Low rank perturbation of regular matrix polynomials*

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Abstract

Let $A(\lambda)$ be a complex regular matrix polynomial of degree ℓ with g elementary divisors corresponding to the finite eigenvalue λ_0. We show that for most complex matrix polynomials $B(\lambda)$ with degree at most ℓ satisfying $\operatorname{rank} B(\lambda_0) < g$ the perturbed polynomial $(A+B)(\lambda)$ has exactly $g - \operatorname{rank} B(\lambda_0)$ elementary divisors corresponding to λ_0, and we determine their degrees. If $\operatorname{rank} B(\lambda_0) + \operatorname{rank}(B(\lambda) - B(\lambda_0))$ does not exceed the number of λ_0-elementary divisors of $A(\lambda)$ with degree greater than 1, then the λ_0-elementary divisors of $(A+B)(\lambda)$ are the $g - \operatorname{rank} B(\lambda_0) - \operatorname{rank}(B(\lambda) - B(\lambda_0))$ elementary divisors of $A(\lambda)$ corresponding to λ_0 with smallest degree, together with $\operatorname{rank}(B(\lambda) - B(\lambda_0))$ linear λ_0-elementary divisors. Otherwise, the degree of all the λ_0-elementary divisors of $(A+B)(\lambda)$ is one. This behavior happens for any matrix polynomial $B(\lambda)$ except those in a proper algebraic submanifold in the set of matrix polynomials of degree at most ℓ. If $A(\lambda)$ has an infinite eigenvalue, the corresponding result follows from considering the zero eigenvalue of the perturbed dual polynomial.

Key words. Regular matrix polynomials, elementary divisors, low rank perturbations, matrix spectral perturbation theory

AMS subject classification. 15A18, 15A21

1 Introduction

It is well known that a matrix polynomial of degree ℓ, $A(\lambda) = A_0 + \lambda A_1 + \ldots + \lambda^\ell A_\ell$ with $A_0, \ldots, A_\ell \in \mathbb{C}^{n\times n}$ and $A_\ell \neq 0$, can be transformed by equivalence into diagonal form

$$P(\lambda)A(\lambda)Q(\lambda) = \operatorname{diag}(h_1(\lambda), \ldots, h_r(\lambda), 0, \ldots, 0)\,, \qquad (1)$$

where $P(\lambda)$ and $Q(\lambda)$ are unimodular matrix polynomials, i.e., matrix polynomials with nonzero constant determinants, and $h_1(\lambda), \ldots, h_r(\lambda)$ are polynomials with complex coefficients satisfying the divisibility chain $h_r(\lambda)|h_{r-1}(\lambda)|\ldots|h_1(\lambda)$. As usual, $h_r(\lambda)|h_{r-1}(\lambda)$ means that $h_r(\lambda)$ divides $h_{r-1}(\lambda)$. The diagonal form (1) is known as the *Smith normal form* of $A(\lambda)$ [4, Chapter VI]. The polynomials $h_1(\lambda), \ldots, h_r(\lambda)$ are called the *invariant factors* of $A(\lambda)$. If, for $\lambda_0 \in \mathbb{C}$, we factorize each invariant factor $h_k(\lambda) = (\lambda - \lambda_0)^{d_k}\widetilde{h}_k(\lambda)$, where $\widetilde{h}_k(\lambda)$ is a polynomial such that $\widetilde{h}_k(\lambda_0) \neq 0$, $k = 1, \ldots, r$, the polynomials $(\lambda-\lambda_0)^{d_1}, \ldots, (\lambda-\lambda_0)^{d_r}$ that are different from one are the *elementary divisors* of $A(\lambda)$ associated with λ_0. In this work, the matrix polynomial $A(\lambda)$ will be *regular*, i.e., $\det A(\lambda)$ is nonzero as a polynomial in λ. In this case $r = n$ and a *finite eigenvalue* of $A(\lambda)$ is a complex number λ_0 such that $\det A(\lambda_0) = 0$. If λ_0 is a finite eigenvalue of $A(\lambda)$,

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there is at least one elementary divisor of $A(\lambda)$ associated with λ_0. We will assume throughout this paper that $A(\lambda)$ has exactly g λ_0-elementary divisors with degrees $0 < d_g \leq d_{g-1} \leq \ldots \leq d_1$. These degrees are known as the *partial multiplicities* of $A(\lambda)$ at λ_0 [5]. Note that g is the *geometric multiplicity* of λ_0, i.e., $g = \dim\ker A(\lambda_0)$, where ker denotes the null space, and that $d_1 + \ldots + d_g$ is the *algebraic multiplicity* of λ_0 in $\det A(\lambda)$. [1]

If the regular matrix polynomial $A(\lambda)$ is perturbed by another polynomial $B(\lambda)$ to obtain $(A+B)(\lambda)$, then, for most perturbations $B(\lambda)$, $(A+B)(\lambda)$ is regular, and all its eigenvalues are different from those of $A(\lambda)$. However, if $\operatorname{rank} B(\lambda_0)$ is small enough then λ_0 is still an eigenvalue of $(A+B)(\lambda)$, because the well-known inequality

$$\operatorname{rank}(A(\lambda_0) + B(\lambda_0)) \leq \operatorname{rank} A(\lambda_0) + \operatorname{rank} B(\lambda_0),$$

gives rise to

$$g - \operatorname{rank} B(\lambda_0) \leq \dim\ \ker(A(\lambda_0) + B(\lambda_0)). \tag{2}$$

Therefore, whenever

$$\operatorname{rank} B(\lambda_0) < g, \tag{3}$$

the eigenvalue λ_0 of $A(\lambda)$ stays as an eigenvalue of the perturbed polynomial

$$(A+B)(\lambda). \tag{4}$$

As a consequence, by "low" rank perturbation we will mean in what follows that $B(\lambda)$ satisfies (3), a condition which depends on the particular eigenvalue λ_0 we are considering. Assuming that (4) is still regular, equation (2) implies that the perturbation $B(\lambda)$ can destroy at most $\operatorname{rank} B(\lambda_0)$ elementary divisors of $A(\lambda)$ associated with λ_0. This does not fix the number and degrees of the elementary divisors of $(A+B)(\lambda)$ associated with λ_0, and to describe these elementary divisors in terms of the λ_0-elementary divisors of $A(\lambda)$ for generic low rank perturbations $B(\lambda)$ is the goal of this work.

The result we present depends on two quantities *for each eigenvalue* λ_0, namely

$$\rho_0 = \operatorname{rank} B(\lambda_0) \quad \text{and} \quad \rho_1 = \operatorname{rank}(B(\lambda) - B(\lambda_0)).$$

Note that the first quantity is the usual rank of a constant matrix, whereas the second one is the rank of a matrix polynomial, i.e., the dimension of its largest non-identically zero minor considered as a polynomial in λ [4, Chapter VI]. Assuming that condition (3) holds, we will prove that *for generic matrix polynomials* $B(\lambda)$ there are precisely $g - \rho_0$ elementary divisors of $(A+B)(\lambda)$ associated with λ_0. Moreover, if $\rho_0 + \rho_1$ is less than or equal to the number of nonlinear λ_0-elementary divisors of $A(\lambda)$, then the λ_0-elementary divisors of $(A+B)(\lambda)$ are the $g - \rho_0 - \rho_1$ lowest degree λ_0-elementary divisors of $A(\lambda)$, together with ρ_1 linear λ_0-elementary divisors. Otherwise, the degree of all the λ_0-elementary divisors of $(A+B)(\lambda)$ is one.

We often use the word *generic* in this work, so it is convenient to establish its precise meaning. The set of complex $n \times n$ matrix polynomials of degree at most ℓ is isomorphic to $\mathbb{C}^{(\ell+1)n^2}$. Thus, given two nonnegative integers $\rho_0\,(< g)$ and $\rho_1\,(\leq n)$, the set of matrix polynomials $B(\lambda) = \sum_{j=0}^{\ell} B_j \lambda^j$ satisfying $\operatorname{rank} B(\lambda_0) \leq \rho_0$ and $\operatorname{rank}(B(\lambda) - B(\lambda_0)) \leq \rho_1$ is an algebraic manifold $\mathcal{C} \subset \mathbb{C}^{(\ell+1)n^2}$, i.e., it is the set of common zeros of some multivariate polynomials in the entries of $B_0, \ldots, B_\ell$. The algebraic manifold $\mathcal{C}$ is the set of allowable perturbations we will consider. We will prove that the behavior described in the previous paragraph happens for any perturbation in $\mathcal{C}$ except those in a proper algebraic submanifold $\mathcal{M}$ of $\mathcal{C}$. This fact allows us to call this behavior *generic*, and to term the perturbations in $\mathcal{C}$ for which it occurs as *generic*. The algebraic submanifold $\mathcal{M}$ includes, among others, all polynomials such that $\operatorname{rank} B(\lambda_0) < \rho_0$.

[1] A matrix polynomial $A(\lambda)$ with degree ℓ may also have an *infinite eigenvalue*. This is the case when the *dual polynomial* $A^\sharp(\lambda) \equiv \lambda^\ell A(1/\lambda)$ has a zero eigenvalue. The partial multiplicities of the infinite eigenvalue of $A(\lambda)$ are precisely the partial multiplicities of the zero eigenvalue in $A^\sharp(\lambda)$. In this paper we will deal with finite eigenvalues, but results for the infinite eigenvalue can be easily obtained by considering the zero eigenvalue of the dual polynomials.

Note that in our notion of genericity, we are considering that the degree of the perturbation polynomial $B(\lambda)$ is less than or equal to the degree of the unperturbed polynomial $A(\lambda)$, i.e., ℓ. This is the relevant case in applications, because if, for instance, we are dealing with a vibrational problem related to a quadratic matrix polynomial $A(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2$, then perturbations in the parameters of the problem cannot lead to polynomials with higher degree. However, from a mathematical point of view, one can think in perturbations with degree less than or equal to a fixed number $s > \ell$. The genericity results we present remain valid in this case simply by considering $A(\lambda)$ as a formal polynomial of degree s by defining the coefficients $A_{\ell+1} = \cdots = A_s = 0$.

The generic behavior under low rank perturbations of canonical forms, and so of elementary divisors, of matrices and matrix pencils has received considerable attention in the last years [2, 3, 6, 9, 10, 11, 12], but the problem for polynomials remained open. The results presented in this work include, as particular cases, previous results for matrices and regular pencils. In fact, the first two results we present for matrix polynomials, Lemma 1 and Theorem 2, correspond to results proved in [3] only for matrix pencils by using essentially the same procedure.

On the other hand, this paper is connected to classical results on the change of the invariant factors of matrix polynomials under perturbations of low rank, and the related modifications of row and/or columns prolongations [8, 13, 14]. This interesting line of research has been continued is several works, see for instance [1, 7, 15]. In particular, we will take the main result in [14] as our starting point. However, this type of results shows important differences with respect to the ones we present: in [8, 13, 14] all the possible changes are described, but nothing is said about the generic change; in addition, the low rank condition is on the whole polynomial perturbation $B(\lambda)$, and not on the polynomial evaluated on an specific eigenvalue λ_0 of the unperturbed polynomial, as it happens in (3).

The paper is organized as follows: in Section 2 we briefly outline the main result in [14], and prove, as a direct consequence, Lemma 1 that is used in the next section. Section 3 includes the main results, summarized in Theorem 3.

2 Thompson's Result and consequences

As a consequence of results in [13], the following result is presented in [14].

Theorem 1 [14, Theorem 1] *Let $L(\lambda)$ be an $n \times n$ matrix polynomial with invariant factors $h_n(L)|h_{n-1}(L)|\dots|h_1(L)$, $Z(\lambda)$ be another matrix polynomial with* $\operatorname{rank} Z(\lambda) \leq 1$, *and $M(\lambda) = L(\lambda) + Z(\lambda)$. Then the achievable invariant factors $h_n(M)|h_{n-1}(M)|\dots|h_1(M)$ of $M(\lambda)$ as $Z(\lambda)$ ranges over all matrix polynomials with* $\operatorname{rank} Z(\lambda) \leq 1$ *are precisely those polynomials that satisfy*

$$h_n(L)|h_{n-1}(M)|h_{n-2}(L)|h_{n-3}(M)|\dots$$
$$h_n(M)|h_{n-1}(L)|h_{n-2}(M)|h_{n-3}(L)|\dots$$

Thompson proved this result in the more general setting of matrices with entries in an arbitrary principal ideal domain. As a corollary of Theorem 1 we obtain Lemma 1.

Lemma 1 *Let $A(\lambda)$ be a complex regular matrix polynomial and $B(\lambda)$ be another complex polynomial of the same dimension with rank at most r. Let λ_0 be an eigenvalue of $A(\lambda)$ with g associated elementary divisors of degrees $d_1 \geq \dots \geq d_g > 0$. If $(A+B)(\lambda)$ is also a regular matrix polynomial and $r \leq g$ then the polynomial $(A+B)(\lambda)$ has at least $g-r$ elementary divisors associated with λ_0 of degrees $\beta_{r+1} \geq \dots \geq \beta_g$, such that $\beta_i \geq d_i$ for $r+1 \leq i \leq g$.*

Proof. First, let us assume that the rank of $B(\lambda)$ is exactly r. Then, by using (1), we can write down $B(\lambda)$ as the sum of r singular matrix polynomials of rank one:

$$B(\lambda) = B_1(\lambda) + \dots + B_r(\lambda),$$

where $\operatorname{rank} B_i(\lambda) = 1$ for $1 \leq i \leq r$. Now, consider the sequence of polynomials $A(\lambda), A(\lambda) + B_1(\lambda), A(\lambda)+B_1(\lambda)+B_2(\lambda), \dots, A(\lambda)+B(\lambda)$, and note that each of them is a rank one perturbation of the preceding one. Applying Theorem 1 on this sequence leads to

$$\begin{array}{l}(\lambda-\lambda_0)^{d_{r+1}}|h_{r+1}(A)|h_r(A+B_1)|\dots|h_1(A+B),\\(\lambda-\lambda_0)^{d_{r+2}}|h_{r+2}(A)|h_{r+1}(A+B_1)|\dots|h_2(A+B),\\ \qquad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \quad \vdots\\(\lambda-\lambda_0)^{d_g}|h_g(A)|h_{g-1}(A+B_1)|\dots|h_{g-r}(A+B),\end{array}$$

where $h_1(A+B)\neq 0$ because the polynomial $(A+B)(\lambda)$ is regular. These divisibility chains mean that the polynomial $(A+B)(\lambda)$ has, *at least*, $g-r$ elementary divisors associated with λ_0 of degrees $\beta_{r+1}\geq\ldots\geq\beta_g$ such that $d_i\leq\beta_i$ for $r+1\leq i\leq g$.

If the rank of $B(\lambda)$ is $r_1<r$, the result we have just proved can be applied to show that the perturbed polynomial $(A+B)(\lambda)$ has at least $g-r_1>g-r$ elementary divisors associated with λ_0 whose degrees satisfy $\beta_i\geq d_i$, $i=r_1+1,\ldots,g$, and the result follows. □

3 Generic change of elementary divisors under low rank perturbations

Throughout this section we denote by

$$a_{L(\lambda)}(\lambda_0) \tag{5}$$

the algebraic multiplicity of the eigenvalue λ_0 in the regular matrix polynomial $L(\lambda)$. Our aim is to determine the generic degrees of the elementary divisors of the matrix polynomial $(A+B)(\lambda)$ associated with λ_0 in terms of the degrees of the elementary divisors of the unperturbed polynomial $A(\lambda)$. In the first result, Theorem 2, we obtain a lower bound on the algebraic multiplicity of λ_0 in $(A+B)(\lambda)$, and, more important, we show that this lower bound is attained if and only if the degrees of the λ_0-elementary divisors of $(A+B)(\lambda)$ are the ones corresponding to the behavior described in the Introduction.

Theorem 2 *Let λ_0 be a finite eigenvalue of the complex regular matrix polynomial $A(\lambda)$, and $d_1\geq\ldots\geq d_g>0$ be the degrees of its elementary divisors associated with λ_0. Let $B(\lambda)$ be any complex polynomial such that $(A+B)(\lambda)$ is regular and $g\geq\operatorname{rank}B(\lambda_0)$. Set $\rho_0=\operatorname{rank}B(\lambda_0)$, $\rho_1=\operatorname{rank}(B(\lambda)-B(\lambda_0))$, $\rho=\rho_0+\rho_1$ and $d_m=1$ for any $m=g+1,\ldots,\rho$. Then*

$$a_{(A+B)(\lambda)}(\lambda_0)\geq a_{A(\lambda)}(\lambda_0)+\rho_1-d_1-\ldots-d_\rho, \tag{6}$$

where the notation in (5) is used. Moreover, equality in this inequality holds if and only if the degrees of the elementary divisors of $(A+B)(\lambda)$ associated with λ_0 are obtained by removing the first ρ members in the list $d_1,\ldots,d_g,\underbrace{1,\ldots,1}_{\rho_1}$.

Proof. Notice that

$$\operatorname{rank}B(\lambda)=\operatorname{rank}(B(\lambda_0)+B(\lambda)-B(\lambda_0))\leq\operatorname{rank}B(\lambda_0)+\operatorname{rank}(B(\lambda)-B(\lambda_0))=\rho.$$

So, in the case $\rho<g$, Lemma 1 guarantees the existence of $g-\rho$ elementary divisors of $(A+B)(\lambda)$ associated with λ_0 with degrees $\beta_{\rho+1}\geq\ldots\geq\beta_g$, such that $\beta_{\rho+1}\geq d_{\rho+1},\ldots,\beta_g\geq d_g$. Moreover, the left hand side in the inequality (2) implies that there are at least ρ_1 additional elementary divisors of degrees $\alpha_1\geq 1,\ldots,\alpha_{\rho_1}\geq 1$ associated with λ_0. Thus,

$$a_{(A+B)(\lambda)}(\lambda_0)\geq\beta_{\rho+1}+\ldots+\beta_g+\alpha_1+\ldots+\alpha_{\rho_1}\geq d_{\rho+1}+\ldots+d_g+\rho_1.$$

Obviously, this inequality is (6). If $g\leq\rho$, inequality (6) becomes $a_{(A+B)(\lambda)}(\lambda_0)\geq g-\operatorname{rank}B(\lambda_0)$. This is true because of inequality (2) and the inequality

$$a_{(A+B)(\lambda)}(\lambda_0)\geq\dim\ker(A(\lambda_0)+B(\lambda_0)),$$

that is satisfied because $(A+B)(\lambda)$ is regular. Finally, notice that the previous inequalities become equalities if and only if the degrees of the elementary divisors of $(A+B)(\lambda)$ associated with λ_0 are those appearing in the statement of Theorem 2. $\square$

Remark 1 *Note that in Theorem 2 the results are independent of ρ_1 whenever ρ is greater than or equal to the number e_0 of nonlinear elementary divisors of $A(\lambda)$ associated with λ_0: the lower bound in (6), i.e., the right hand side, is simply $g - \operatorname{rank} B(\lambda_0)$, and the equality in (6) holds if and only if $(A+B)(\lambda)$ has $g - \operatorname{rank} B(\lambda_0)$ linear elementary divisors associated with λ_0. As a consequence, note that the lower bound $a_{A(\lambda)}(\lambda_0) + \rho_1 - d_1 - \ldots - d_\rho$ increases as ρ_0 decreases, increases as ρ_1 decreases when $\rho \leq e_0$, and remains constant as ρ_1 decreases when $\rho > e_0$.*

In the rest of this section, we will prove that equality in (6) and the corresponding degrees of the λ_0-elementary divisors are *generic* in the precise sense explained in this paragraph. Let us assume that ℓ is the degree of the $n \times n$ polynomial $A(\lambda)$ in Theorem 2, and that a couple of nonnegative integers ρ_0 and ρ_1, such that $\rho_0 < g$ and $\rho_1 \leq n$, are given. Let us define

$$\widetilde{a} = a_{A(\lambda)}(\lambda_0) + \rho_1 - d_1 - \ldots - d_\rho,$$

where $\rho = \rho_0 + \rho_1$, i.e., the right hand side in (6). Then, for every perturbation $B(\lambda)$ of $A(\lambda)$ in the set

$$\mathcal{C} = \{B(\lambda) \,:\, \operatorname{degree}(B(\lambda)) \leq \ell,\, \operatorname{rank} B(\lambda_0) \leq \rho_0,\, \operatorname{rank}(B(\lambda) - B(\lambda_0)) \leq \rho_1\}\,, \tag{7}$$

Theorem 2 implies that

$$\det{(A+B)(\lambda)} = (\lambda - \lambda_0)^{\widetilde{a}}\, q(\lambda), \tag{8}$$

where $q(\lambda)$ is a polynomial. Therefore, if $(A+B)(\lambda)$ is regular, $q(\lambda_0) \neq 0$ if and only if the algebraic multiplicity of λ_0 in this polynomial is exactly $\widetilde{a}$. This may happen only for elements of $\mathcal{C}$ such that $\operatorname{rank} B(\lambda_0) = \rho_0$ (see Remark 1) and $\operatorname{rank}(B(\lambda) - B(\lambda_0)) = \rho_1$ when $\rho \leq e_0$, while $\operatorname{rank}(B(\lambda) - B(\lambda_0))$ may be smaller than ρ_1 when $\rho > e_0$. Anyway, according to Theorem 2, the algebraic multiplicity of λ_0 is $\widetilde{a}$ if and only if the degrees of the λ_0-elementary divisors of $(A+B)(\lambda)$ are the ones obtained by removing the first ρ members in the list $d_1, \ldots, d_g, 1, \ldots, 1$, where the number of 1$s$ is ρ_1. Clearly, once $A(\lambda)$ and λ_0 are fixed, $q(\lambda_0)$ is a multivariate polynomial in the entries of the coefficient matrices of $B(\lambda)$, and $q(\lambda_0) = 0$ defines an algebraic submanifold of $\mathbb{C}^{(\ell+1)n^2}$ whose intersection with $\mathcal{C}$ is the algebraic submanifold $\mathcal{M} \subseteq \mathcal{C}$ for which the behavior described in the Introduction does not happen. Now, it remains to show that the algebraic submanifold $\mathcal{M}$ is proper or, in other words, that $q(\lambda_0) \neq 0$ for some perturbations $B(\lambda) \in \mathcal{C}$. This is proved in the next Lemma.

Lemma 2 *Let λ_0 be a finite eigenvalue of $A(\lambda)$, a complex $n \times n$ regular matrix polynomial of degree $\ell \geq 1$, and $d_1 \geq \ldots \geq d_g > 0$ be the degrees of the elementary divisors of $A(\lambda)$ associated with λ_0. Let ρ_0 and ρ_1 be two nonnegative integers such that $\rho_0 \leq g$ and $\rho_1 \leq n$. Then, there exists a complex matrix polynomial $B(\lambda)$ with degree at most ℓ,*

$$\operatorname{rank} B(\lambda_0) \leq \rho_0\,, \qquad \operatorname{rank}(B(\lambda) - B(\lambda_0)) \leq \rho_1\,,$$

such that $(A+B)(\lambda)$ is regular, and the algebraic multiplicity of λ_0 in the polynomial $(A+B)(\lambda)$ is exactly $a_{A(\lambda)}(\lambda_0) + \rho_1 - d_1 - \ldots - d_\rho$, where $\rho = \rho_0 + \rho_1$ and $d_m = 1$ for $m = g+1, \ldots, \rho$.

Proof. We will prove that there exists a linear matrix polynomial $B(\lambda) = B_0 + \lambda B_1$, i.e., a pencil, satisfying the conditions of the statement. Note that in this linear case the rank conditions are:

$$\operatorname{rank}(B_0 + \lambda_0 B_1) \leq \rho_0\,, \qquad \operatorname{rank} B_1 \leq \rho_1\,. \tag{9}$$

For simplicity, we set, as previously, $\widetilde{a} = a_{A(\lambda)}(\lambda_0) + \rho_1 - d_1 - \ldots - d_\rho$. We will reduce the proof to find the required perturbation pencil in the following two cases: **Case 1:** $\rho_0 = 1\,, \rho_1 = 0\,$; and, **Case 2:** $\rho_0 = 0\,, \rho_1 = 1$. Note that, for nonzero perturbations, $B(\lambda)$ in **Case 1** is just a constant rank one matrix, whereas in **Case 2** the pencil is of the type $(\lambda - \lambda_0)B_1$ with $\operatorname{rank} B_1 = 1$.

Once the result is proved in these two simple cases, we can find the perturbation pencil $B(\lambda)$ for arbitrary nonnegative integers ρ_0 and ρ_1, such that $\rho_0 \leq g$ and $\rho_1 \leq n$, by applying iteratively ρ_0 times the case 1, and ρ_1 times the case 2. To be more precise, the perturbation pencil will be of the type

$$B(\lambda) = R_1 + \cdots + R_{\rho_0} + (\lambda - \lambda_0)(S_1 + \cdots + S_{\rho_1}), \tag{10}$$

where $R_1, \ldots, R_{\rho_0}$ are rank one constant matrices corresponding to the ρ_0 cases of type 1, and $S_1, \ldots, S_{\rho_1}$ are rank one constant matrices corresponding to the ρ_1 cases of type 2. Note that we are applying iteratively the cases 1 and 2 to the unperturbed polynomials $A(\lambda), A(\lambda)+R_1, \ldots, A(\lambda)+R_1+\cdots+R_{\rho_0}, A(\lambda)+R_1+\cdots+R_{\rho_0}+(\lambda-\lambda_0)S_1, \ldots, A(\lambda)+R_1+\cdots+R_{\rho_0}+(\lambda-\lambda_0)S_1+\cdots+(\lambda-\lambda_0)S_{\rho_1}$. Notice that the perturbation pencil $B(\lambda)$ given by (10) satisfies the required conditions in the statement. So, let us prove the cases 1 and 2.

Case 1: We must find a rank one constant matrix B such that

$$\det(A(\lambda) + B) = (\lambda - \lambda_0)^{\widetilde{a}} q(\lambda),$$

where $q(\lambda)$ is a polynomial with $q(\lambda_0) \neq 0$, and $\widetilde{a} = d_2 + \ldots + d_g$. Taking into account the Smith normal form of $A(\lambda)$ given by (1), we have, for some nonzero constant c, that: 1) $\det A(\lambda) = c \cdot h_1(\lambda) \cdots h_n(\lambda) = (\lambda - \lambda_0)^{d_1 + \widetilde{a}} q_A(\lambda)$, with $q_A(\lambda_0) \neq 0$; and, 2) $h_2(\lambda) \cdots h_n(\lambda) = (\lambda - \lambda_0)^{\widetilde{a}} \widetilde{q}(\lambda)$, with $\widetilde{q}(\lambda_0) \neq 0$. Note that every function of λ appearing in the previous equations is a polynomial. Now, recall that the product $h_2(\lambda) \cdots h_n(\lambda)$ is the greatest common divisor of all $(n-1) \times (n-1)$ minors of $A(\lambda)$ [4, Chapter VI]. Then there exists at least one $(n-1) \times (n-1)$ minor of $A(\lambda)$, $\widetilde{M}_{ij}(\lambda)$ (complementary of the (i,j) entry, $a_{ij}(\lambda)$, of $A(\lambda)$), such that

$$\widetilde{M}_{ij}(\lambda) = (\lambda - \lambda_0)^{\widetilde{a}} q_{ij}(\lambda)\,,$$

with $q_{ij}(\lambda_0) \neq 0$. If we denote the cofactors of $A(\lambda)$ as $M_{ik}(\lambda) \equiv (-1)^{i+k} \widetilde{M}_{ik}(\lambda)$, the Laplace expansion of $\det A(\lambda)$ by the ith row gives rise to

$$\det A(\lambda) = a_{i1}(\lambda) M_{i1}(\lambda) + \ldots + a_{ij}(\lambda) M_{ij}(\lambda) + \ldots + a_{in}(\lambda) M_{in}(\lambda)\,. \tag{11}$$

Let us write[2] $a_{ik}(\lambda) = a_{ik} + O(\lambda - \lambda_0)$, where $a_{ik} \in \mathbb{C}$, and $M_{ik}(\lambda) = m_{ik}(\lambda - \lambda_0)^{\widetilde{a}} + O((\lambda - \lambda_0)^{\widetilde{a}+1})$, with $m_{ik} \in \mathbb{C}$, for $k = 1, \ldots, n$, and $m_{ij} \neq 0$. Then

$$\det A(\lambda) = (a_{i1} m_{i1} + \ldots + a_{in} m_{in})(\lambda - \lambda_0)^{\widetilde{a}} + O((\lambda - \lambda_0)^{\widetilde{a}+1})\,,$$

where $a_{i1} m_{i1} + \ldots + a_{in} m_{in} = 0$, because $\det A(\lambda) = (\lambda - \lambda_0)^{d_1 + \widetilde{a}} q_A(\lambda)$. Since $m_{ij} \neq 0$, we have that for every nonzero number ε

$$a_{i1} m_{i1} + \ldots + (a_{ij} + \varepsilon) m_{ij} + \ldots + a_{in} m_{in} \neq 0\,.$$

Choose one particular ε and let $B = (b_{kl})_{k,l=1}^{n}$ be the rank one matrix defined by

$$b_{kl} = \begin{cases} 0 & \text{if} \quad (k,l) \neq (i,j) \\ \varepsilon & \text{if} \quad (k,l) = (i,j) \end{cases}\,.$$

Then

$$\det(A(\lambda) + B) = a_{i1}(\lambda) M_{i1}(\lambda) + \ldots + (a_{ij}(\lambda) + \varepsilon) M_{ij}(\lambda) + \ldots + a_{in}(\lambda) M_{in}(\lambda) =$$

$$= (a_{i1} m_{i1} + \ldots + (a_{ij} + \varepsilon) m_{ij} + \ldots + a_{in} m_{in})(\lambda - \lambda_0)^{\widetilde{a}} + O((\lambda - \lambda_0)^{\widetilde{a}+1})\,,$$

with $a_{i1} m_{i1} + \ldots + (a_{ij} + \varepsilon) m_{ij} + \ldots + a_{in} m_{in} \neq 0$, so B is the required perturbation.

Case 2: We must find a perturbation pencil of the type $B(\lambda) = (\lambda - \lambda_0) B_1$, where B_1 is a rank one constant matrix, such that

$$\det(A + B)(\lambda) = (\lambda - \lambda_0)^{\widetilde{a}} q(\lambda),$$

[2] In this proof big-O expressions of the type $O((\lambda - \lambda_0)^k)$ are in fact polynomials of degree greater than or equal to k in $(\lambda - \lambda_0)$.

with $q(\lambda_0) \neq 0$ and, in this case, $\widetilde{a} = d_2 + \ldots + d_g + 1$. Arguments similar to those in Case 1 show that: 1) $\det A(\lambda) = (\lambda - \lambda_0)^{d_1+\cdots+d_g} q_A(\lambda)$, with $q_A(\lambda_0) \neq 0$; and 2) $h_2(\lambda)\cdots h_n(\lambda) = (\lambda - \lambda_0)^{d_2+\cdots+d_g}\widetilde{q}(\lambda)$, with $\widetilde{q}(\lambda_0) \neq 0$. Then there exists an entry $a_{ij}(\lambda)$ of $A(\lambda)$ such that the complementary $(n-1)\times(n-1)$ cofactor $M_{ij}(\lambda)$ of $A(\lambda)$ can be written as

$$M_{ij}(\lambda) = (\lambda - \lambda_0)^{d_2+\ldots+d_g} q_{ij}(\lambda) = (\lambda - \lambda_0)^{d_2+\ldots+d_g} \left(m_{ij} + O(\lambda - \lambda_0)\right),$$

with $q_{ij}(\lambda_0) = m_{ij} \neq 0$. Let us write $a_{ik}(\lambda) = a_{ik} + a_{ik}^1\,(\lambda - \lambda_0) + O((\lambda-\lambda_0)^2)$, where $a_{ik}, a_{ik}^1 \in \mathbb{C}$, for $k = 1, \ldots, n$. Let us expand $\det A(\lambda)$ by the ith row as in (11), to get

$$\det A(\lambda) = (a_{i1}m_{i1} + \ldots + a_{in}m_{in})(\lambda - \lambda_0)^{\widetilde{a}-1} + (a_{ij}^1 m_{ij} + y)\,(\lambda - \lambda_0)^{\widetilde{a}} + O((\lambda - \lambda_0)^{\widetilde{a}+1}),$$

where y is independent of a_{ij}^1. As in Case 1 $(a_{i1}m_{i1} + \ldots + a_{in}m_{in}) = 0$. Since $m_{ij} \neq 0$, if ε is any nonzero number such that $\varepsilon \neq -(y + a_{ij}^1 m_{ij})/m_{ij}$ then

$$(a_{ij}^1 + \varepsilon)m_{ij} + y \neq 0\,.$$

Let $B(\lambda) = (b_{kl}(\lambda))_{k,l=1}^n$ be the rank one matrix pencil defined as

$$b_{kl}(\lambda) = \begin{cases} 0 & \text{if} \quad (k,l) \neq (i,j) \\ \varepsilon(\lambda - \lambda_0) & \text{if} \quad (k,l) = (i,j) \end{cases}\,.$$

Then

$$\det(A+B)(\lambda) = a_{i1}(\lambda)M_{i1}(\lambda) + \ldots + (a_{ij}(\lambda) + \varepsilon(\lambda - \lambda_0))\, M_{ij}(\lambda) + \ldots + a_{in}(\lambda)M_{in}(\lambda) =$$

$$= ((a_{ij}^1 + \varepsilon)m_{ij} + y)(\lambda - \lambda_0)^{\widetilde{a}} + O((\lambda - \lambda_0)^{\widetilde{a}+1}),$$

where $(a_{ij}^1 + \varepsilon)m_{ij} + y \neq 0$. So $B(\lambda)$ is the required perturbation. □

Remark 2 *Note that the proof we have presented of Lemma 2 allows us to guarantee that the polynomial $B(\lambda)$ can always be chosen with degree less than or equal to one, whatever the degree of $A(\lambda)$ is.*

As a consequence of the results proved in this section, we can state Theorem 3 on the generic behavior of elementary divisors under low rank perturbations.

Theorem 3 *Let λ_0 be a finite eigenvalue of $A(\lambda)$, a complex $n \times n$ regular matrix polynomial of degree $\ell \geq 1$, and $d_1 \geq \ldots \geq d_g > 0$ be the degrees of the elementary divisors of $A(\lambda)$ associated with λ_0. Let ρ_0 and ρ_1 be two nonnegative integers such that $\rho_0 \leq g$ and $\rho_1 \leq n$, $\rho = \rho_0 + \rho_1$, and let us define the algebraic manifold of $n \times n$ matrix polynomials*

$$\mathcal{C} = \{B(\lambda) : \textit{degree}(B(\lambda)) \leq \ell,\ \operatorname{rank} B(\lambda_0) \leq \rho_0\,,\ \operatorname{rank}(B(\lambda) - B(\lambda_0)) \leq \rho_1\}.$$

Then, for every polynomial $B(\lambda)$ in $\mathcal{C}$, except those in a proper algebraic submanifold of $\mathcal{C}$, the polynomial $(A+B)(\lambda)$ is regular, λ_0 is an eigenvalue of $(A+B)(\lambda)$, and the degrees of its elementary divisors associated with λ_0 are obtained by removing the first ρ members in the list $d_1, \ldots, d_g, \underbrace{1, \ldots, 1}_{\rho_1}$. Note that this means, in particular, that $(A+B)(\lambda)$ has $g - \rho_0$ elementary divisors associated with λ_0.

It should be noticed that if $\operatorname{rank} B(\lambda_0) < \rho_0$ then (2) implies that the number of elementary divisors of $(A+B)(\lambda)$ associated with λ_0 is greater than $g - \rho_0$, so all the polynomials in $\mathcal{C}$ for which the generic behavior happens satisfy $\operatorname{rank} B(\lambda_0) = \rho_0$.

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