\min})$ be the s th mosaic Toeplitz matrices of degrees $d_{\min} - 1$*

and d_{\min} , respectively, associated with a complete set of right connection polynomials of P and Q . If

$$(37) \quad A_s(d_{\min} - 1) \quad \text{has full column rank or is the empty matrix, and}$$

$$(38) \quad A_s(d_{\min}) \quad \text{has full row rank,}$$

then $(P + Q)(\lambda)$ has the following $p - \rho$ column minimal indices:

$$(39) \quad \underbrace{d_{\min} = \cdots = d_{\min}}_{s - \rho - \gamma_s} < \underbrace{(d_{\min} + 1) = \cdots = (d_{\min} + 1)}_{\gamma_s} \leq \varepsilon_{s+1} \leq \cdots \leq \varepsilon_p,$$

where γ_s is the remainder in the integer division of $\sum_{i=1}^s \varepsilon_i + \tilde{\varepsilon}$ by $s - \rho$.

Proof. In the first place, let us notice that the ordering appearing in (39) is a consequence of Lemma 5.7.6. Also notice that the number of column minimal indices of $P + Q$ is $p - \rho$; this is a simple consequence of (4) and $\text{rank}(P + Q) = \text{rank}(P) + \text{rank}(Q)$. For the rest of the proof, it is convenient to bear in mind Lemma 2.4 applied to $P + Q$, and the way ROMBs of $P + Q$ are constructed (see the first paragraph in the proof of Lemma 2.4).

Let us begin by proving that there are no column minimal indices of $P + Q$ smaller than d_{\min} . Lemma 5.5.1 and Lemma 5.7.6 guarantee that every right null space vector of $P + Q$ with degree $d < d_{\min}$ is a linear combination of the type $x(\lambda) = \alpha_1(\lambda)x_1(\lambda) + \cdots + \alpha_k(\lambda)x_k(\lambda)$ for some $k \leq s$. In this situation, the matrix $A_k(d)$ appearing in (28) has full column rank in the case $A_s(d_{\min} - 1)$ has full column rank, by Lemma 5.6.4. The system (28) has only the zero solution and $x(\lambda) = 0$. In the case $A_s(d_{\min} - 1)$ is the empty matrix $d_{\min} = \varepsilon_k$ whenever $1 \leq k \leq s$, and there are no nonzero linear combinations of $\{x_1(\lambda), \dots, x_p(\lambda)\}$ of degree smaller than $\varepsilon_1 = d_{\min}$, because otherwise the smallest column minimal index of P would be less than ε_1 .

Our next step is to prove that d_{\min} is a column minimal index of $P + Q$. The system (28) with coefficient matrix $A_s(d_{\min})$ necessarily has nonzero solutions because $A_s(d_{\min})$ has more columns than rows by Lemma 5.7.7. Besides, there are no nonzero solutions with $\alpha_{1, d_{\min} - \varepsilon_1} = \cdots = \alpha_{s, d_{\min} - \varepsilon_s} = 0$, because otherwise the solutions of (28) correspond to right null space vectors of $P + Q$ of degree less than d_{\min} , and we already know that they do not exist. This proves that d_{\min} is the smallest column minimal index of $P + Q$.

To see that there are precisely $s - \rho - \gamma_s$ column minimal indices of $P + Q$ equal to d_{\min} , we need to find $s - \rho - \gamma_s$ linearly independent right null space vectors of $P + Q$ of degree d_{\min} , and to prove that there are no more. Again, Lemma 5.5.1 and Lemma 5.7.6 guarantee that every right null space vector of $P + Q$ with degree d_{\min} is a linear combination of the type

$$(40) \quad x(\lambda) = \alpha_1(\lambda)x_1(\lambda) + \cdots + \alpha_s(\lambda)x_s(\lambda).$$

Notice that the set of solutions of (28) with $A_s(d_{\min})$ as coefficient matrix can be described in terms of a number of free parameters equal to the difference between the number of columns and the number of rows of $A_s(d_{\min})$, i.e.,

$$s(d_{\min} + 1) - \sum_{i=1}^s \varepsilon_i - \rho(d_{\min} + 1) - \tilde{\varepsilon} = s - \rho - \gamma_s,$$

where Lemma 5.6.1 and Lemma 5.6.2 have been used. This means that the system of linear equations (28) with $A_s(d_{\min})$ has $s - \rho - \gamma_s$ linearly independent solutions, and,

by Lemma 5.5, that they correspond to $s - \rho - \gamma_s$ right null space vectors of $P + Q$ of the form (40) of degree d_{\min} . Let us denote these vectors by

$$(41) \quad \{z_1(\lambda), z_2(\lambda), \dots, z_{\beta_s}(\lambda)\}, \quad \text{with } \beta_s \equiv s - \rho - \gamma_s.$$

It is clear that any other solution of (28) corresponds to right null space vectors of degree d_{\min} that are linear combinations of (41) with constant coefficients; however, we still need to prove that the vectors $\{z_1(\lambda), z_2(\lambda), \dots, z_{\beta_s}(\lambda)\}$ can be chosen to be linearly independent in $\mathbb{C}^n(\lambda)$. To see this, notice that the β_s free parameters of (28) with $A_s(d_{\min})$ may be taken among the $\alpha_{1,d_{\min}-\varepsilon_1}, \dots, \alpha_{s,d_{\min}-\varepsilon_s}$ variables, because the columns of $A_s(d_{\min})$ that do not correspond to these variables are linearly independent, as we have already seen in the paragraph proving that d_{\min} is the smallest minimal index of $P + Q$. By setting the l th of these β_s variables equal to 1 and the rest equal to 0, and repeating this process for $l = 1, \dots, \beta_s$, a set \mathcal{S} of β_s linearly independent solutions of (28) may be obtained. Let us denote by

$$(42) \quad a_l = [\alpha_{1,d_{\min}-\varepsilon_1}^l, \dots, \alpha_{s,d_{\min}-\varepsilon_s}^l]^T, \quad l = 1, \dots, \beta_s,$$

a vector containing the shown entries of the l th solution of (28) in \mathcal{S} . The vectors $\{a_1, \dots, a_{\beta_s}\}$ are obviously linearly independent. If (27) and (40) are recalled, the coefficients of the highest degree terms of the vectors (41) corresponding to the β_s solutions of (28) in \mathcal{S} are

$$(43) \quad z_{l,d_{\min}} = \alpha_{1,d_{\min}-\varepsilon_1}^l x_{1,\varepsilon_1} + \dots + \alpha_{s,d_{\min}-\varepsilon_s}^l x_{s,\varepsilon_s}, \quad \text{for } l = 1, \dots, \beta_s,$$

where x_{i,ε_i} is the highest degree coefficient of $x_i(\lambda)$. The vectors $\{x_{1,\varepsilon_1}, \dots, x_{s,\varepsilon_s}\}$ are linearly independent in \mathbb{C}^n , because $x_1(\lambda), \dots, x_s(\lambda)$ are part of an ROMB and [6, Main Theorem, Item 2, p. 495] can be applied. Therefore, $\{z_{1,d_{\min}}, \dots, z_{\beta_s,d_{\min}}\}$ is a linearly independent set, because $[z_{1,d_{\min}}, \dots, z_{\beta_s,d_{\min}}] = [x_{1,\varepsilon_1}, \dots, x_{s,\varepsilon_s}][a_1, \dots, a_{\beta_s}]$ and the two matrices in the right-hand side have full column rank. Finally, Lemma 2.6.2 implies that $\{z_1(\lambda), z_2(\lambda), \dots, z_{\beta_s}(\lambda)\}$ are linearly independent, and that there are precisely $\beta_s \equiv s - \rho - \gamma_s$ column minimal indices of $P + Q$ equal to d_{\min} .

In this paragraph, we prove that there are γ_s column minimal indices of $P + Q$ equal to $d_{\min} + 1$. At present, we have found a set $\mathcal{C}_1 = \{z_1(\lambda), z_2(\lambda), \dots, z_{\beta_s}(\lambda)\}$ of $s - \rho - \gamma_s$ linearly independent right null space vectors of $P + Q$ of the form (40) and degree d_{\min} . However, the fact that $A_s(d_{\min})$ has full row rank, Lemma 5.6.6 with $j = s$, and Lemma 5.2 imply that a maximal linearly independent set of right null space vectors of $P + Q$ of the form (40) has $s - \rho$ vectors. We will prove that the remaining γ_s vectors can be chosen to be of degree $d_{\min} + 1$. Let us consider the system (28) with coefficient matrix $A_s(d_{\min} + 1)$. The matrix $A_s(d_{\min} + 1)$ has full row rank because $A_s(d_{\min})$ has full row rank, and Lemma 5.6.5 can be applied. This means that $\text{rank}(A_s(d_{\min} + 1)) = \text{rank}(A_s(d_{\min})) + \rho$. Remember that $A_s(d_{\min} + 1)$ can be obtained from $A_s(d_{\min})$ by adding one row and one column in the last positions of each block. Therefore, among the s columns of $A_s(d_{\min} + 1)$ that are in the last positions of the column blocks, $s - \rho$ are linear combinations of the remaining columns of $A_s(d_{\min} + 1)$. Thus, the corresponding variables in the system (28) with $A_s(d_{\min} + 1)$ can be taken as some of the free parameters to solve this system.² This

²The reader should notice that the difference between the number of columns and rows of $A_s(d_{\min} + 1)$ is $2(s - \rho) - \gamma_s$. Therefore, the system (28) with matrix $A_s(d_{\min} + 1)$ has $2(s - \rho) - \gamma_s$ linearly independent solutions, while there are only $s - \rho$ linearly independent right null space vectors of the form (40). This means that linearly independent solutions of (28) do not always correspond to linearly independent right null space vectors (26).

implies that $s - \rho$ free parameters to solve (28) with $A_s(d_{\min} + 1)$ may be taken among the $\alpha_{1,(d_{\min}+1)-\varepsilon_1}, \dots, \alpha_{s,(d_{\min}+1)-\varepsilon_s}$ variables. If we proceed with these parameters as in the previous paragraph (arguments around (41)–(43)), we can find a set \mathcal{C}_2 of $(s - \rho)$ linearly independent right null space vectors of $P + Q$ of degree exactly $d_{\min} + 1$, and of the form (40). Therefore, we can join the set \mathcal{C}_1 of $s - \rho - \gamma_s$ vectors of degree d_{\min} with some γ_s vectors of \mathcal{C}_2 , to get a maximal linearly independent set of right null space vectors of $P + Q$ of the form (40). This proves that there exist γ_s column minimal indices of $P + Q$ equal to $d_{\min} + 1$.

Our last task in proving Theorem 5.8 is to show that the remaining column minimal indices of $P + Q$ are $\varepsilon_{s+1} \leq \dots \leq \varepsilon_p$. The right null space vectors of $P + Q$ that we have already found, corresponding to minimal indices equal to d_{\min} and to $d_{\min} + 1$, constitute a maximal linearly independent set of right null space vectors of $P + Q$ of the form (40). This fact implies that any right null space vector $x(\lambda)$ of $P + Q$ corresponding to the next smallest minimal index has to be necessarily of the form (22) with at least one of the coefficients $\alpha_{s+1}(\lambda), \dots, \alpha_p(\lambda)$ different from zero. Otherwise, it would depend linearly on the right null space vectors corresponding to the minimal indices d_{\min} and $d_{\min} + 1$. Thus, according to [6, Main Theorem, p. 495], $\deg(x) = \max_{1 \leq i \leq p} \{\varepsilon_i + \deg(\alpha_i) : \alpha_i(\lambda) \neq 0\} \geq \varepsilon_{s+1} \geq d_{\min} + 1$, where the last inequality is a consequence of Lemma 5.7.6. Then, the least candidate to the next minimal index is ε_{s+1} . To show that, in fact, ε_{s+1} is the next column minimal index, we will prove that there is a right null space vector of $P + Q$ of the form

$$(44) \quad x(\lambda) = \alpha_1(\lambda)x_1(\lambda) + \dots + \alpha_{s+1}(\lambda)x_{s+1}(\lambda),$$

with $\alpha_{s+1}(\lambda) \neq 0$ and $\deg(x) = \varepsilon_{s+1}$, i.e., with $\alpha_{s+1}(\lambda)$ a nonzero constant. This is equivalent to proving that the linear system (28) with coefficient matrix $A_{s+1}(\varepsilon_{s+1})$ has solutions with the last entry different from zero. Notice that $A_{s+1}(\varepsilon_{s+1})$ has full row rank because $A_s(d_{\min})$ has full row rank, $\varepsilon_{s+1} > d_{\min}$ by Lemma 5.7.6, and Lemma 5.6.5 can be applied. Besides, the matrices in the last columns of blocks of $A_{s+1}(\varepsilon_{s+1})$ have only one column. Therefore, if the last column of $A_{s+1}(\varepsilon_{s+1})$ is removed, then $A_s(\varepsilon_{s+1})$ is obtained. However,

$$(45) \quad \text{number of rows of } A_{s+1}(\varepsilon_{s+1}) = \text{number of rows of } A_s(\varepsilon_{s+1}),$$

and $A_s(\varepsilon_{s+1})$ has also full row rank by the same argument; then

$$(46) \quad \text{rank}(A_{s+1}(\varepsilon_{s+1})) = \text{rank}(A_s(\varepsilon_{s+1})).$$

This implies that the last column of $A_{s+1}(\varepsilon_{s+1})$ is a linear combination of its remaining columns. As a consequence the last variable in the linear system (28) with coefficient matrix $A_{s+1}(\varepsilon_{s+1})$ may be considered as free parameter, and therefore it may be different from zero. This proves that the ε_{s+1} is the next minimal index.

Notice that assumption (38) and Lemma 5.6.6 imply that a maximal linearly independent set of right null space vectors of $P + Q$ of the form (44) has $s + 1 - \rho$ vectors. Therefore, a maximal linearly independent set of this type has already been found in the previous paragraphs. With this remark in mind, the proof that ε_{s+2} is the next smallest column minimal index follows step-by-step the proof presented in the previous paragraph for ε_{s+1} with the corresponding changes of indices. The same holds for proving that $\varepsilon_{s+3}, \dots, \varepsilon_p$ are the remaining column minimal indices of $P + Q$. \square

5.4. Application of Theorem 5.8 to an example. Let us show with an example how to apply Theorem 5.8. Let $P(\lambda)$ be the 5×5 matrix pencil

$$P(\lambda) = \text{diag}(L_0, L_1, L_1, L_0^T, L_0^T, L_0^T) = \begin{bmatrix} 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

with $\varepsilon_1 = 0, \varepsilon_2 = \varepsilon_3 = 1$. An ROMB of P is given by

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ -\lambda \\ 0 \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -\lambda \end{bmatrix}.$$

Consider an arbitrary perturbation Q of P with $\rho = 2$ and $\tilde{\varepsilon} = 1$. This means that a right decomposition of Q (see (7)) is of the form

$$Q(\lambda) = v_1 w_1^T + v_2 w_2^T,$$

where

$$w_1 = \begin{bmatrix} b_1 + \lambda c_1 \\ b_2 + \lambda c_2 \\ b_3 + \lambda c_3 \\ b_4 + \lambda c_4 \\ b_5 + \lambda c_5 \end{bmatrix}, \quad w_2 = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix},$$

and $b_i, c_i, d_i \in \mathbb{C}$ for $i = 1, \dots, 5$. In addition, $\deg(v_1) = 0$ and $\deg(v_2) \leq 1$. Notice that in this example $p - \rho = 3 - 2 = 1$, and so the sequence $\{d_{\rho+1}, \dots, d_p\}$ has only the element d_3 . This means that in the conditions of Theorem 5.8 *the pencil $P + Q$ has only one column minimal index, which is precisely*

$$d_{\min} = d_3 = \left\lfloor \frac{0 + 1 + 1 + 1}{3 - 2} \right\rfloor = 3.$$

The matrices $A_s(d_{\min} - 1)$ and $A_s(d_{\min})$ are in this case $A_3(2)$ and $A_3(3)$, respectively. The right connection polynomials associated with the previous data are given by

$$\begin{aligned} a_{11}(\lambda) &= w_1(\lambda)^T x_1 = b_1 + \lambda c_1, & a_{21}(\lambda) &= w_2^T x_1 = d_1, \\ a_{12}(\lambda) &= w_1(\lambda)^T x_2 = b_2 + \lambda(c_2 - b_3) - \lambda^2 c_3, & a_{22}(\lambda) &= w_2^T x_2 = d_2 - \lambda d_3, \\ a_{13}(\lambda) &= w_1(\lambda)^T x_3 = b_4 + \lambda(c_4 - b_5) - \lambda^2 c_5, & a_{23}(\lambda) &= w_2^T x_3 = d_4 - \lambda d_5, \end{aligned}$$

and then the mosaic Toeplitz matrix $A_3(3)$ is the 9×10 matrix

$$A_3(3) = \left[\begin{array}{cccc|ccc|ccc} b_1 & 0 & 0 & 0 & b_2 & 0 & 0 & b_4 & 0 & 0 \\ c_1 & b_1 & 0 & 0 & c_2 - b_3 & b_2 & 0 & c_4 - b_5 & b_4 & 0 \\ 0 & c_1 & b_1 & 0 & -c_3 & c_2 - b_3 & b_2 & -c_5 & c_4 - b_5 & b_4 \\ 0 & 0 & c_1 & b_1 & 0 & -c_3 & c_2 - b_3 & 0 & -c_5 & c_4 - b_5 \\ 0 & 0 & 0 & c_1 & 0 & 0 & -c_3 & 0 & 0 & -c_5 \\ \hline d_1 & 0 & 0 & 0 & d_2 & 0 & 0 & d_4 & 0 & 0 \\ 0 & d_1 & 0 & 0 & -d_3 & d_2 & 0 & -d_5 & d_4 & 0 \\ 0 & 0 & d_1 & 0 & 0 & -d_3 & d_2 & 0 & -d_5 & d_4 \\ 0 & 0 & 0 & d_1 & 0 & 0 & -d_3 & 0 & 0 & -d_5 \end{array} \right].$$

It can be numerically checked using MATLAB that this matrix has full row rank for random values of b_i, c_i , and d_i . The 7×7 matrix $A_3(2)$ is constructed in a similar way, and it can be numerically checked that $A_3(2)$ has full column rank.

5.5. On the genericity of the assumptions of Theorem 5.8. The relevance of Theorem 5.8 depends on the *genericity* of its hypotheses, i.e., whether they are satisfied in a dense open subset of the considered set of perturbations. The meaning and genericity of the condition $\text{rank}(P+Q) = \text{rank}(P) + \text{rank}(Q) < n$ was discussed in depth in section 3. The other two essential hypotheses in Theorem 5.8 are (37)–(38). We have checked numerically with MATLAB on a sample of more than 50000 mosaic Toeplitz matrices (Definition 5.4), constructed on random polynomials, that these matrices have full rank. We have run experiments with matrices with more rows than columns, and vice versa. Then to see that (37)–(38) are indeed generic assumptions that hold for almost all perturbations, it remains only to justify that the connection polynomials of P and Q are *random* for random perturbations Q . In this process the natural assumption

$$(47) \quad \text{rank}(Q) \leq \text{rank}(P),$$

noted in section 3, plays a relevant role. Let us remember Definition 5.3. We can assume in the following argument, without loss of generality, that P is given in KCF. Taking into account that the right null space vector of a column singular block L_ε can be chosen to be $[1, -\lambda, \lambda^2, \dots, (-\lambda)^\varepsilon]^T$, the vectors $\{x_1(\lambda), \dots, x_p(\lambda)\}$ of the ROMB of P can be chosen with the following property: if $(x_j(\lambda))_k \neq 0$ for some j , then $(x_{j'}(\lambda))_k = 0$ for $j' \neq j$; i.e., the nonzero entries of every vector correspond to zero entries of the remaining vectors. With this in mind, it seems at a first glance that the coefficients of the connection polynomials (24) are random, because Q being a random perturbation, the vectors $\{w_1(\lambda), \dots, w_\rho(\lambda)\}$ should be also random. But, according to (7), the vectors $w_i(\lambda)$ are of degree at most one. This means that, putting together the zero and first order coefficients of each $w_i(\lambda)$, we get a set \mathcal{W} with $\rho + \tilde{\varepsilon}$ vectors of \mathbb{C}^n . Notice that $\text{rank}(P+Q) = \text{rank}(P) + \text{rank}(Q) \leq \min\{m, n\}$ and (47) imply that $\rho + \tilde{\varepsilon} \leq n$, and then the vectors in \mathcal{W} are linearly independent for almost all Q , and the coefficients of the connection polynomials are really random. But, if $\rho + \tilde{\varepsilon} > n$, then the set \mathcal{W} is linearly dependent, and some linear dependence may appear among the coefficients of the connection polynomials.³

5.6. When the only information available on the perturbation is its rank. Theorem 5.8 determines the generic whole set of column minimal indices of the perturbed pencil $(P+Q)(\lambda)$. This set depends on $\tilde{\varepsilon}$, i.e., the sum of the column minimal indices of the perturbation $Q(\lambda)$. The reason for this dependence can be traced back to the expansion (7), because the properties of (7) depend on $\tilde{\varepsilon}$. This fact is related to a deeper mathematical result: the set of singular matrix pencils of rank ρ has exactly $\rho + 1$ maximal irreducible components, each of them corresponding to a value of $\tilde{\varepsilon}$, for $\tilde{\varepsilon} = 0, \dots, \rho$ [3]. However, one may want to get some partial information if only the rank of the perturbation $Q(\lambda)$ is known. This partial information is presented in Theorem 5.10. To prove this theorem the following lemma is needed.

LEMMA 5.9. *Let $0 \leq \varepsilon_1 \leq \dots \leq \varepsilon_p$ be p integer numbers, and ρ and $\tilde{\varepsilon}$ be integer numbers such that $0 < \rho < p$ and $0 \leq \tilde{\varepsilon} \leq \rho$. Let us consider for each value of $\tilde{\varepsilon}$ the*

³Notice that the argument of this paragraph holds if the assumption (47) is replaced by $\rho + \tilde{\varepsilon} \leq n$, which is fulfilled by a wider set of perturbations. However, this condition is not natural and requires knowing $\tilde{\varepsilon}$ apart from the rank of the perturbation.

sequence of integer numbers

$$(48) \quad d_k(\tilde{\varepsilon}) = \left\lfloor \frac{\sum_{i=1}^k \varepsilon_i + \tilde{\varepsilon}}{k - \rho} \right\rfloor \quad \text{for } k = \rho + 1, \dots, p.$$

Let $d_{\min}(\tilde{\varepsilon}) \equiv \min_{\rho+1 \leq k \leq p} d_k(\tilde{\varepsilon})$, and let $s(\tilde{\varepsilon})$ be the largest index such that $d_{s(\tilde{\varepsilon})}(\tilde{\varepsilon}) = d_{\min}(\tilde{\varepsilon})$ and $d_{s(\tilde{\varepsilon})}(\tilde{\varepsilon}) \geq \varepsilon_{s(\tilde{\varepsilon})}$. Then

$$d_{\min}(\rho) \geq d_{\min}(\rho - 1) \geq \dots \geq d_{\min}(0) \quad \text{and} \quad s(\rho) \geq s(\rho - 1) \geq \dots \geq s(0).$$

Proof. Let us prove that $d_{\min}(\tilde{\varepsilon}') \geq d_{\min}(\tilde{\varepsilon})$ and $s(\tilde{\varepsilon}') \geq s(\tilde{\varepsilon})$, whenever $\tilde{\varepsilon}' > \tilde{\varepsilon}$. Notice that $d_k(\tilde{\varepsilon}') \geq d_k(\tilde{\varepsilon})$ for $k = \rho + 1, \dots, p$. Therefore $d_{\min}(\tilde{\varepsilon}') \geq d_{\min}(\tilde{\varepsilon})$. According to Lemma 5.7.6, $\varepsilon_{s(\tilde{\varepsilon}')} + 1 > d_{s(\tilde{\varepsilon}')}'(\tilde{\varepsilon}') \geq d_{s(\tilde{\varepsilon})}(\tilde{\varepsilon}) \geq \varepsilon_{s(\tilde{\varepsilon})}$. This means that $s(\tilde{\varepsilon}') + 1 > s(\tilde{\varepsilon})$. \square

By combining Theorem 5.8 and Lemma 5.9, we can state the generic theorem, Theorem 5.10. We name Theorem 5.10 as generic, because the precise assumptions needed in the theorem are (37)–(38), and they depend on the sum of the column minimal indices of the perturbation Q , information that is not available. The only requirement for proving Theorem 5.10 is to notice that if ρ is the rank of Q and $\tilde{\varepsilon}$ is the sum of the column minimal indices of Q , then $0 \leq \tilde{\varepsilon} \leq \rho$.

THEOREM 5.10. *Let $P(\lambda)$ and $Q(\lambda)$ be two $m \times n$ complex matrix pencils such that $\text{rank}(P + Q) = \text{rank}(P) + \text{rank}(Q) < n$ and $\text{rank}(Q) \leq \text{rank}(P)$. Let us define $\rho \equiv \text{rank}(Q)$. Let $\varepsilon_1 \leq \dots \leq \varepsilon_p$ be the column minimal indices of P , and*

$$(49) \quad d'_k = \left\lfloor \frac{\sum_{i=1}^k \varepsilon_i + \rho}{k - \rho} \right\rfloor \quad \text{for } k = \rho + 1, \dots, p.$$

Let d'_{\min} be the minimum of the sequence $\{d'_k\}$, and s' be the largest index such that $d'_{s'} = d'_{\min}$ and $d'_{s'} \geq \varepsilon_{s'}$. Then, for generic pencils $Q(\lambda)$ with rank ρ , $(P + Q)(\lambda)$ has exactly $p - \rho$ column minimal indices and

1. the $p - s'$ largest column minimal indices of $(P + Q)(\lambda)$ are $\varepsilon_{s'+1} \leq \dots \leq \varepsilon_p$;
2. the $s' - \rho$ smallest column minimal indices of $(P + Q)(\lambda)$, $\hat{\varepsilon}_1 \leq \dots \leq \hat{\varepsilon}_{s'-\rho}$, satisfy $\varepsilon_{\rho+j} \leq \hat{\varepsilon}_j$ for $j = 1, \dots, s' - \rho$ and $\hat{\varepsilon}_1 \leq d'_{\min}$.

6. The Kronecker canonical form of perturbed pencils without full rank. The results presented so far remain valid whenever $\text{rank}(P + Q) = \text{rank}(P) + \text{rank}(Q) \leq \min\{m, n\}$. This assumption includes the limit case $\text{rank}(P + Q) = \text{rank}(P) + \text{rank}(Q) = \min\{m, n\}$, i.e., the case of perturbed pencils $(P + Q)(\lambda)$ with full rank. In this full rank case $P + Q$ does not have row minimal indices if $\text{rank}(P + Q) = m$, and $P + Q$ does not have column minimal indices if $\text{rank}(P + Q) = n$, according to (4). If $\text{rank}(P + Q) = m < n$, the generic column minimal indices of $P + Q$ are described by Theorem 5.8, and if $\text{rank}(P + Q) = n < m$, the generic row minimal indices of $P + Q$ are described by Theorem 5.8 applied on $(P + Q)^T$. Theorem 4.4 also holds in the full rank case and gives partial information on the regular part of $P + Q$.

The purpose of this section is to show that complete information on the generic KCF of $P + Q$ can be obtained for perturbed pencils without full rank, i.e.,

$$\text{rank}(P + Q) = \text{rank}(P) + \text{rank}(Q) < \min\{m, n\}.$$

We will gather the information obtained in Theorems 4.4 and 5.8, together with the counterpart version of Theorem 5.8 for row minimal indices, to fully describe the

generic KCF of $(P + Q)(\lambda)$, in terms of the sums of the row and column minimal indices and of the regular structure of the perturbation $Q(\lambda)$. This KCF will be presented in Theorem 6.2. In addition, Theorem 6.3 presents some generic partial information on the KCF of $P + Q$ when $\text{rank}(Q)$ is the only information available on the perturbation.

Lemma 6.1 will allow us to avoid certain redundancy in the hypotheses.

LEMMA 6.1. *Let $P(\lambda)$ and $Q(\lambda)$ be two $m \times n$ matrix pencils such that $\text{rank}(P) + \text{rank}(Q) < \min\{m, n\}$. Let $A_s(d_{\min})$ be the mosaic Toeplitz matrix associated with a complete set of right connection polynomials of P and Q appearing in Theorem 5.8, and $B_t(h_{\min})$ be the corresponding matrix associated with a complete set of left connection polynomials, i.e., the matrix in Theorem 5.8 if it is applied to P^T and Q^T . If $A_s(d_{\min})$ and $B_t(h_{\min})$ have full row rank, then $\text{rank}(P + Q) = \text{rank}(P) + \text{rank}(Q)$.*

Proof. From elementary linear algebra we know $\text{rank}(P + Q) \leq \text{rank}(P) + \text{rank}(Q)$. Let us consider right decompositions of P and Q of the kind appearing in (7):

$$\begin{aligned} P(\lambda) &= v'_1(\lambda)w'_1(\lambda)^T + \cdots + v'_r(\lambda)w'_r(\lambda)^T, \\ Q(\lambda) &= v_1(\lambda)w_1(\lambda)^T + \cdots + v_\rho(\lambda)w_\rho(\lambda)^T, \end{aligned}$$

where $r \equiv \text{rank}(P)$ and $\rho \equiv \text{rank}(Q)$. Therefore

$$P + Q = [v'_1, \dots, v'_r, v_1, \dots, v_\rho] [w'_1, \dots, w'_r, w_1, \dots, w_\rho]^T,$$

where the dependence on λ has been omitted. This means that $P + Q$ is the product of an $m \times (r + \rho)$ matrix times an $(r + \rho) \times n$ matrix, with $(r + \rho) < \min\{m, n\}$. Therefore $\text{rank}(P + Q) = \text{rank}(P) + \text{rank}(Q)$ if and only if

$$\text{rank}[v'_1, \dots, v'_r, v_1, \dots, v_\rho] = r + \rho \quad \text{and} \quad \text{rank}[w'_1, \dots, w'_r, w_1, \dots, w_\rho] = r + \rho.$$

Let us prove that if $A_s(d_{\min})$ has full row rank, then $\text{rank}[w'_1, \dots, w'_r, w_1, \dots, w_\rho] = r + \rho$. If $\text{rank}[w'_1, \dots, w'_r, w_1, \dots, w_\rho] < r + \rho$, there exists an index i such that $w_i(\lambda)$ is a linear combination of $\{w'_1(\lambda), \dots, w'_r(\lambda), w_1(\lambda), \dots, w_{i-1}(\lambda)\}$ in $\mathbb{C}^n(\lambda)$, i.e.,

$$w_i(\lambda) = \beta'_1(\lambda)w'_1(\lambda) + \cdots + \beta'_r(\lambda)w'_r(\lambda) + \beta_1(\lambda)w_1(\lambda) + \cdots + \beta_{i-1}(\lambda)w_{i-1}(\lambda),$$

for some rational functions $\beta'_1(\lambda), \dots, \beta_{i-1}(\lambda)$. Let us recall (24) and $P(\lambda)x_j(\lambda) = 0$, i.e., $w'_k(\lambda)^T x_j(\lambda) = 0$ for all k . Then the right connection polynomials of P and Q satisfy

$$a_{ij}(\lambda) = \beta_1(\lambda)a_{1j}(\lambda) + \cdots + \beta_{i-1}(\lambda)a_{i-1,j}(\lambda) \quad \text{for } j = 1, \dots, p,$$

and the matrix $[a_{kl}(\lambda)]_{\substack{1 \leq l \leq p \\ 1 \leq k \leq \rho}}$ does not have full row rank. But the fact that $A_s(d_{\min})$ has full row rank implies that $\text{rank}[a_{kl}(\lambda)]_{\substack{1 \leq l \leq p \\ 1 \leq k \leq \rho}} = \rho$, by Lemma 5.6.6.

An analogous argument shows that if $B_t(h_{\min})$ has full row rank, then $\text{rank}[v'_1, \dots, v'_r, v_1, \dots, v_\rho] = r + \rho$. \square

We do not explicitly impose $\text{rank}(P + Q) = \text{rank}(P) + \text{rank}(Q)$ in Theorem 6.2 because of Lemma 6.1.

THEOREM 6.2. *Let $P(\lambda)$ and $Q(\lambda)$ be two $m \times n$ complex matrix pencils such that $\text{rank}(P) + \text{rank}(Q) < \min\{m, n\}$ and $\rho \equiv \text{rank}(Q)$. Let $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ and $\eta_1 \leq \cdots \leq \eta_q$ be, respectively, the column and row minimal indices of P , and \mathcal{J}_P be the regular structure of the KCF of P . Let $\tilde{\varepsilon}$ and $\tilde{\eta}$ be, respectively, the sum of the*

column minimal indices and the sum of the row minimal indices of Q , and \mathcal{J}_Q be the regular structure of the KCF of Q . Let us consider the sequences

$$d_k = \left\lfloor \frac{\sum_{i=1}^k \varepsilon_i + \tilde{\varepsilon}}{k - \rho} \right\rfloor \quad \text{for } k = \rho + 1, \dots, p \quad \text{and}$$

$$h_l = \left\lfloor \frac{\sum_{i=1}^l \eta_i + \tilde{\eta}}{l - \rho} \right\rfloor \quad \text{for } l = \rho + 1, \dots, q.$$

Let $d_{\min} = \min_{\rho+1 \leq k \leq p} \{d_k\}$, and s be the largest index such that $d_s = d_{\min}$ and $d_s \geq \varepsilon_s$. Let $h_{\min} = \min_{\rho+1 \leq l \leq q} \{h_l\}$, and t be the largest index such that $h_t = h_{\min}$ and $h_t \geq \eta_t$. Finally, let $A_s(d_{\min} - 1)$ and $A_s(d_{\min})$ ($B_t(h_{\min} - 1)$ and $B_t(h_{\min})$) be the s th (t th) mosaic Toeplitz matrices of degrees $d_{\min} - 1$ and d_{\min} ($h_{\min} - 1$ and h_{\min}), respectively, associated with a complete set of right (left) connection polynomials of P and Q . If

$A_s(d_{\min} - 1)$ and $B_t(h_{\min} - 1)$ have full column rank or are empty matrices, and $A_s(d_{\min})$ and $B_t(h_{\min})$ have full row rank,

then

1. $(P + Q)(\lambda)$ has exactly $p - \rho$ column minimal indices that are

$$(50) \quad \underbrace{d_{\min} = \dots = d_{\min}}_{s - \rho - \gamma_s} < \underbrace{(d_{\min} + 1) = \dots = (d_{\min} + 1)}_{\gamma_s} \leq \varepsilon_{s+1} \leq \dots \leq \varepsilon_p,$$

where γ_s is the remainder in the integer division of $\sum_{i=1}^s \varepsilon_i + \tilde{\varepsilon}$ by $s - \rho$;

2. $(P + Q)(\lambda)$ has exactly $q - \rho$ row minimal indices that are

$$(51) \quad \underbrace{h_{\min} = \dots = h_{\min}}_{t - \rho - \mu_t} < \underbrace{(h_{\min} + 1) = \dots = (h_{\min} + 1)}_{\mu_t} \leq \eta_{t+1} \leq \dots \leq \eta_q,$$

where μ_t is the remainder in the integer division of $\sum_{i=1}^t \eta_i + \tilde{\eta}$ by $t - \rho$; and

3. $\mathcal{J}_P \oplus \mathcal{J}_Q$ is the regular structure of the KCF of $(P + Q)(\lambda)$.

This fully determines the KCF of $(P + Q)(\lambda)$.

Remark 5. We noted in subsection 5.5 that the additional assumption $\text{rank}(Q) \leq \text{rank}(P)$ is sufficient for considering that the KCF of $(P + Q)(\lambda)$ found in Theorem 6.2 is generic.

Proof of Theorem 6.2. Theorem 5.8 applied to P and Q proves (50) and applied to P^T and Q^T proves (51). Theorem 4.4 proves that for every complex number λ_0 , including the infinite, $\mathcal{S}_{P+Q}(\lambda_0) \geq \mathcal{S}_{P \oplus Q}(\lambda_0)$. To prove that, in fact, $\mathcal{S}_{P+Q}(\lambda_0) = \mathcal{S}_{P \oplus Q}(\lambda_0)$, we will simply show that the direct sum of $\mathcal{J}_P \oplus \mathcal{J}_Q$ plus the column and row singular blocks corresponding to (50) and to (51) is an $m \times n$ pencil. Let us call this direct sum $Z(\lambda)$.

Let the matrix \mathcal{J}_P be $r_1 \times r_1$, and \mathcal{J}_Q be $r_2 \times r_2$. Notice that, in this situation, the following identities hold:

$$\varepsilon + \eta + q + r_1 = m, \quad \varepsilon + p + \eta + r_1 = n, \quad \tilde{\varepsilon} + \tilde{\eta} + r_2 = \rho,$$

where ε (η) is the sum of the column (row) minimal indices of P . Thus, the number of rows of $Z(\lambda)$ is

$$\begin{aligned} & [d_{\min}(s - \rho - \gamma_s) + (d_{\min} + 1)\gamma_s + \varepsilon_{s+1} + \dots + \varepsilon_p] \\ & + [(h_{\min} + 1)(t - \rho - \mu_t) + (h_{\min} + 2)\mu_t + (\eta_{t+1} + 1) + \dots + (\eta_q + 1)] + r_1 + r_2 \\ & = [\varepsilon + \tilde{\varepsilon}] + [\eta + q + \tilde{\eta} - \rho] + r_1 + r_2 = m + \tilde{\varepsilon} + \tilde{\eta} + r_2 - \rho = m. \end{aligned}$$

An analogous computation shows that the number of columns of $Z(\lambda)$ is n , and, therefore, that $Z(\lambda)$ is the KCF of $(P + Q)(\lambda)$. \square

Example 2. Let us apply Theorem 6.2 to the pencil $P(\lambda)$ of the example in subsection 5.4. In that subsection, we considered a perturbation $Q(\lambda)$ with $\rho = 2$ and $\tilde{\varepsilon} = 1$. Now, let us assume also that $\tilde{\eta} = 0$ and that Q has a simple eigenvalue $\mu = 1$. The generic column minimal index of $P + Q$ predicted by Theorem 6.2 was computed in subsection 5.4 and is 3. Let us compute the generic row minimal indices of $P + Q$. In this example $\eta_1 = \eta_2 = \eta_3 = 0$. Thus, the number of row minimal indices of $P + Q$ is $q - \rho = 3 - 2 = 1$. Therefore, $h_{\min} = h_3 = 0$ is the generic row minimal index of $P + Q$. The generic KCF of $P + Q$ is

$$\begin{bmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda - 1 \end{bmatrix}.$$

In the case that the only information available on the perturbation $Q(\lambda)$ is its rank, Theorem 4.4 can be combined with Theorem 5.10, and the corresponding counterpart version for row minimal indices, to produce Theorem 6.3, that gives partial information of the KCF of $(P + Q)(\lambda)$.

THEOREM 6.3. *Let $P(\lambda)$ and $Q(\lambda)$ be two $m \times n$ complex matrix pencils such that $\text{rank}(P) + \text{rank}(Q) < \min\{m, n\}$ and $\text{rank}(Q) \leq \text{rank}(P)$. Let us define $\rho \equiv \text{rank}(Q)$. Let $\varepsilon_1 \leq \dots \leq \varepsilon_p$ and $\eta_1 \leq \dots \leq \eta_q$ be, respectively, the column and row minimal indices of P , and \mathcal{J}_P be the regular structure of the KCF of P . Let us consider the sequences*

$$d'_k = \left\lfloor \frac{\sum_{i=1}^k \varepsilon_i + \rho}{k - \rho} \right\rfloor \quad \text{for } k = \rho + 1, \dots, p \quad \text{and}$$

$$h'_l = \left\lfloor \frac{\sum_{i=1}^l \eta_i + \rho}{l - \rho} \right\rfloor \quad \text{for } l = \rho + 1, \dots, q.$$

Let d'_{\min} (h'_{\min}) be the minimum of the sequence $\{d'_k\}$ ($\{h'_l\}$), and s' (t') be the largest index such that $d'_{s'} = d'_{\min}$ ($h'_{t'} = h'_{\min}$) and $d'_{s'} \geq \varepsilon_{s'}$ ($h'_{t'} \geq \eta_{t'}$). Then, for generic pencils $Q(\lambda)$ with rank ρ , $(P + Q)(\lambda)$ has exactly $p - \rho$ column minimal indices and $q - \rho$ row minimal indices and the following hold:

1. The $p - s'$ largest column minimal indices of $(P + Q)(\lambda)$ are $\varepsilon_{s'+1} \leq \dots \leq \varepsilon_p$.
2. The $s' - \rho$ smallest column minimal indices of $(P + Q)(\lambda)$, $\hat{\varepsilon}_1 \leq \dots \leq \hat{\varepsilon}_{s'-\rho}$, satisfy $\varepsilon_{\rho+j} \leq \hat{\varepsilon}_j$ for $j = 1, \dots, s' - \rho$ and $\hat{\varepsilon}_1 \leq d'_{\min}$.
3. The $q - t'$ largest row minimal indices of $(P + Q)(\lambda)$ are $\eta_{t'+1} \leq \dots \leq \eta_q$.
4. The $t' - \rho$ smallest row minimal indices of $(P + Q)(\lambda)$, $\hat{\eta}_1 \leq \dots \leq \hat{\eta}_{t'-\rho}$, satisfy $\eta_{\rho+j} \leq \hat{\eta}_j$ for $j = 1, \dots, t' - \rho$ and $\hat{\eta}_1 \leq h'_{\min}$.
5. The regular part of the KCF of $(P + Q)(\lambda)$ contains \mathcal{J}_P .

7. Conclusions and open problems. The results presented in this paper are, as far as we know, the first contribution in the area of *generic* low rank perturbations of singular matrix pencils, but they do not solve all the problems of this kind.

A first interesting problem is to consider unperturbed pencils $P(\lambda)$ with full rank, i.e., $\text{rank}(P) = \min\{m, n\}$. The full rank square case, $m = n$, corresponds to unperturbed regular pencils. In this case the KCF of $P(\lambda)$ does not have singular blocks,

and it is called the Weierstrass canonical form. This problem has been solved in [4]. The full rank rectangular case, $m \neq n$, is an open problem. In this case the KCF of $P(\lambda)$ has only one type of singular blocks: $n - m$ column or right singular blocks if $m < n$, and $m - n$ row or left singular blocks if $m > n$. Generically the same holds for the perturbed pencil $(P + Q)(\lambda)$, but the dimensions of the singular blocks may change. A first important task in this setting is to define the precise meaning of *low rank perturbation*.

A second open problem is to consider unperturbed pencils $P(\lambda)$ without full rank, but *perturbations whose rank does not satisfy (1)*. For instance, if $P(\lambda)$ is a 100×200 pencil with $\text{rank}(P) = 98$ and the rank of the perturbations is $\rho \equiv \text{rank}(Q) = 3$, then the perturbations $Q(\lambda)$ are, intuitively, low rank perturbations of $P(\lambda)$. The solution of this kind of problem is naturally connected with the results presented in this work and with the first open problem we have discussed in the previous paragraph. In our specific example, the right decomposition of Q in (7) allows us to write $Q(\lambda) = Q_1(\lambda) + Q_2(\lambda)$, where $\text{rank}(Q_1) = 2$ and $\text{rank}(Q_2) = 1$. Thus, we can split the original perturbation problem, $P + Q = P + Q_1 + Q_2$, into two perturbation problems of smaller rank, $P + Q_1$ and $(P + Q_1) + Q_2$. The first one is of the type considered in this work, and in the second one the unperturbed pencil $P + Q_1$ has generically full rank.

A final open problem has to do with the fact that in some situations the information given by the results presented in this paper for the *limit case*, i.e., $\text{rank}(P + Q) = \text{rank}(P) + \text{rank}(Q) = \min\{m, n\}$, of *square pencils* is irrelevant. Notice that in the rectangular case—let us assume $m < n$ without loss of generality—our results say that $P + Q$ does not have row minimal indices, and Theorems 5.8 and 5.10 determine the generic column minimal indices. Additionally, Theorem 4.4 gives information on the regular part of $P + Q$. However, in the square case, although our results are still true, they may produce irrelevant information. Let us illustrate this with two examples. The first example is

$$(52) \quad \underbrace{\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & 0 \end{bmatrix}}_P + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \lambda \end{bmatrix}}_Q = \underbrace{\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 2 \\ 0 & 0 & \lambda \end{bmatrix}}_{P+Q}.$$

Notice that P has $\text{rank}(P) = 2$, minimal indices $\varepsilon_1 = 2$ and $\eta_1 = 0$, and no eigenvalues because its rank is 2 for all the values of λ . The same holds for the dual pencil. Thus, P has no regular part. The pencil Q has $\text{rank}(Q) = 1$, minimal indices $\varepsilon_1 = \varepsilon_2 = 0$ and $\eta_1 = 0, \eta_2 = 1$, and no eigenvalues. However, $P + Q$ has $\text{rank}(P + Q) = 3$; i.e., it is a regular pencil and does not have minimal indices, neither row nor column minimal indices. This is predicted by our theory; see Corollary 3.2. In addition, $P + Q$ has $\mu = 0$ as a triple eigenvalue with only one associated Jordan block. Notice that the information given by Theorem 4.4 is true— $\mathcal{S}_{P+Q}(0) = (3, 0, \dots)$, while $\mathcal{S}_{P \oplus Q}(0) = (0, 0, \dots)$ —but irrelevant, because *there is not any relationship between the (nonexistent) regular parts of P and Q and the regular part of $P + Q$* . This first example illustrates a type of perturbation that destroys all the singular information of the unperturbed pencil and creates a regular part in $P + Q$ that does not exist at all in P . Therefore the *regular part* of $P + Q$ is created from *singular parts* of P and Q . Notice that this example is not particular, because once P is fixed and the rank of the perturbations is fixed to be one, it is generic that $\text{rank}(P + Q) = 3$ (see Theorem 3.1), and $P + Q$ has no minimal indices but only a regular part. The second example is the following:

$$(53) \quad \underbrace{\begin{bmatrix} \lambda - 1 & 0 \\ 0 & 0 \end{bmatrix}}_P + \underbrace{\begin{bmatrix} \lambda & 1 + 2\lambda \\ 2\lambda & 2 + 4\lambda \end{bmatrix}}_Q = \underbrace{\begin{bmatrix} 2\lambda - 1 & 1 + 2\lambda \\ 2\lambda & 2 + 4\lambda \end{bmatrix}}_{P+Q}.$$

In this example, the pencil P has $\text{rank}(P) = 1$, minimal indices $\varepsilon_1 = 0$ and $\eta_1 = 0$, and one simple eigenvalue equal to 1. The pencil Q has $\text{rank}(Q) = 1$, minimal indices $\varepsilon_1 = 1$ and $\eta_1 = 0$, and no regular part. The pencil $P + Q$ is regular with determinant $\det(P + Q) = 2(1 + 2\lambda)(\lambda - 1)$; this means that $P + Q$ has two simple eigenvalues equal to $-1/2$ and 1. Notice that in this case, $\mu = 1$ is an eigenvalue of P and also of $P + Q$. This is guaranteed by Theorem 4.4, and it is not a coincidence. But the new eigenvalue appearing in $P + Q$, i.e., $-1/2$, is not related to the regular structure of P . In both examples, (52) and (53), it seems difficult to say something generic on the regular part of $P + Q$ beyond Theorem 4.4, except that the new eigenvalues appearing in $P + Q$ will be *generically* different from those of P . However, to find precise conditions for this behavior to hold needs delicate algebraic work and still remains as an open problem.

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