FIEDLER COMPANION LINEARIZATIONS AND THE RECOVERY OF MINIMAL INDICES *

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Abstract. A standard way of dealing with a matrix polynomial $P(\lambda)$ is to convert it into an equivalent matrix pencil – a process known as linearization. For any regular matrix polynomial, a new family of linearizations generalizing the classical first and second Frobenius companion forms has recently been introduced by Antoniou and Vologiannidis, extending some linearizations previously defined by Fiedler for scalar polynomials. We prove that these pencils are linearizations even when $P(\lambda)$ is a singular square matrix polynomial, and show explicitly how to recover the left and right minimal indices and minimal bases of the polynomial $P(\lambda)$ from the minimal indices and bases of these linearizations. In addition, we provide a simple way to recover the eigenvectors of a regular polynomial from those of any of these linearizations, without any computational cost. The existence of an eigenvector recovery procedure is essential for a linearization to be relevant for applications.

Key words. singular matrix polynomials, matrix pencils, minimal indices, minimal bases, linearization, recovery of eigenvectors, Fiedler pencils, companion forms

AMS subject classifications. 65F15, 15A18, 15A21, 15A22

1. Introduction.

Throughout this work we consider $n \times n$ matrix polynomials with degree $k \geq 2$ of the form

$$P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i\,, \qquad A_0, \ldots, A_k \in \mathbb{F}^{n\times n}, \quad A_k \neq 0\,, \tag{1.1}$$

where $\mathbb{F}$ is an arbitrary field and λ is a scalar variable in $\mathbb{F}$. Our main focus is on singular matrix polynomials, although new results are also obtained for regular polynomials. An $n\times n$ polynomial $P(\lambda)$ is said to be *singular* if $\det P(\lambda)$ is identically zero, i.e., if all its coefficients are zero, otherwise it is *regular*. Square singular polynomials appear in practice in a number of contexts; one well-known example is the study of differential-algebraic equations (see for instance [7] and the references therein). Other sources of problems involving singular matrix polynomials are control theory and linear systems theory [18, 29, 39], where the problem of computing minimal polynomial bases of null spaces of singular matrix polynomials continues to be the subject of intense research (see [3] and the references therein for an updated bibliography).

The standard way to numerically solve polynomial eigenvalue problems for *regular* polynomials $P(\lambda)$ is to first linearize $P(\lambda)$ into a matrix pencil $L(\lambda) = \lambda X + Y$ with $X, Y \in \mathbb{F}^{nk\times nk}$, and then compute the eigenvalues and eigenvectors of $L(\lambda)$ using well-established algorithms for matrix pencils [22]. The classical approach [21] uses the *first and second companion forms* (3.5) and (3.6), sometimes known as the

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Frobenius companion forms of $P(\lambda)$, as linearizations. However, these companion forms usually do not share any algebraic structure that $P(\lambda)$ might have. For example, if $P(\lambda)$ is symmetric, Hermitian, alternating, or palindromic, then the companion forms won't retain any of these structures. Consequently, the rounding errors inherent to numerical computations may destroy qualitative aspects of the spectrum. This has motivated intense activity towards the development of new classes of linearizations. Several classes have been introduced in [4, 5] and [33], generalizing the Frobenius companion forms in a number of different ways. Other classes of linearizations were introduced and studied in [1, 2], motivated by the use of non-monomial bases for the space of polynomials. The numerical properties of the linearizations in [33] have been analyzed in [24, 25, 28], while the exploitation of these linearizations for the preservation of structure in a wide variety of contexts has been extensively developed in [16, 26, 27, 32, 34].

The linearizations introduced in [2], [4], and [33] were originally studied only for *regular* matrix polynomials. Very recently, though, the pencils in the vector spaces $\mathbb{L}_1(P)$ and $\mathbb{L}_2(P)$ defined in [33] were considered in [11] as potential linearizations for square *singular* $P(\lambda)$; it was shown in [11] that almost all pencils in $\mathbb{L}_1(P)$ and $\mathbb{L}_2(P)$ are linearizations even when $P(\lambda)$ is singular. These linearizations were also shown to allow the easy recovery of the *complete eigenstructure* of $P(\lambda)$, i.e., the finite and infinite elementary divisors together with the left and right minimal indices [18, 29], and also the easy recovery of the corresponding minimal bases. Note that the results in [11] apply to the important cases of the first and second companion forms of $P(\lambda)$.

In this paper we study the pencils introduced in [4], with emphasis on the case when $P(\lambda)$ is singular. Since these pencils arise from the companion matrices introduced by Fiedler [17] in the same way that the classical first and second companion forms arise from the companion matrices of Frobenius, we refer to them as the *Fiedler companion pencils*, or *Fiedler pencils* for short.

There are *three main results* in this work. The first is to show that the family of Fiedler pencils, investigated in [4] only for regular matrix polynomials $P(\lambda)$, are still linearizations when $P(\lambda)$ is a *singular* square matrix polynomial. This requires very different techniques from those used in [4] for the regular case. Second we show how these linearizations can be used to immediately recover the complete eigenstructure of $P(\lambda)$. Finally, we develop simple procedures to recover the eigenvectors of a regular polynomial $P(\lambda)$ from those of any Fiedler pencil, without any computational cost. Recovery procedures for eigenvectors were not addressed in [4], but are very important for practical applications, as well as in any numerical algorithm for polynomial eigenvalue problems based on linearizations.

The results in this work expand the arena in which to look for linearizations having additional useful properties. For singular polynomials P that are symmetric, Hermitian, alternating, or palindromic, it was shown in [11] that *none* of the pencils in $\mathbb{L}_1(P)$ or $\mathbb{L}_2(P)$ with structure corresponding to that of P (see [27, 34]) is *ever* a linearization when $P(\lambda)$ is singular. Hence for singular polynomials an expanded palette of linearizations is essential for preserving structure. Using pencils closely related to the Fiedler pencils, it is possible to develop structured linearizations for at least some large classes of structured singular matrix polynomials [12, 35].

Apart from the preservation of structure, there is another property that potentially may be useful; some Fiedler pencils have a much smaller bandwidth than the classical Frobenius companion forms [17], e.g., see Example 3.2. It may be possible to exploit this band structure to develop fast algorithms to compute the complete eigenstructure of high degree matrix polynomials. As far as we know, though, this has not yet been addressed either for regular or for singular polynomials.

The numerical computation of minimal indices and bases is a difficult problem that can be addressed in several different ways [3]; one of the most reliable of these methods from a numerical point of view uses one of the Frobenius companion linearizations [6, 38]. The results in this paper, together with those in [11], now allow many other linearizations to be used for this purpose.

We begin in Section 2 by recalling some basic concepts that are used throughout the paper, followed in Section 3 by the fundamental definitions and notation needed for working effectively with Fiedler pencils. Section 4 then proves that Fiedler pencils are always strong linearizations, even for singular matrix polynomials. In Section 5 we show how to recover the minimal indices and bases of a singular square matrix polynomial from those of any Fiedler pencil; as a consequence, we are then able in Section 6 to characterize which Fiedler pencils are strictly equivalent and which are not. Section 7 provides a very simple recipe for recovering, without any computational cost, the eigenvectors of a regular matrix polynomial from the eigenvectors of any of its Fiedler companion linearizations. Finally, we wrap up in Section 8 with some conclusions and discussion of ongoing related work.

2. Basic concepts. We present some basic concepts related to matrix polynomials (singular or not), referring the reader to [11, Section 2] for a more complete treatment. Note that 0_d and I_d are used to denote the $d \times d$ zero and identity matrices, respectively. We emphasize that any equation in this paper involving expressions in λ is to be understood as a formal algebraic identity, and not just as an equality of functions on the field $\mathbb{F}$. For finite fields $\mathbb{F}$ this distinction is important, and we will always intend the stronger meaning of a formal algebraic identity.

Let $\mathbb{F}(\lambda)$ denote the field of rational functions with coefficients in $\mathbb{F}$, so that $\mathbb{F}(\lambda)^n$ is the vector space of column n-tuples with entries in $\mathbb{F}(\lambda)$. The *normal rank* of a matrix polynomial $P(\lambda)$, denoted $\operatorname{nrank} P(\lambda)$, is the rank of $P(\lambda)$ considered as a matrix with entries in $\mathbb{F}(\lambda)$, or equivalently, the size of the largest non-identically zero minor of $P(\lambda)$ [19]. A *finite eigenvalue* of $P(\lambda)$ is an element $\lambda_0 \in \mathbb{F}$ such that

$$\operatorname{rank} P(\lambda_0) < \operatorname{nrank} P(\lambda) .$$

We say that $P(\lambda)$ with degree k has an *infinite eigenvalue* if the *reversal polynomial*

$$\operatorname{rev} P(\lambda) := \lambda^k P(1/\lambda) = \sum_{i=0}^{k} \lambda^i A_{k-i} \tag{2.1}$$

has zero as an eigenvalue.

An $n \times n$ singular matrix polynomial $P(\lambda)$ has *right* (*column*) and *left* (*row*) *null vectors*, that is, vectors $x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1}$ and $y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times n}$ such that $P(\lambda)x(\lambda) \equiv 0$ and $y(\lambda)^T P(\lambda) \equiv 0$, where $y(\lambda)^T$ denotes the transpose of $y(\lambda)$. This leads to the following definition.

DEFINITION 2.1. *The* right *and* left nullspaces *of the $n \times n$ matrix polynomial $P(\lambda)$, denoted by $\mathcal{N}_r(P)$ and $\mathcal{N}_\ell(P)$ respectively, are the following subspaces:*

$$\mathcal{N}_r(P) := \left\{ x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} \, : \, P(\lambda)x(\lambda) \equiv 0 \right\} ,$$
$$\mathcal{N}_\ell(P) := \left\{ y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times n} \, : \, y(\lambda)^T P(\lambda) \equiv 0^T \right\} .$$

Note that we have the identity

$$\operatorname{nrank}(P) = n - \dim \mathcal{N}_r(P) = n - \dim \mathcal{N}_\ell(P), \tag{2.2}$$

and, in particular, $\dim \mathcal{N}_r(P) = \dim \mathcal{N}_\ell(P)$.

It is well known that the *elementary divisors* of $P(\lambda)$ corresponding to its finite eigenvalues (see definition in [19]), as well as the dimensions of $\mathcal{N}_r(P)$ and $\mathcal{N}_\ell(P)$, are invariant under *unimodular equivalence* [19], i.e., under pre- and post-multiplication of $P(\lambda)$ by *unimodular matrices* (matrix polynomials with nonzero constant determinant). The elementary divisors of $P(\lambda)$ corresponding to the infinite eigenvalue are defined as the elementary divisors corresponding to the zero eigenvalue of the reversal polynomial [23, Definition 1].

Next we recall the definition of linearization as introduced in [21], and also the related notion of strong linearization introduced in [20] and named in [31].

Definition 2.2. *A matrix pencil* $L(\lambda) = \lambda X + Y$ *with* $X, Y \in \mathbb{F}^{nk\times nk}$ *is a* linearization *of an* $n\times n$ *matrix polynomial* $P(\lambda)$ *of degree* k *if there exist two unimodular* $nk \times nk$ *matrices* $U(\lambda)$ *and* $V(\lambda)$ *such that*

$$U(\lambda)L(\lambda)V(\lambda) = \begin{bmatrix} I_{(k-1)n} & 0 \\ 0 & P(\lambda) \end{bmatrix}, \tag{2.3}$$

i.e., if $L(\lambda)$ *is* unimodularly equivalent *to* $\operatorname{diag}\left[I_{(k-1)n},\, P(\lambda)\right]$. *A linearization* $L(\lambda)$ *is called a* strong linearization *if* $\operatorname{rev} L(\lambda)$ *is also a linearization of* $\operatorname{rev} P(\lambda)$.

These definitions were introduced in [20, 21] only for regular polynomials, and were extended in [11, Section 2] to square singular matrix polynomials. Lemma 2.3 shows why linearizations and strong linearizations are relevant in the study of both regular and singular matrix polynomials. Note that this result appeared in [11] for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, but with only slight modifications the proof given in [11] also holds for matrix polynomials over an arbitrary field $\mathbb{F}$.

Lemma 2.3. [11, Lemma 2.3] *Let* $P(\lambda)$ *be a regular or singular* $n \times n$ *matrix polynomial of degree* k*, over an arbitrary field* $\mathbb{F}$*, and let* $L(\lambda)$ *be an* $nk \times nk$ *matrix pencil over* $\mathbb{F}$*. Consider the following conditions on* $L(\lambda)$ *and* $P(\lambda)$:

(a) $\dim \mathcal{N}_r(L) = \dim \mathcal{N}_r(P)$,

(b) *the finite elementary divisors of* $L(\lambda)$ *and* $P(\lambda)$ *are identical,*

(c) *the infinite elementary divisors of* $L(\lambda)$ *and* $P(\lambda)$ *are identical.*

Then $L(\lambda)$ *is*

- *a linearization of* $P(\lambda)$ *if and only if conditions* (a) *and* (b) *hold,*
- *a strong linearization of* $P(\lambda)$ *if and only if conditions* (a), (b) *and* (c) *hold.*

Note that the issues addressed in Lemma 2.3 for general $n \times n$ matrix polynomials were already considered for regular polynomials in [30].

We mention briefly that linearizations with smaller size than the ones in Definition 2.2 have been introduced recently in [7], and that their minimal possible size has been determined in [10].

A *vector polynomial* is a vector whose entries are polynomials in the variable λ. For any subspace of $\mathbb{F}(\lambda)^n$, it is always possible to find a basis consisting entirely of vector polynomials; simply take an arbitrary basis and multiply each vector by the denominators of its entries. The *degree* of a vector polynomial is the greatest degree of its components, and the *order* of a polynomial basis is defined as the sum of the degrees of its vectors [18, p. 494]. Then the following definition makes sense.

Definition 2.4. [18] *Let* $\mathcal{V}$ *be a subspace of* $\mathbb{F}(\lambda)^n$*. A* minimal basis *of* $\mathcal{V}$ *is any polynomial basis of* $\mathcal{V}$ *with least order among all polynomial bases of* $\mathcal{V}$*.*

It can be shown [18] that for any given subspace $\mathcal{V}$ of $\mathbb{F}(\lambda)^n$, the ordered list of degrees of the vector polynomials in any minimal basis of $\mathcal{V}$ is always the same. These degrees are then called the *minimal indices* of $\mathcal{V}$. Specializing $\mathcal{V}$ to be the left and right nullspaces of a singular matrix polynomial gives Definition 2.5; here $\deg(p(\lambda))$ denotes the degree of the vector polynomial $p(\lambda)$.

DEFINITION 2.5. *Let $P(\lambda)$ be a square singular matrix polynomial, and let the sets* $\{y_1(\lambda)^T, \ldots, y_p(\lambda)^T\}$ *and* $\{x_1(\lambda), \ldots, x_p(\lambda)\}$ *be minimal bases of, respectively, the left and right nullspaces of $P(\lambda)$, ordered such that* $\deg(y_1) \leq \deg(y_2) \leq \cdots \leq \deg(y_p)$ *and* $\deg(x_1) \leq \deg(x_2) \leq \cdots \leq \deg(x_p)$. *Let* $\eta_i = \deg(y_i)$ *and* $\varepsilon_i = \deg(x_i)$ *for* $i = 1, \ldots, p$. *Then* $\eta_1 \leq \eta_2 \leq \cdots \leq \eta_p$ *and* $\varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_p$ *are, respectively, the* left and right minimal indices of $P(\lambda)$. *For the sake of brevity, we call minimal bases of the left and right nullspaces of $P(\lambda)$ simply* left and right minimal bases of $P(\lambda)$.

It is not hard to see that the minimal indices of a singular polynomial $P(\lambda)$ are invariant under *strict equivalence*, i.e., under pre- and post-multiplication of $P(\lambda)$ by nonsingular constant matrices. However, unimodular equivalence can change any (even all) of the minimal indices, as illustrated by the results in [11] and in this paper.

In the case of matrix pencils, the left (right) minimal indices can be read off from the sizes of the left (right) singular blocks of the Kronecker canonical form of the pencil [19, Chap. XII]. Consequently, the minimal indices of a pencil can be stably computed through unitary transformations that lead to the GUPTRI form [36, 8, 9, 15]. Therefore it is natural to look for relationships between the minimal indices of a singular polynomial P and the minimal indices of a given linearization of P, since this would lead to a numerical method for computing the minimal indices of P. Such relationships were found in [11] for the pencils introduced in [33], and will be developed in this work for the Fiedler pencils introduced in [4]. Note in this context that Lemma 2.3 implies only that linearizations of P have the same *number* of minimal indices as P, but does not provide the values of the minimal indices of P in terms of the minimal indices of a linearization. In fact, it is known [11] that different linearizations of the same polynomial P may have different minimal indices. For this reason each family of linearizations of a singular polynomial requires a separate study in order to establish the relationships (if any) between the minimal indices of the polynomial and those of the linearizations in that family.

In this paper we adopt the following definition, which was introduced in [11, Section 2] as an extension to matrix polynomials of the one given in [37] for pencils.

DEFINITION 2.6. *The* complete eigenstructure *of a matrix polynomial consists of its finite and infinite elementary divisors, together with its left and right minimal indices.*

3. Definition of Fiedler companion pencils. Let $P(\lambda)$ be the matrix polynomial in (1.1). From the coefficients of $P(\lambda)$, we first define the $nk \times nk$ matrices

$$M_k := \begin{bmatrix} A_k & \\ & I_{(k-1)n} \end{bmatrix}, \qquad M_0 := \begin{bmatrix} I_{(k-1)n} & \\ & -A_0 \end{bmatrix}, \tag{3.1}$$

$$\text{and} \qquad M_i := \begin{bmatrix} I_{(k-i-1)n} & & & \\ & -A_i & I_n & \\ & I_n & 0 & \\ & & & I_{(i-1)n} \end{bmatrix}, \quad i = 1, \ldots, k-1, \tag{3.2}$$

which are the basic factors used to build all the Fiedler pencils. In [4] these pencils are constructed as

$$\lambda M_k - M_{i_0} M_{i_1} \cdots M_{i_{k-1}},$$

where $(i_0, i_1 \ldots, i_{k-1})$ is any possible permutation of the n-tuple $(0, 1, \ldots, k-1)$. In order to better express certain key properties of this permutation and the resulting Fiedler pencil, we have found it useful to index the product of the M_i-factors in a slightly different way, as described in the following definition.

DEFINITION 3.1 (Fiedler Pencils). *Let $P(\lambda)$ be the matrix polynomial in* (1.1), *and let M_i for $i = 0, \dots, k$ be the matrices defined in* (3.1) *and* (3.2). *Given any bijection $\sigma : \{0, 1, \dots, k-1\} \to \{1, \dots, k\}$, the* Fiedler pencil of $P(\lambda)$ associated with σ *is the $nk \times nk$ matrix pencil*

$$F_\sigma(\lambda) := \lambda M_k - M_{\sigma^{-1}(1)} \cdots M_{\sigma^{-1}(k)} . \tag{3.3}$$

Note that $\sigma(i)$ describes the position of the factor M_i in the product $M_{\sigma^{-1}(1)} \cdots M_{\sigma^{-1}(k)}$ defining the zero-degree term in (3.3), *i.e., $\sigma(i) = j$ means that M_i is the jth factor in the product. For brevity, we denote this product by*

$$M_\sigma := M_{\sigma^{-1}(1)} \cdots M_{\sigma^{-1}(k)} , \tag{3.4}$$

so that $F_\sigma(\lambda) = \lambda M_k - M_\sigma$.

Sometimes we will write the bijection σ using the array notation $\sigma = (\sigma(0), \sigma(1), \dots, \sigma(k-1))$. Unless otherwise stated, the matrices M_i, $i = 0, \dots, k$, M_σ, and the Fiedler pencil $F_\sigma(\lambda)$ refer to the matrix polynomial $P(\lambda)$ in (1.1). When necessary, we will explicitly indicate the dependence on a certain polynomial $Q(\lambda)$ by writing $M_i(Q)$, $M_\sigma(Q)$ and $F_\sigma(Q)$. In this situation, the dependence on λ is dropped in the Fiedler pencil $F_\sigma(Q)$ for simplicity. Since matrix polynomials will always be denoted by capital letters, there is no risk of confusion between $F_\sigma(\lambda)$ and $F_\sigma(Q)$.

The set of Fiedler pencils includes the well-known *first* and *second companion forms* [21] of the polynomial in (1.1). They are

$$C_1(\lambda) := \lambda \begin{bmatrix} A_k & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix} + \begin{bmatrix} A_{k-1} & A_{k-2} & \cdots & A_0 \\ -I_n & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -I_n & 0 \end{bmatrix} \tag{3.5}$$

and

$$C_2(\lambda) := \lambda \begin{bmatrix} A_k & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix} + \begin{bmatrix} A_{k-1} & -I_n & & 0 \\ A_{k-2} & 0 & \ddots & \\ \vdots & \vdots & \ddots & -I_n \\ A_0 & 0 & \cdots & 0 \end{bmatrix} . \tag{3.6}$$

More precisely, $C_1(\lambda) = F_{\sigma_1}(\lambda)$ where $\sigma_1 = (k, k-1, \dots, 2, 1)$, and $C_2(\lambda) = F_{\sigma_2}(\lambda)$ where $\sigma_2 = (1, 2, \dots, k-1, k)$. These companion forms are well known to be strong linearizations for any $P(\lambda)$ [20, Prop. 1.1], [11].

The set of Fiedler pencils also includes several block-pentadiagonal pencils [17], which have a smaller bandwidth than $C_1(\lambda)$ and $C_2(\lambda)$, indeed a *much* smaller bandwidth if the degree of the polynomial is high. For these pencils the M_σ matrix (3.4) is constructed as in the following example.

EXAMPLE 3.2 (Low bandwidth Fiedler pencils). *Let $\mathcal{O} = M_1 M_3 \cdots$ be the product of the odd M_i factors, and let $\mathcal{E} = M_2 M_4 \cdots$ be the product of the even M_i factors, excluding M_0 and M_k. Clearly $\mathcal{O}$ and $\mathcal{E}$ are block-tridiagonal and M_0 is block-diagonal, so the product of $\mathcal{O}$, $\mathcal{E}$, and M_0 in* any *order yields a block-pentadiagonal M_σ. For example if* $\deg P = 6$, *then $M_\sigma = \mathcal{O} M_0 \mathcal{E} = M_1 M_3 M_5 M_0 M_2 M_4$ is*

$$M_\sigma = \begin{bmatrix} -A_5 & -A_4 & I_n & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & -A_3 & 0 & -A_2 & I_n & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -A_1 & 0 & -A_0 \\ 0 & 0 & 0 & I_n & 0 & 0 \end{bmatrix} .$$

*Note that the matrices $\mathcal{O}$ and $\mathcal{E}$ are nonsingular with simple block-tridiagonal inverses, so appropriately pre- or post-multiplying one of these block-pentadiagonal Fiedler pencils by one of either $\mathcal{E}^{-1}$ or $\mathcal{O}^{-1}$ will yield a block-*tridiagonal *pencil that is strictly equivalent to a Fiedler pencil. For example*: $F_\sigma^{\mathcal{P}}(\lambda) = \lambda M_k - \mathcal{O}M_0\mathcal{E}$ *is strictly equivalent to the block-tridiagonal pencils* $\lambda M_k\mathcal{E}^{-1} - \mathcal{O}M_0$ *and* $\lambda\mathcal{O}^{-1}M_k - M_0\mathcal{E}$.

The commutativity relations

$$M_iM_j = M_jM_i \quad \text{for} \quad |i-j| \neq 1, \tag{3.7}$$

are easily checked. They imply that some Fiedler pencils associated with different bijections σ are equal. For instance, for $k = 3$, the Fiedler pencils $\lambda M_3 - M_0M_2M_1$ and $\lambda M_3 - M_2M_0M_1$ are equal. These relations suggest that the relative positions of the matrices M_i and M_{i+1} in the product M_σ are of fundamental interest in studying Fiedler pencils. This motivates Definition 3.3.

DEFINITION 3.3. *Let* $\sigma : \{0, 1, \ldots, k-1\} \to \{1, \ldots, k\}$ *be a bijection.*

(a) *For* $i = 0, \ldots, k-2$, *we say that* σ *has a* consecution *at* i *if* $\sigma(i) < \sigma(i+1)$ *and that* σ *has an* inversion *at* i *if* $\sigma(i) > \sigma(i+1)$.

(b) *We denote by* $\mathfrak{c}(\sigma)$ *the total number of consecutions in* σ, *and by* $\mathfrak{i}(\sigma)$ *the total number of inversions in* σ.

(c) *The consecution-inversion structure sequence of* σ, *denoted by* CISS(σ), *is the tuple* $(c_1, i_1, c_2, i_2, \ldots, c_\ell, i_\ell)$, *where* σ *has* c_1 *consecutive consecutions at* $0, 1, \ldots, c_1 - 1$; i_1 *consecutive inversions at* $c_1, c_1+1, \ldots, c_1+i_1-1$ *and so on, up to* i_ℓ *inversions at* $k-1-i_\ell, \ldots, k-2$.

REMARK 3.4. The following simple observations on Defn. 3.3 will be used freely:

1. σ has a consecution at i if and only if M_i is to the *left* of M_{i+1} in M_σ, while σ has an inversion at i if and only if M_i is to the *right* of M_{i+1} in M_σ.
2. Either c_1 or i_ℓ in CISS(σ) may be zero (in the first case σ has an inversion at 0, in the second it has a consecution at $k-2$), but $i_1, c_2, i_2, \ldots, i_{\ell-1}, c_\ell$ are all strictly positive. These conditions uniquely determine CISS(σ) and, in particular, the parameter ℓ.
3. $\mathfrak{c}(\sigma) = \sum_{j=1}^{\ell} c_j$, $\mathfrak{i}(\sigma) = \sum_{j=1}^{\ell} i_j$, and $\mathfrak{c}(\sigma) + \mathfrak{i}(\sigma) = k-1$.

EXAMPLE 3.5. *For the block-pentadiagonal Fiedler pencil* $F_\sigma^{\mathcal{P}}(\lambda) = \lambda M_6 - \mathcal{O}M_0\mathcal{E}$ *defined in* Example 3.2 *we have* $\sigma^{-1} = (1,3,5,0,2,4)$, $\sigma = (4,1,5,2,6,3)$ *and structure sequence* CISS(σ) $= (0,1,1,1,1,1)$. *By contrast the block-pentadiagonal Fiedler pencil* $F_\tau^{\mathcal{P}}(\lambda) = \lambda M_6 - M_0\mathcal{O}\mathcal{E}$ *has* $\tau^{-1} = (0,1,3,5,2,4)$, $\tau = (1,2,5,3,6,4)$ *and structure sequence* CISS(τ) $= (2,1,1,1)$.

In Section 5 we will use the concept of *reversal bijection*: the reversal $\operatorname{rev}\sigma$ of a given bijection $\sigma : \{0, 1, \ldots, k-1\} \to \{1, 2, \ldots, k\}$ is another bijection from $\{0, 1, \ldots, k-1\}$ into $\{1, 2, \ldots, k\}$, defined by $\operatorname{rev}\sigma(i) := k+1-\sigma(i)$ for $0 \le i \le k-1$. Note that $\operatorname{rev}\sigma$ reverses the order of the factors M_j in the zero degree term M_σ of the Fiedler pencil $F_\sigma(\lambda)$ in (3.3). More precisely, the pencil $F_{\operatorname{rev}\sigma}(\lambda) = \lambda M_k - M_{\operatorname{rev}\sigma}$ satisfies

$$M_{\operatorname{rev}\sigma} = M_{\sigma^{-1}(k)}M_{\sigma^{-1}(k-1)} \cdots M_{\sigma^{-1}(1)} \,. \tag{3.8}$$

We will also use the block-transpose operation. More information on this operation can be found in [32, Chapter 3]. Here we simply recall the definition.

DEFINITION 3.6. *Let* $A = (A_{ij})$ *be a block* $r \times s$ *matrix with* $m \times n$ *blocks* A_{ij}. *The* block transpose *of* A *is the block* $s \times r$ *matrix* $A^{\mathcal{B}}$ *with* $m \times n$ *blocks defined by* $\left(A^{\mathcal{B}}\right)_{ij} = A_{ji}$.

REMARK 3.7. It is interesting to note that $F_{\operatorname{rev}\sigma}(\lambda) \equiv F_\sigma^{\mathcal{B}}(\lambda)$. Since this fact is not needed for the development in this paper, though, it will not be proved here.

4. Fiedler pencils are strong linearizations. We prove in this section that every Fiedler pencil $F_\sigma(\lambda)$ of a square matrix polynomial $P(\lambda)$, regular or singular, is a strong linearization for $P(\lambda)$. This fact was proved *only for regular polynomials* in [4]. A general proof including the singular case requires different techniques, in particular the systematic use of the *Horner shifts* of $P(\lambda)$, which are fundamental throughout the rest of the paper.

DEFINITION 4.1. *Let $P(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^k A_k$ be a matrix polynomial of degree k. For $d = 0, \ldots, k$, the* degree d Horner shift of $P(\lambda)$ *is the matrix polynomial $P_d(\lambda) := A_{k-d} + \lambda A_{k-d+1} + \cdots + \lambda^d A_k$. Observe that these Horner shifts satisfy*:

$$\begin{aligned} P_0(\lambda) &= A_k\,, \\ P_{d+1}(\lambda) &= \lambda P_d(\lambda) + A_{k-d-1} \quad \textit{for}\ \ 0 \le d \le k-1\,, \\ P_k(\lambda) &= P(\lambda)\,. \end{aligned} \tag{4.1}$$

Our explicit construction of the unimodular equivalence (2.3) showing $F_\sigma(\lambda)$ to be a linearization of $P(\lambda)$ proceeds in a stepwise fashion. Keeping M_σ in factored form, we eliminate one factor M_j at a time, going from highest to lowest index j, while at the same time building up higher degree Horner shifts of $P(\lambda)$ in the λ-term, finally ending up with $\operatorname{diag}\big[\,I,\ P(\lambda)\,\big]$. The tools needed to implement this proof strategy are introduced and developed in the next section.

4.1. Auxiliary matrices and equivalences. We begin with the auxiliary matrices that appear repeatedly throughout the following development.

DEFINITION 4.2. *For an $n \times n$ matrix polynomial $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$, let $P_i(\lambda)$ be the degree i Horner shift of $P(\lambda)$. For $1 \le i \le k-1$, we define the following $nk \times nk$ matrix polynomials*:

$$Q_i(\lambda) := \begin{bmatrix} I_{(i-1)n} & & & \\ & I_n & \lambda I_n & \\ & 0_n & I_n & \\ & & & I_{(k-i-1)n} \end{bmatrix},$$

$$R_i(\lambda) := \begin{bmatrix} I_{(i-1)n} & & & \\ & 0_n & I_n & \\ & I_n & P_i(\lambda) & \\ & & & I_{(k-i-1)n} \end{bmatrix} = R_i^{\mathcal{B}}(\lambda)\,,$$

$$T_i(\lambda) := \begin{bmatrix} 0_{(i-1)n} & & & \\ & 0_n & \lambda P_{i-1}(\lambda) & \\ & \lambda I_n & \lambda^2 P_{i-1}(\lambda) & \\ & & & 0_{(k-i-1)n} \end{bmatrix},$$

$$D_i(\lambda) := \begin{bmatrix} 0_{(i-1)n} & & & \\ & P_{i-1}(\lambda) & 0_n & \\ & 0_n & I_n & \\ & & & I_{(k-i-1)n} \end{bmatrix},$$

$$\textit{and} \quad D_k(\lambda) := \operatorname{diag}\big[\,0_{(k-1)n}, P_{k-1}(\lambda)\,\big]\,.$$

For the sake of brevity, we will sometimes omit the dependence on λ in these matrices and write just Q_i, R_i, T_i or D_i. Note that $D_1(\lambda) = M_k$, and that $Q_i(\lambda)$ and $R_i(\lambda)$ are unimodular for all $i = 1, \ldots, k-1$.

The following unimodular equivalences are now easily verified by straightforward computations and use of the recurrence relation (4.1) for Horner shifts.

LEMMA 4.3. *Let Q_i, R_i, T_i and D_i be the matrices introduced in* Definition 4.2, *and M_j the matrices in* (3.1) *and* (3.2). *Then the following unimodular equivalences hold for $i = 1, \ldots, k-1$.*

(a) $Q_i^{\mathcal{B}}\big(\lambda D_i\big) R_i \;=\; \lambda D_{i+1} + T_i\,,$ *and* $Q_i^{\mathcal{B}}\big(M_{k-(i+1)} M_{k-i}\big) R_i \;=\; M_{k-(i+1)} + T_i\,.$

(b) $R_i^{\mathcal{B}}\big(\lambda D_i\big) Q_i \;=\; \lambda D_{i+1} + T_i^{\mathcal{B}},$ *and* $R_i^{\mathcal{B}}\big(M_{k-i} M_{k-(i+1)}\big) Q_i \;=\; M_{k-(i+1)} + T_i^{\mathcal{B}}.$

The following relations also hold for $i = 1, \ldots, k-1$.

(c) $T_i M_j \;=\; M_j T_i \;=\; T_i$ *and* $T_i^{\mathcal{B}} M_j \;=\; M_j T_i^{\mathcal{B}} \;=\; T_i^{\mathcal{B}}$ *for all* $j \le k-i-2$.

The next definition introduces the pencils which will form the intermediate steps in the unimodular transformation of a Fiedler pencil $F_\sigma(\lambda)$ into $\operatorname{diag}\big[I_{(k-1)n},\, P(\lambda)\big]$.

DEFINITION 4.4. *Let* $P(\lambda) \;=\; \sum_{i=0}^{k} \lambda^i A_i$, *and let* $F_\sigma(\lambda) \;=\; \lambda M_k \,-\, M_\sigma$ *be the Fiedler pencil of $P(\lambda)$ associated with the bijection σ. For $i = 1, \ldots, k$ define*

$$M_\sigma^{(i)} := \prod_{\sigma^{-1}(j)\,\le\, k-i} M_{\sigma^{-1}(j)}\,,$$

where the factors $M_{\sigma^{-1}(j)}$ are in the same relative order as they are in M_σ. Equivalently, $M_\sigma^{(i)}$ is obtained from M_σ by deleting all factors M_ℓ with index $\ell > k-i$. Note that $M_\sigma^{(1)} = M_\sigma$ and $M_\sigma^{(k)} = M_0$. Also for $i = 1, \ldots, k$ define the $nk \times nk$ pencils

$$F_\sigma^{(i)}(\lambda) \;:=\; \lambda D_i(\lambda) \;-\; M_\sigma^{(i)}\,.$$

Observe that $F_\sigma^{(1)}(\lambda) = F_\sigma(\lambda)$ and $F_\sigma^{(k)}(\lambda) = \operatorname{diag}\big[\,-I_{(k-1)n},\, P(\lambda)\big]$.

The final technical lemma shows explicitly how to transform each $F_\sigma^{(i)}(\lambda)$ into $F_\sigma^{(i+1)}(\lambda)$ by unimodular transformations.

LEMMA 4.5. *For each $i = 1, \ldots, k-1$, $F_\sigma^{(i)}(\lambda)$ is unimodularly equivalent to $F_\sigma^{(i+1)}(\lambda)$. Specifically, if Q_i and R_i are the unimodular matrices introduced in* Definition 4.2*, then*:

(a) *If σ has a* consecution *at $k-i-1$, then* $F_\sigma^{(i+1)}(\lambda) \;=\; Q_i^{\mathcal{B}}\, F_\sigma^{(i)}(\lambda)\, R_i,$

(b) *If σ has an* inversion *at $k-i-1$, then* $F_\sigma^{(i+1)}(\lambda) \;=\; R_i^{\mathcal{B}}\, F_\sigma^{(i)}(\lambda)\, Q_i.$

Proof. We prove only part (a), using Lemma 4.3(a,c); part (b) is proved similarly using Lemma 4.3(b,c). Suppose, then, that σ has a consecution at $k-i-1$, i.e., M_{k-i-1} is to the *left* of M_{k-i} in the product $M_\sigma^{(i)}$. Since M_{k-i} has the highest index among all factors in $M_\sigma^{(i)}$, by the commutativity relations (3.7) we may shift M_{k-i} leftwards until it is adjacent to M_{k-i-1}, that is $M_\sigma^{(i)} = \cdots M_{k-i-1}\, M_{k-i} \cdots$. Now $Q_i^{\mathcal{B}}$ and R_i commute with M_j for all $j \le k-i-2$, so we have

$$\begin{aligned} Q_i^{\mathcal{B}} M_\sigma^{(i)} R_i \;&=\; \cdots \big(Q_i^{\mathcal{B}} M_{k-i-1} M_{k-i} R_i\big) \cdots \\ &=\; \cdots \big(M_{k-i-1} + T_i\big) \cdots && \text{by Lemma 4.3(a)}\,, \\ &=\; M_\sigma^{(i+1)} + T_i && \text{by Lemma 4.3(c)}\,. \end{aligned}$$

Hence

$$\begin{aligned} Q_i^{\mathcal{B}}\, F_\sigma^{(i)}(\lambda)\, R_i \;&=\; Q_i^{\mathcal{B}}\, \big(\lambda D_i \,-\, M_\sigma^{(i)}\big)\, R_i \\ &=\; Q_i^{\mathcal{B}}\, \big(\lambda D_i\big) R_i \,-\, Q_i^{\mathcal{B}} \big(M_\sigma^{(i)}\big)\, R_i \\ &=\; \big(\lambda D_{i+1} \,+\, T_i\big) \,-\, \big(M_\sigma^{(i+1)} \,+\, T_i\big) && \text{by Lemma 4.3(a)}\,, \\ &=\; \lambda D_{i+1} \,-\, M_\sigma^{(i+1)} \;=\; F_\sigma^{(i+1)}(\lambda)\,, \end{aligned}$$

as desired. ∎

4.2. Strong linearizations via explicit unimodular equivalences. We are now in a position to prove the main result of Section 4, i.e., that every Fiedler pencil is a strong linearization. Note that the explicit unimodular equivalence constructed for this purpose is extracted from the proof and presented as Corollary 4.7, to facilitate the development of recovery formulas for minimal indices and bases in Section 5.

THEOREM 4.6. *Let $P(\lambda)$ be an $n \times n$ matrix polynomial (singular or regular). Then any Fiedler companion pencil $F_\sigma(\lambda)$ of $P(\lambda)$ is a strong linearization for $P(\lambda)$.*

Proof. To prove that $F_\sigma(\lambda)$ is a linearization for $P(\lambda)$, we need a unimodular equivalence of the form (2.3) between $F_\sigma(\lambda)$ and diag $\left[I_{(k-1)n},\, P(\lambda)\right]$. Such an equivalence can be explicitly constructed from Lemma 4.5 as the composition of a sequence of $k-1$ unimodular equivalences

$$F_\sigma(\lambda) = F_\sigma^{(1)}(\lambda) \longrightarrow F_\sigma^{(2)}(\lambda) \longrightarrow \cdots \longrightarrow F_\sigma^{(k)}(\lambda) = \operatorname{diag}\left[\,-I_{(k-1)n},\, P(\lambda)\right] \tag{4.2}$$

where $$F_\sigma^{(i+1)}(\lambda) \;=\; \begin{cases} Q_i^{\mathcal{B}}\, F_\sigma^{(i)}(\lambda)\, R_i & \text{if } \sigma \text{ has a } \textit{consecution} \text{ at } k-i-1 \\ R_i^{\mathcal{B}}\, F_\sigma^{(i)}(\lambda)\, Q_i & \text{if } \sigma \text{ has an } \textit{inversion} \text{ at } k-i-1\,, \end{cases}$$

together with a final pre-multiplication by diag $\left[\,-I_{(k-1)n},\, I_n\right]$. This completes the proof that $F_\sigma(\lambda)$ is a linearization for $P(\lambda)$.

To see why $F_\sigma(\lambda)$ is a *strong* linearization for $P(\lambda)$, all that remains is to prove that $\operatorname{rev} F_\sigma(\lambda)$ is a linearization for $\operatorname{rev} P(\lambda)$. The crux of the argument is to show that the pencil $-\operatorname{rev} F_\sigma(\lambda)$ is strictly equivalent to one of the Fiedler pencils of the polynomial $-\operatorname{rev} P(\lambda)$. By the argument in the first paragraph it would then follow that $-\operatorname{rev} F_\sigma(\lambda)$ is unimodularly equivalent to diag $\left[-I_{(k-1)n}, -\operatorname{rev} P(\lambda)\right]$, and hence that $\operatorname{rev} F_\sigma(\lambda)$ is unimodularly equivalent to diag $\left[I_{(k-1)n}, \operatorname{rev} P(\lambda)\right]$, thus completing the proof of the theorem.

So let us consider $-\operatorname{rev} F_\sigma(\lambda)$. If $F_\sigma(\lambda)$ is given by (3.3), then we may write

$$-\operatorname{rev} F_\sigma(\lambda) \;=\; \lambda(M_{\sigma^{-1}(1)} \cdots M_{\sigma^{-1}(s-1)} M_0 M_{\sigma^{-1}(s+1)} \cdots M_{\sigma^{-1}(k)}) - M_k\,,$$

where $s = \sigma(0)$. Pre and post-multiplying in the appropriate order by the inverses of $M_1, M_2, \ldots, M_{k-1}$, we see that $-\operatorname{rev} F_\sigma(\lambda)$ is strictly equivalent to

$$\lambda M_0 - \left(M_{\sigma^{-1}(s-1)}^{-1} \cdots M_{\sigma^{-1}(1)}^{-1} M_k M_{\sigma^{-1}(k)}^{-1} \cdots M_{\sigma^{-1}(s+1)}^{-1}\right),$$

which in turn is strictly equivalent to

$$\lambda(BM_0B) - B\left(M_{\sigma^{-1}(s-1)}^{-1} \cdots M_{\sigma^{-1}(1)}^{-1} M_k M_{\sigma^{-1}(k)}^{-1} \cdots M_{\sigma^{-1}(s+1)}^{-1}\right)B\,, \tag{4.3}$$

where
$$B := \begin{bmatrix} & & I_n \\ & \iddots & \\ I_n & & \end{bmatrix}$$

is the block $k \times k$ backwards "identity" matrix, satisfying $B^2 = I_{kn}$. Now define

$$\widehat{M_0} := BM_0B = \begin{bmatrix} -A_0 & \\ & I_{(k-1)n} \end{bmatrix}, \qquad \widehat{M_k} := BM_kB = \begin{bmatrix} I_{(k-1)n} & \\ & A_k \end{bmatrix},$$

and
$$\widehat{M_i} := BM_i^{-1}B = \begin{bmatrix} I_{(i-1)n} & & & \\ & A_i & I_n & \\ & I_n & 0 & \\ & & & I_{(k-i-1)n} \end{bmatrix}, \text{ for } i = 1, \ldots, k-1\,.$$

With these definitions, the pencil (4.3) can be written as

$$\lambda \widehat{M}_0 - \left(\widehat{M}_{\sigma^{-1}(s-1)} \cdots \widehat{M}_{\sigma^{-1}(1)} \widehat{M}_k \widehat{M}_{\sigma^{-1}(k)} \cdots \widehat{M}_{\sigma^{-1}(s+1)}\right),$$

which is a Fiedler pencil for the polynomial $-\mathrm{rev}\, P(\lambda)$. This completes the proof. ▯

In Corollary 4.7 we accumulate the $k-1$ unimodular equivalences in (4.2), as a first step in developing recovery formulas for minimal indices and bases. Note that the (somewhat unexpected?) indexing of the U_i and V_i factors in $U(\lambda)$ and $V(\lambda)$ has been chosen here in order to simplify the notation in Section 5.

Corollary 4.7. *Let $P(\lambda)$ be the matrix polynomial in* (1.1)*, let $F_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ associated with the bijection σ, and let Q_i and R_i for $i = 1, \ldots, k-1$ be the matrices introduced in* Definition 4.2*. Then*

$$U(\lambda) F_\sigma(\lambda) V(\lambda) = \begin{bmatrix} -I_{(k-1)n} & 0 \\ 0 & P(\lambda) \end{bmatrix}, \tag{4.4}$$

where $U(\lambda)$ and $V(\lambda)$ are the following $nk \times nk$ unimodular matrix polynomials:

$$U(\lambda) := U_0 U_1 \cdots U_{k-3} U_{k-2}\,, \quad \textit{with} \quad U_i = \begin{cases} Q^{\mathcal{B}}_{k-(i+1)}\,, & \textit{if } \sigma \textit{ has a consecution at } i \\ R^{\mathcal{B}}_{k-(i+1)}\,, & \textit{if } \sigma \textit{ has an inversion at } i\,, \end{cases}$$

$$V(\lambda) := V_{k-2} V_{k-3} \cdots V_1 V_0\,, \quad \textit{with} \quad V_i = \begin{cases} R_{k-(i+1)}\,, & \textit{if } \sigma \textit{ has a consecution at } i \\ Q_{k-(i+1)}\,, & \textit{if } \sigma \textit{ has an inversion at } i\,. \end{cases}$$

Example 4.8. *Consider again the block-pentadiagonal pencil $F^{\mathcal{P}}_\tau(\lambda)$ defined in* Example 3.5*. By* Corollary 4.7*, a unimodular equivalence that will transform this pencil to the block-diagonal form in* (4.4) *is* $\left(Q_5^{\mathcal{B}} Q_4^{\mathcal{B}} R_3^{\mathcal{B}} Q_2^{\mathcal{B}} R_1^{\mathcal{B}}\right) F^{\mathcal{P}}_\tau(\lambda) \left(Q_1 R_2 Q_3 R_4 R_5\right)$.

5. Recovery of minimal indices and bases. In this section we deal only with singular square matrix polynomials, since regular polynomials do not have any minimal indices or bases. The recovery of the minimal indices and bases of a singular polynomial from those of any of its Fiedler pencils is based on Lemma 5.1, which is *valid for any linearization*, not just for Fiedler pencils.

Lemma 5.1. *Let the $nk \times nk$ pencil $L(\lambda)$ be a linearization of an $n \times n$ matrix polynomial $P(\lambda)$, and let $U(\lambda)$ and $V(\lambda)$ be unimodular matrices such that*

$$U(\lambda) L(\lambda) V(\lambda) = \begin{bmatrix} \pm I_{(k-1)n} & 0 \\ 0 & P(\lambda) \end{bmatrix}. \tag{5.1}$$

Viewing $U(\lambda)$ and $V(\lambda)$ as block $k \times k$ matrices with $n \times n$ blocks, let $U^L = U^L(\lambda)$ be the last block-row of $U(\lambda)$, and $V^R = V^R(\lambda)$ the last block-column of $V(\lambda)$. Then:

(a) *The linear map*
$$\begin{aligned} \mathcal{L} :\ \mathcal{N}_\ell(P) &\longrightarrow \mathcal{N}_\ell(L) \\ w^T(\lambda) &\longmapsto w^T(\lambda) \cdot U^L \end{aligned}$$
is an isomorphism of $\mathbb{F}(\lambda)$-vector spaces.

(b) *The linear map*
$$\begin{aligned} \mathcal{R} :\ \mathcal{N}_r(P) &\longrightarrow \mathcal{N}_r(L) \\ v(\lambda) &\longmapsto V^R \cdot v(\lambda) \end{aligned}$$
is an isomorphism of $\mathbb{F}(\lambda)$-vector spaces.

Proof. We prove only (b); part (a) is similar. For $v \in \mathcal{N}_r(P)$, let $\widetilde{v} := [0^T v^T]^T \in \mathbb{F}(\lambda)^{nk}$ and $\widetilde{P}(\lambda) := \mathrm{diag}\left[\pm I_{(k-1)n},\, P(\lambda)\right]$. Then the map $\mathcal{R}$ is well-defined because $P(\lambda) v = 0 \Rightarrow \widetilde{P}(\lambda)\widetilde{v} = 0 \Rightarrow U(\lambda)L(\lambda)V(\lambda)\widetilde{v} = 0 \Rightarrow L(\lambda)(V^R v) = 0$. The columns of V^R are linearly independent in $\mathbb{F}(\lambda)^{nk}$ since $V(\lambda)$ is unimodular, which implies that $\mathcal{R}$ is injective. Since $\dim \mathcal{N}_r(P) = \dim \mathcal{N}_r(L)$ from (5.1), $\mathcal{R}$ is an isomorphism. ▯

Lemma 5.1 implies that every basis of $\mathcal{N}_r(L)$ is of the form $\mathcal{B}_r = \{V^R v_1, \ldots, V^R v_p\}$, where $\mathcal{E}_r = \{v_1, \ldots, v_p\}$ is a basis of $\mathcal{N}_r(P)$ that is uniquely determined by $\mathcal{B}_r$. However, this does not in general mean either that $\mathcal{E}_r$ may be easily obtained from $\mathcal{B}_r$, or that $\mathcal{E}_r$ is a minimal basis of $P(\lambda)$ whenever $\mathcal{B}_r$ is a minimal basis of $L(\lambda)$, or that the minimal indices of $P(\lambda)$ are simply related to those of $L(\lambda)$. In the particular case of Fiedler linearizations, though, we will see that $\mathcal{E}_r$ *is* immediately recoverable from $\mathcal{B}_r$, because one of the blocks of $V^R(\lambda)$ will always be equal to I_n. Furthermore, we will prove that $\mathcal{E}_r$ *is* a minimal basis whenever $\mathcal{B}_r$ is, and we will show that the minimal indices of $P(\lambda)$ *are* simply obtained from the ones of $F_\sigma(\lambda)$, by the uniform substraction of a constant that is easily determined from σ. To show all of this will require a careful analysis of the last block column $V^R(\lambda)$ of the unimodular matrix $V(\lambda)$ specified in Corollary 4.7. Analogous results also hold for left minimal indices and bases.

5.1. The last block column of $V(\lambda)$. Corollary 4.7 shows that $V(\lambda)$ is built up out of products of Q_i and R_i matrices. Lemma 5.2 is a first step in seeing what such products can look like, focusing on the especially simple case of products of Q_i's (or R_i's) with *consecutive* indices. The proof is a straightforward induction on the number of factors, and so is omitted. Note that in this section, identity and zero blocks of size $n \times n$ will be denoted simply by I and 0.

LEMMA 5.2. *Let $P_d(\lambda)$ for $d = 0, \ldots, k$ be the Horner shifts of the matrix polynomial $P(\lambda)$ in* (1.1)*, and let Q_i and R_i for $i = 1, \ldots, k-1$ be the matrices introduced in* Definition 4.2*. Then for each $i = 1, \ldots, k-1$ and $j = 1, \ldots, k-i$, the product of j consecutively indexed Q's starting at Q_i is*

$$\mathcal{Q}(i,j) := Q_i Q_{i+1} \cdots Q_{i+j-1} = \left[\begin{array}{c|cccc|c} I_{(i-1)n} & & & & & \\ \hline & I & \lambda I & \ldots & \lambda^j I & \\ & & \ddots & \ddots & \vdots & \\ & & & \ddots & \lambda I & \\ & & & & I & \\ \hline & & & & & I_{(k-(i+j))n} \end{array}\right],$$

while the product of j consecutively indexed R's starting at R_i is

$$\mathfrak{R}(i,j) := R_i R_{i+1} \cdots R_{i+j-1} = \left[\begin{array}{c|cccccc|c} I_{(i-1)n} & & & & & & \\ \hline & 0 & 0 & \ldots & 0 & I & \\ & I & 0 & \ldots & 0 & P_i(\lambda) & \\ & & \ddots & \ddots & \vdots & \vdots & \\ & & & \ddots & 0 & P_{i+j-2}(\lambda) & \\ & & & & I & P_{i+j-1}(\lambda) & \\ \hline & & & & & & I_{(k-(i+j))n} \end{array}\right].$$

By grouping together consecutively indexed runs of Q_i's (and R_i's) in the product expression for $V(\lambda)$, we can analyze the last block column of $V(\lambda)$ in terms of the simple block matrices in Lemma 5.2. Each Q_i (resp., R_i) factor in $V(\lambda)$ corresponds to an inversion (resp., consecution) in the bijection σ, so the pattern of consecutively indexed runs of Q_i's and R_i's in $V(\lambda)$ is encoded in the consecution-inversion structure sequence introduced in Definition 3.3; the CISS(σ) will thus be used frequently in this section. In particular, the individual entries $(c_1, i_1, \ldots, c_\ell, i_\ell)$ as well as the partial sums

$$s_0 := 0, \quad s_j := \sum_{p=1}^{j} (c_p + i_p) \quad \text{for } j = 1, \ldots, \ell$$

will play a key role. Recall that $s_\ell = k-1$. We will also need the quantities

$$m_0 := 0\,, \qquad m_j := i_1 + i_2 + \cdots + i_j\,, \;\; \text{for} \;\; j = 1, \ldots, \ell\,. \tag{5.2}$$

Note that $m_\ell = \mathfrak{i}(\sigma)$, that is, the total number of inversions in σ.

In order to write down a reasonably simple formula for the last block column of $V(\lambda)$ from Corollary 4.7, we need to define some block column matrices associated with the matrix polynomial $P(\lambda)$ in (1.1) and the bijection σ. These matrices are denoted $\Lambda_{\sigma,j}(P)$ for $j = 1, \ldots, \ell$, and $\widehat{\Lambda}_{\sigma,j}(P)$ for $j = 1, \ldots, \ell-1$, and are defined in terms of the Horner shifts of $P(\lambda)$ and $\text{CISS}(\sigma) = (c_1, i_1, \ldots, c_\ell, i_\ell)$ as follows:

$$\Lambda_{\sigma,j}(P) := \begin{bmatrix} \lambda^{i_j} I \\ \vdots \\ \lambda I \\ I \\ P_{k-s_{j-1}-c_j} \\ \vdots \\ P_{k-s_{j-1}-2} \\ P_{k-s_{j-1}-1} \end{bmatrix} \quad \text{and} \quad \widehat{\Lambda}_{\sigma,j}(P) := \begin{bmatrix} \lambda^{i_j-1} I \\ \vdots \\ \lambda I \\ I \\ P_{k-s_{j-1}-c_j} \\ \vdots \\ P_{k-s_{j-1}-2} \\ P_{k-s_{j-1}-1} \end{bmatrix} \quad \text{if } c_1 \geq 1\,, \tag{5.3}$$

but if $c_1 = 0$ then $\Lambda_{\sigma,1}(P) := [\lambda^{i_1} I, \ldots, \lambda I, I]^{\mathcal{B}}$, $\widehat{\Lambda}_{\sigma,1}(P) := [\lambda^{i_1-1} I, \ldots, \lambda I, I]^{\mathcal{B}}$, with $\Lambda_{\sigma,j}(P)$, $\widehat{\Lambda}_{\sigma,j}(P)$ as in (5.3) for $j > 1$. Here for simplicity we omit λ from the Horner shifts $P_d(\lambda)$. Note that $\Lambda_{\sigma,j}(P)$ and $\widehat{\Lambda}_{\sigma,j}(P)$ are associated with the entries (c_j, i_j) of $\text{CISS}(\sigma)$, and that $\widehat{\Lambda}_{\sigma,j}(P)$ is just a "truncated" version of $\Lambda_{\sigma,j}(P)$, with one less block at the top. Note also that $\widehat{\Lambda}_{\sigma,j}(P)$ is defined only for $j < \ell$, so that $i_j - 1 \geq 0$.

LEMMA 5.3. *Let $P(\lambda)$ be the matrix polynomial in* (1.1)*, let $F_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ associated with the bijection σ, and let $V(\lambda)$ be the $nk \times nk$ unimodular matrix polynomial in Corollary* 4.7*, viewed as a $k \times k$ block matrix with $n \times n$ blocks. If* $\text{CISS}(\sigma) = (c_1, i_1, \ldots, c_\ell, i_\ell)$*, then the last block-column $V^R(\lambda)$ of $V(\lambda)$ is*

$$\Lambda^R_\sigma(P) := \begin{bmatrix} \lambda^{m_{\ell-1}} \Lambda_{\sigma,\ell}(P) \\ \lambda^{m_{\ell-2}} \widehat{\Lambda}_{\sigma,\ell-1}(P) \\ \vdots \\ \lambda^{m_1} \widehat{\Lambda}_{\sigma,2}(P) \\ \widehat{\Lambda}_{\sigma,1}(P) \end{bmatrix} \quad \text{if } \ell > 1\,, \tag{5.4}$$

and $V^R(\lambda) = \Lambda_{\sigma,1}(P) =: \Lambda^R_\sigma(P)$ if $\ell = 1$.

Proof. First observe that, using the information in $\text{CISS}(\sigma) = (c_1, i_1, \ldots, c_\ell, i_\ell)$, the factors defining $V(\lambda)$ in Corollary 4.7 can be grouped together into the form $V(\lambda) = \widetilde{V}_\ell \cdots \widetilde{V}_2 \widetilde{V}_1$, where

$$\widetilde{V}_j \;:=\; \mathcal{Q}(k - s_j\,, i_j) \cdot \mathfrak{R}(k - s_j + i_j\,, c_j)$$

is associated with the pair $(c_j\,, i_j)$ from $\text{CISS}(\sigma)$, and consists of i_j consecutive Q-factors and c_j consecutive R-factors as in Lemma 5.2. A direct multiplication gives

$$\widetilde{V}_j \;=\; \left[\begin{array}{c|c|c} I_{(k-s_j-1)n} & & \\ \hline & * \;\; * \;\; \cdots \;\; * \;\; \Lambda_{\sigma,j}(P) & \\ \hline & & I_{(s_{j-1})\,n} \end{array}\right]. \tag{5.5}$$

In the block-diagonal partitioning of $\widetilde{V}_j$ in (5.5), let us from now on refer to the large non-identity diagonal block $\begin{bmatrix} * \;\; * \;\; \cdots \;\; * \;\; \Lambda_{\sigma,j}(P) \end{bmatrix}$ as the "central part" of $\widetilde{V}_j$. Then the central part of $\widetilde{V}_j$ is a $(c_j + i_j + 1) \times (c_j + i_j + 1)$ block matrix with $n \times n$ blocks. Note that the first $c_j + i_j$ block-columns are of no relevance to the last block column of $V(\lambda)$, and so are denoted by $*$.

Next observe that the central part of $\widetilde{V}_j$ overlaps the central part of the adjacent grouped factors $\widetilde{V}_{j+1}$ and $\widetilde{V}_{j-1}$. For example, the last block row and last block column of the central part of $\widetilde{V}_j$ have the same block index as the first block row and first block column of the central part of $\widetilde{V}_{j-1}$. This overlap causes some nontrivial interaction when multiplying all the $\widetilde{V}_j$ factors together to get the last block column of $V(\lambda)$. Multiplying out this product from right to left, i.e.,

$$V(\lambda) \;=\; \widetilde{V}_\ell\Big(\widetilde{V}_{\ell-1}\cdots\big(\widetilde{V}_3(\widetilde{V}_2\widetilde{V}_1)\big)\cdots\Big),$$

using (5.5), and taking into account the overlap of the central parts, it is straightforward to see (inductively) that for $j = 1, \dots, \ell$ we have

$$\widetilde{V}_j\widetilde{V}_{j-1}\cdots\widetilde{V}_1 = \left[\begin{array}{c|cccc} I_{(k-s_j-1)n} & & & & \\ \hline & * & \dots & * & \lambda^{m_{j-1}}\Lambda_{\sigma,j}(P) \\ & * & \dots & * & \lambda^{m_{j-2}}\widehat{\Lambda}_{\sigma,j-1}(P) \\ & \vdots & & \vdots & \vdots \\ & * & \dots & * & \lambda^{m_1}\widehat{\Lambda}_{\sigma,2}(P) \\ & * & \dots & * & \widehat{\Lambda}_{\sigma,1}(P) \end{array}\right].$$

The desired result now follows by taking $j = \ell$ in this identity. □

REMARK 5.4. It is important to highlight two key features of the matrix $\Lambda^R_\sigma(P)$ in (5.4) that are essential in the recovery of right minimal indices and bases of $P(\lambda)$ from those of $F_\sigma(\lambda)$:

(a) $\Lambda^R_\sigma(P)$ always has *exactly one block equal to* I_n; it resides in the block segment $\widehat{\Lambda}_{\sigma,1}(P)$ (or in $\Lambda_{\sigma,1}(P)$ if $\ell = 1$) at block index $k - c_1$, i.e., in the $(c_1 + 1)^{\text{th}}$ block counting from the bottom of $\Lambda^R_\sigma(P)$.

(b) The *topmost block* of $\Lambda^R_\sigma(P)$ is always equal to $\lambda^{\mathfrak{i}(\sigma)} I_n$. This is because the topmost block of $\lambda^{m_{\ell-1}}\Lambda_{\sigma,\ell}(P)$ is $\lambda^{m_{\ell-1}}\lambda^{i_\ell} I_n = \lambda^{m_\ell} I_n = \lambda^{\mathfrak{i}(\sigma)} I_n$.

These features of $\Lambda^R_\sigma(P)$ can be clearly seen in the following example.

EXAMPLE 5.5. *Consider the pencil $F^{\mathcal{P}}_\tau(\lambda)$ from* Examples 3.5 *and* 4.8 *for a polynomial $P(\lambda)$ of degree $k = 6$. This pencil has* $\mathrm{CISS}(\tau) = (c_1, i_1, c_2, i_2) = (2, 1, 1, 1)$, *so $\ell = 2$, $s_{\ell-1} = s_1 = 3$, $m_{\ell-1} = m_1 = i_1 = 1$, and $\mathfrak{i}(\tau) = 2$. Thus we have* $\widehat{\Lambda}_{\tau,1}(P) = \big[\, I_n \; P_{k-2} \; P_{k-1} \,\big]^{\mathcal{B}}$, $\Lambda_{\tau,2}(P) = \big[\, \lambda I_n \; I_n \; P_{k-4} \,\big]^{\mathcal{B}}$, *and hence*

$$\Lambda^R_\tau(P) \;=\; \begin{bmatrix} \lambda^{m_1}\Lambda_{\tau,2}(P) \\ \widehat{\Lambda}_{\tau,1}(P) \end{bmatrix} \;=\; \big[\, \lambda^2 I_n \;\; \lambda I_n \;\; \lambda P_2(\lambda) \;\; I_n \;\; P_4(\lambda) \;\; P_5(\lambda) \,\big]^{\mathcal{B}}.$$

5.2. Horner shifts of singular polynomials. One final technical lemma is needed to establish the relationship between the right minimal indices and bases of a matrix polynomial and those of its Fiedler pencils. This result concerns the action of the Horner shifts $P_d(\lambda)$ of a *singular* matrix polynomial $P(\lambda)$ on any right null vector $v(\lambda)$ of $P(\lambda)$. We show that either $v(\lambda)$ is also a null vector of $P_d(\lambda)$, or else that at the very least the action of $P_d(\lambda)$ *reduces* the degree by at least one.

LEMMA 5.6. *Let $P(\lambda)$ be an $n \times n$ matrix polynomial of degree k as in* (1.1), *with degree j Horner shift $P_j(\lambda)$. Suppose $v(\lambda) \in \mathcal{N}_r(P)$ is a vector polynomial such that $P_j(\lambda)v(\lambda) \neq 0$. Then*

$$\deg\big(P_j(\lambda)v(\lambda)\big) \;\le\; \deg\big(v(\lambda)\big) - 1\,. \tag{5.6}$$

Proof. If $\deg\big(v(\lambda)\big) = 0$, then $v(\lambda) = v$ would be a constant vector such that $v \in \mathcal{N}_r(A_i)$ for all $0 \le i \le k$, and so $P_j(\lambda)v = 0$ for all $0 \le j \le k$. Thus $P_j(\lambda)v(\lambda) \ne 0$ implies that $\deg\big(v(\lambda)\big) \ge 1$, and hence that the right-hand side of (5.6) is non-negative. Also note that $P_j(\lambda)v(\lambda) \ne 0$ implies that $j < k$.

By Definition 4.1 we have $P_j(\lambda)v(\lambda) = \big(\lambda^j A_k + \lambda^{j-1}A_{k-1} + \cdots + A_{k-j}\big)v(\lambda)\,,$ and so

$$\begin{aligned} \lambda^{k-j}P_j(\lambda)v(\lambda) &= \big(\lambda^{k-j}P_j(\lambda) - P(\lambda)\big)v(\lambda) \\ &= -\big(\lambda^{k-j-1}A_{k-j-1} + \cdots + \lambda A_1 + A_0\big)v(\lambda)\,. \end{aligned}$$

Since $P_j(\lambda)v(\lambda) \ne 0$ we have

$$\begin{aligned} (k-j) + \deg\big(P_j(\lambda)v(\lambda)\big) &= \deg\big(\lambda^{k-j}P_j(\lambda)v(\lambda)\big) \\ &= \deg\big((\lambda^{k-j-1}A_{k-j-1} + \cdots + \lambda A_1 + A_0)v(\lambda)\big) \\ &\le (k-j-1) + \deg\big(v(\lambda)\big)\,, \end{aligned}$$

and hence $\deg\big(P_j(\lambda)v(\lambda)\big) \le \deg\big(v(\lambda)\big) - 1$, as desired. ☐

5.3. Right minimal indices and bases. We first establish a degree-shifting isomorphism between $\mathcal{N}_r(P)$ and $\mathcal{N}_r(F_\sigma)$ in Theorem 5.7. Then as an immediate consequence, we obtain in Corollary 5.8 the simple recovery formulas for right minimal indices and bases, one of the main results in this paper.

THEOREM 5.7. *Let $P(\lambda)$ be an $n \times n$ matrix polynomial as in* (1.1)*, let $F_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ associated with bijection σ, let $\mathfrak{i}(\sigma)$ be the total number of inversions of σ, and let $\Lambda_\sigma^R(P)$ be the $nk \times n$ matrix defined in* (5.4)*. Then the linear map*

$$\begin{array}{rcll} \mathcal{R}_\sigma : \mathcal{N}_r(P) & \longrightarrow & \mathcal{N}_r(F_\sigma) \\ v & \longmapsto & \Lambda_\sigma^R(P)\,v \end{array} \tag{5.7}$$

is an isomorphism of $\mathbb{F}(\lambda)$-vector spaces with uniform degree-shift $\mathfrak{i}(\sigma)$ on the vector polynomials in $\mathcal{N}_r(P)$. More precisely, $\mathcal{R}_\sigma$ induces a bijection between the subsets of vector polynomials in $\mathcal{N}_r(P)$ and $\mathcal{N}_r(F_\sigma)$, with the property that

$$\deg \mathcal{R}_\sigma(v) = \mathfrak{i}(\sigma) + \deg v \tag{5.8}$$

for every nonzero vector polynomial $v \in \mathcal{N}_r(P)$. Furthermore, for any nonzero vector polynomial v, $\deg \mathcal{R}_\sigma(v)$ is attained only in the topmost $n \times 1$ block of $\mathcal{R}_\sigma(v)$.

Proof. The fact that $\mathcal{R}_\sigma$ is an isomorphism is a special case of Lemma 5.1(b), since by Lemma 5.3 the matrix $\Lambda_\sigma^R(P)$ is the last block-column of $V(\lambda)$ in Corollary 4.7.

The form of $\Lambda_\sigma^R(P)$ guarantees that $\mathcal{R}_\sigma(v)$ is a vector polynomial whenever v is, and, because of the identity block in $\Lambda_\sigma^R(P)$ at block index $k - c_1$, that $\mathcal{R}_\sigma(v)$ is a *non*-polynomial vector rational function whenever v is *non*-polynomial. Thus $\mathcal{R}_\sigma$ restricts to a bijection between the vector polynomials in $\mathcal{N}_r(P)$ and those in $\mathcal{N}_r(F_\sigma)$.

To see why the uniform degree-shifting property (5.8) holds, first observe that there are only two different types of blocks in $\Lambda_\sigma^R(P)$:

$$\lambda^p I \text{ with } 0 \le p \le \mathfrak{i}(\sigma)\,, \quad \text{and} \quad \lambda^q P_j(\lambda) \text{ with } 0 \le q \le m_{\ell-1} \le \mathfrak{i}(\sigma)\,.$$

Thus $\mathcal{R}_\sigma(v)$ is made up of blocks of the form $\lambda^p v$ and $\lambda^q P_j(\lambda)v$. Clearly for a nonzero vector polynomial $v \in \mathcal{N}_r(P)$ the maximum degree among all blocks of the form $\lambda^p v$ is $\mathfrak{i}(\sigma) + \deg v$, attained only in the topmost block of $\mathcal{R}_\sigma(v)$. Blocks of the form $\lambda^q P_j(\lambda)v$ are either 0 (if $P_j(\lambda)v = 0$), or by Lemma 5.6 have degree bounded by

$$\deg\big(\lambda^q P_j(\lambda)v\big) \le \mathfrak{i}(\sigma) + (\deg v) - 1 < \mathfrak{i}(\sigma) + \deg v\,.$$

Thus $\deg \mathcal{R}_\sigma(v) = \mathfrak{i}(\sigma) + \deg v$, with equality attained only in the topmost block of $\mathcal{R}_\sigma(v)$. ☐

COROLLARY 5.8 (**Recovery of right minimal indices and bases**). *Let $P(\lambda)$ be an $n\times n$ singular matrix polynomial with degree $k\geq 2$, and let $F_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ associated with a bijection σ having* $\mathrm{CISS}(\sigma)=(c_1,i_1,\ldots,c_\ell,i_\ell)$ *and total number of inversions $\mathfrak{i}(\sigma)$. Also let $nk\times 1$ vectors be partitioned as $k\times 1$ block vectors with $n\times 1$ blocks.*

(a) *If $z(\lambda)\in\mathcal{N}_r(F_\sigma)\subseteq\mathbb{F}(\lambda)^{nk\times 1}$, and $x(\lambda)\in\mathbb{F}(\lambda)^{n\times 1}$ is the $(k-c_1)^{\text{th}}$ block of $z(\lambda)$, then $x(\lambda)\in\mathcal{N}_r(P)$.*

(b) *If $\mathcal{B}_r=\{z_1(\lambda),\ldots,z_p(\lambda)\}$ is a right minimal basis of $F_\sigma(\lambda)$, and $x_j(\lambda)$ is the $(k-c_1)^{\text{th}}$ block of $z_j(\lambda)$ for each $j=1,\ldots,p$, then $\mathcal{E}_r=\{x_1(\lambda),\ldots,x_p(\lambda)\}$ is a right minimal basis of $P(\lambda)$.*

(c) *If $0\leq\varepsilon_1\leq\varepsilon_2\leq\cdots\leq\varepsilon_p$ are the right minimal indices of $P(\lambda)$, then*

$$\varepsilon_1+\mathfrak{i}(\sigma)\ \leq\ \varepsilon_2+\mathfrak{i}(\sigma)\ \leq\cdots\leq\ \varepsilon_p+\mathfrak{i}(\sigma)$$

are the right minimal indices of $F_\sigma(\lambda)$.

Note that these results hold for the first companion form of $P(\lambda)$ by taking $c_1=0$ and $\mathfrak{i}(\sigma)=k-1$, and for the second companion form using $c_1=k-1$ and $\mathfrak{i}(\sigma)=0$.

Proof. Since $\mathcal{R}_\sigma$ in (5.7) is an isomorphism, and the $(k-c_1)^{\text{th}}$ block of $\Lambda_\sigma^R(P)$ is I_n, therefore $x(\lambda)$ in part (a) is just $\mathcal{R}_\sigma^{-1}\big(z(\lambda)\big)$, and hence in $\mathcal{N}_r(P)$. For part (b), $\mathcal{R}_\sigma$ being an isomorphism immediately implies that $\mathcal{E}_r=\mathcal{R}_\sigma^{-1}(\mathcal{B}_r)$ is a vector polynomial basis for $\mathcal{N}_r(P)$, but why is it a *minimal* basis? Suppose there was another polynomial basis $\widetilde{\mathcal{E}}_r=\{\widetilde{x}_1(\lambda),\ldots,\widetilde{x}_p(\lambda)\}$ of $\mathcal{N}_r(P)$ with order lower than $\mathcal{E}_r$. Then the uniform degree-shift property (5.8) would imply that the polynomial basis $\mathcal{R}_\sigma(\widetilde{\mathcal{E}}_r)$ had order lower than $\mathcal{R}_\sigma(\mathcal{E}_r)=\mathcal{B}_r$, a contradiction to $\mathcal{B}_r$ being a minimal basis of $F_\sigma(\lambda)$. Part (c) follows immediately from part (b) together with (5.8). □

We cannot overemphasize the utter simplicity of the final "recipe" for recovering right minimal indices and bases described in Corollary 5.8, as contrasted with the long, hard work needed to develop it. In the end none of the rather complicated structure of $\Lambda_\sigma^R(P)$ is used, and recovery can be achieved almost without effort; only the constants c_1 and $\mathfrak{i}(\sigma)$ from the bijection σ need to be determined.

5.4. Left minimal indices and bases. For the recovery of the left minimal indices and bases of $P(\lambda)$, we develop results analogous to Theorem 5.7 and Corollary 5.8 in Theorem 5.9 and Corollary 5.11, respectively. One obvious strategy for accomplishing this takes its lead from Lemma 5.1(a), and tries to find an expression for the last block-row of $U(\lambda)$ in Corollary 4.7 by doggedly imitating the construction of the last block-column of $V(\lambda)$ in Lemmas 5.2 and 5.3. Instead we adopt a different strategy, one that uses less brute force and gives more insight into the dual nature of left vs. right minimal index/basis recovery. This strategy builds up an appropriate isomorphism between $\mathcal{N}_\ell(P)$ and $\mathcal{N}_\ell(F_\sigma)$ as the composition of simpler maps, one of which is based on applying Theorem 5.7 to relate the *right* nullspaces of $P^T(\lambda)$ and the Fiedler pencil $F_{\operatorname{rev}\sigma}(P^T)$, where $\operatorname{rev}\sigma$ is the reversal bijection of σ introduced in Section 3.

THEOREM 5.9. *Let $P(\lambda)$ be an $n\times n$ matrix polynomial as in* (1.1), *let $F_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ associated with bijection σ, let $\mathfrak{c}(\sigma)$ be the total number of consecutions of σ, and let $\Lambda^R_{\operatorname{rev}\sigma}(P)$ be, for the reversal bijection* $\operatorname{rev}\sigma$, *the $nk\times n$ matrix defined in* (5.4). *Then the linear map*

$$\begin{array}{rccc}\mathcal{L}_\sigma: & \mathcal{N}_\ell(P) & \longrightarrow & \mathcal{N}_\ell(F_\sigma)\\ & u^T & \longmapsto & u^T\Lambda_\sigma^L(P)\,,\end{array}$$

where $\Lambda_\sigma^L(P):=\big[\Lambda^R_{\operatorname{rev}\sigma}(P)\big]^{\mathcal{B}}$, is an isomorphism of $\mathbb{F}(\lambda)$-vector spaces with uniform degree-shift $\mathfrak{c}(\sigma)$ on the vector polynomials in $\mathcal{N}_\ell(P)$. More precisely, $\mathcal{L}_\sigma$ induces a *bijection between the subsets of vector polynomials in* $\mathcal{N}_\ell(P)$ *and* $\mathcal{N}_\ell(F_\sigma)$*, with the property that*

$$\deg \mathcal{L}_\sigma(u^T) \;=\; \mathfrak{c}(\sigma) + \deg(u^T) \tag{5.9}$$

for every nonzero vector polynomial $u^T \in \mathcal{N}_\ell(P)$*. Furthermore, for any nonzero vector polynomial* u^T*,* $\deg \mathcal{L}_\sigma(u^T)$ *is attained only in the leftmost* $1 \times n$ *block of* $\mathcal{L}_\sigma(u^T)$*.*

Proof. We claim that $\mathcal{L}_\sigma$ can be expressed as a composition of three maps

$$\mathcal{N}_\ell(P) \xrightarrow{\;\Psi_1\;} \mathcal{N}_r\big(P^T\big) \xrightarrow{\;\Psi_2\;} \mathcal{N}_r\big(F_{\operatorname{rev}\sigma}(P^T)\big) \xrightarrow{\;\Psi_3\;} \mathcal{N}_\ell\big(F_\sigma(P)\big)\,,$$

each of which is an $\mathbb{F}(\lambda)$-vector space isomorphism that induces a bijection on vector polynomials. Furthermore, each of these polynomial bijections has its own uniform degree-shifting property.

(1) For $u^T \in \mathcal{N}_\ell(P)$, define the first map by $\Psi_1(u^T) := u$. Since $u^T P(\lambda) = 0^T$ if and only if $P^T(\lambda)u = 0$, we see immediately that Ψ_1 is an isomorphism, while the form of Ψ_1 clearly implies that it induces a degree-preserving bijection on vector polynomials.

(2) The map Ψ_2 is obtained by applying Theorem 5.7 to the transpose polynomial $P^T(\lambda)$ and the associated Fiedler pencil defined by the bijection $\operatorname{rev}\sigma$, i.e., $\Psi_2(v) := \Lambda^R_{\operatorname{rev}\sigma}(P^T)v$. Then by Theorem 5.7, Ψ_2 is an isomorphism inducing a bijection on vector polynomials with uniform degree-shift $\mathfrak{i}(\operatorname{rev}\sigma) = \mathfrak{c}(\sigma)$. Furthermore, $\deg \Psi_2(v)$ is attained only in the topmost block of $\Psi_2(v)$.

(3) The third map Ψ_3 is, like Ψ_1, just transpose of vectors, i.e., $\Psi_3(w) := w^T$. The same kind of argument as given above for Ψ_1 in step (1) shows that Ψ_3, viewed as a map $\Psi_3 : \mathcal{N}_r\big(F_{\operatorname{rev}\sigma}(P^T)\big) \longrightarrow \mathcal{N}_\ell\big(\big[F_{\operatorname{rev}\sigma}(P^T)\big]^T\big)$, is an isomorphism inducing a degree-preserving bijection on vector polynomials. All that remains is to see why $\big[F_{\operatorname{rev}\sigma}(P^T)\big]^T$ is the same pencil as $F_\sigma(P)$. From (3.8) we know that

$$F_{\operatorname{rev}\sigma}(P^T) \;=\; \lambda M_k(P^T) - M_{\sigma^{-1}(k)}(P^T)\cdots M_{\sigma^{-1}(2)}(P^T)\, M_{\sigma^{-1}(1)}(P^T)\,,$$

and so

$$\begin{aligned}\big[F_{\operatorname{rev}\sigma}(P^T)\big]^T &= \lambda\big[M_k(P^T)\big]^T - \big[M_{\sigma^{-1}(1)}(P^T)\big]^T\big[M_{\sigma^{-1}(2)}(P^T)\big]^T\cdots\big[M_{\sigma^{-1}(k)}(P^T)\big]^T\\ &= \lambda M_k(P) - M_{\sigma^{-1}(1)}(P)\, M_{\sigma^{-1}(2)}(P)\cdots M_{\sigma^{-1}(k)}(P)\\ &= F_\sigma(P)\,.\end{aligned}$$

Thus we see that the composition $\Psi_3\Psi_2\Psi_1(u^T) = \big[\Lambda^R_{\operatorname{rev}\sigma}(P^T)\,u\big]^T = u^T\big[\Lambda^R_{\operatorname{rev}\sigma}(P^T)\big]^T$ is the desired isomorphism $\mathcal{L}_\sigma$, since $\big[\Lambda^R_{\operatorname{rev}\sigma}(P^T)\big]^T = \big[\Lambda^R_{\operatorname{rev}\sigma}(P)\big]^{\mathcal{B}}$. ◻

To be able to write down the final "recipe" for left minimal index and basis recovery, we first need to find the position of the unique I_n block in $\Lambda^L_\sigma(P) := \big[\Lambda^R_{\operatorname{rev}\sigma}(P)\big]^{\mathcal{B}}$. Although elementary, this requires some care, and is done in the next lemma.

LEMMA 5.10. *Let* $\Lambda^L_\sigma(P) = \big[\Lambda^R_{\operatorname{rev}\sigma}(P)\big]^{\mathcal{B}}$ *be the* $n \times nk$ *matrix in Theorem* 5.9*, viewed as a block* $1 \times k$ *matrix with* $n \times n$ *blocks, and let* $\mathrm{CISS}(\sigma) = (c_1, i_1, \ldots, c_\ell, i_\ell)$*. Then* $\Lambda^L_\sigma(P)$ *has exactly one block equal to* I_n*, residing at block index*

$$\begin{cases} k & \text{if } c_1 > 0\,,\\ k - i_1 & \text{if } c_1 = 0\,.\end{cases}$$

Proof. First recall from Remark 5.4 that $\Lambda^R_\sigma(P)$ has exactly one block equal to I_n at block index $k - c_1$. Observe that σ has a consecution (resp., an inversion) at j if and only if $\operatorname{rev}\sigma$ has an inversion (resp., a consecution) at j for $j = 0, \ldots, k-2$.

Therefore, if $c_1 > 0$ then the c_1 initial consecutions of σ correspond to c_1 initial inversions in $\operatorname{rev}\sigma$, which implies $\text{CISS}(\operatorname{rev}\sigma) = (0, c_1, \ldots)$, and hence that $\Lambda^R_{\operatorname{rev}\sigma}(P)$ has exactly one block equal to I_n at block index k. On the other hand, if $c_1 = 0$ then the i_1 initial inversions of σ correspond to i_1 initial consecutions in $\operatorname{rev}\sigma$, which implies $\text{CISS}(\operatorname{rev}\sigma) = (i_1, \ldots)$, and hence that $\Lambda^R_{\operatorname{rev}\sigma}(P)$ has exactly one block equal to I_n at block index $k - i_1$. □

Combining Theorem 5.9 with Lemma 5.10 now allows us to state the recovery procedures for left minimal indices and bases in Corollary 5.11. Since the proof is similar to that of Corollary 5.8, it is omitted.

COROLLARY 5.11 (**Recovery of left minimal indices and bases**). *Let $P(\lambda)$ be an $n\times n$ singular matrix polynomial with degree $k \geq 2$, and let $F_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ associated with a bijection σ having* $\text{CISS}(\sigma) = (c_1, i_1, \ldots, c_\ell, i_\ell)$ *and total number of consecutions $\mathfrak{c}(\sigma)$. Also let $1 \times nk$ vectors be partitioned as $1 \times k$ block vectors with $1 \times n$ blocks.*

(a) *If $z(\lambda)^T \in \mathcal{N}_\ell(F_\sigma) \subseteq \mathbb{F}(\lambda)^{1\times nk}$, and*

$$y(\lambda)^T \ \text{ is the } \begin{cases} k^{\text{th}} \text{ block of } z(\lambda)^T & \text{if } c_1 > 0 \\ (k-i_1)^{\text{th}} \text{ block of } z(\lambda)^T & \text{if } c_1 = 0\,, \end{cases}$$

then $y(\lambda)^T \in \mathcal{N}_\ell(P)$.

(b) *If $\{z_1(\lambda)^T, \ldots, z_p(\lambda)^T\}$ is a left minimal basis of $F_\sigma(\lambda)$, and*

$$y_j(\lambda)^T \ \text{ is the } \begin{cases} k^{\text{th}} \text{ block of } z_j(\lambda)^T & \text{if } c_1 > 0 \\ (k-i_1)^{\text{th}} \text{ block of } z_j(\lambda)^T & \text{if } c_1 = 0\,, \end{cases}$$

for $j = 1, \ldots, p$, then $\{y_1(\lambda)^T, \ldots, y_p(\lambda)^T\}$ is a left minimal basis of $P(\lambda)$.

(c) *If $0 \leq \eta_1 \leq \eta_2 \leq \cdots \leq \eta_p$ are the left minimal indices of $P(\lambda)$, then*

$$\eta_1 + \mathfrak{c}(\sigma) \ \leq \ \eta_2 + \mathfrak{c}(\sigma) \ \leq \cdots \leq \ \eta_p + \mathfrak{c}(\sigma)$$

are the left minimal indices of $F_\sigma(\lambda)$.

Note that these results also hold for the first companion form using $(c_1, i_1) = (0, k-1)$ and $\mathfrak{c}(\sigma) = 0$, and for the second companion form taking $(c_1, i_1) = (k-1, 0)$ and $\mathfrak{c}(\sigma) = k-1$.

EXAMPLE 5.12. *Here we bring back the pencil $F^{\mathcal{P}}_\tau(\lambda)$ from* Examples 3.5, 4.8, *and* 5.5 *for one final bow, this time to illustrate all that we have proved about left and right minimal index and basis recovery. Recall that $\tau = (1, 2, 5, 3, 6, 4)$, so that* $\text{CISS}(\tau) = (2, 1, 1, 1)$. *Thus $\operatorname{rev}\tau = (6, 5, 2, 4, 1, 3)$, and hence* $\text{CISS}(\operatorname{rev}\tau) = (0, 2, 1, 1, 1, 0)$. *For $\operatorname{rev}\tau$ we then have $\ell = 3$, $s_1 = 2$, $s_{\ell-1} = s_2 = 4$, and $m_1 = 2$, $m_{\ell-1} = m_2 = 3$. For a polynomial $P(\lambda)$ of degree $k = 6$, this then results in*

$$\begin{aligned} \Lambda^L_\tau(P) \ = \ \left[\Lambda^R_{\operatorname{rev}\tau}(P)\right]^{\mathcal{B}} \ &= \ \left[\ \lambda^3 I_n \ \ \lambda^3 P_1(\lambda) \ \ \lambda^2 I_n \ \ \lambda^2 P_3(\lambda) \ \ \lambda I_n \quad I_n \ \right] \\ \text{and} \qquad \Lambda^R_\tau(P) \ &= \ \left[\ \lambda^2 I_n \quad \lambda I_n \quad \lambda P_2(\lambda) \quad I_n \quad P_4(\lambda) \ \ P_5(\lambda)\right]^{\mathcal{B}} . \end{aligned}$$

Observe how complementary Horner shifts of $P(\lambda)$ appear in complementary positions in $\Lambda^L_\tau(P)$ and $\Lambda^R_\tau(P)$.

The relationships between the minimal indices and bases of $F^{\mathcal{P}}_\tau(\lambda)$ and those of $P(\lambda)$ may now be summarized as follows:

- *Right minimal indices of $F^{\mathcal{P}}_\tau(\lambda)$ are shifted from those of $P(\lambda)$ by $\mathfrak{i}(\tau) = 2$.*
- *Left minimal indices of $F^{\mathcal{P}}_\tau(\lambda)$ are shifted from those of $P(\lambda)$ by $\mathfrak{c}(\tau) = 3$.*
- *A right minimal basis of $P(\lambda)$ is recovered from the $4^{\text{th}} = (k-c_1)^{\text{th}}$ blocks (of size $n \times 1$) of any right minimal basis of $F^{\mathcal{P}}_\tau(\lambda)$.*
- *A left minimal basis of $P(\lambda)$ is recovered from the $6^{\text{th}} = k^{\text{th}}$ blocks (of size $1 \times n$) of any left minimal basis of $F^{\mathcal{P}}_\tau(\lambda)$.* □

6. Strict equivalence of Fiedler pencils. It is now very simple to determine which Fiedler pencils of a square *singular* matrix polynomial are strictly equivalent and which are not, as a consequence of the relationships we have established between the minimal indices of a polynomial and those of its Fiedler pencils.

THEOREM 6.1. *Let $P(\lambda)$ be a singular square matrix polynomial of degree $k \geq 2$. Then two Fiedler pencils $F_{\sigma_1}(\lambda)$ and $F_{\sigma_2}(\lambda)$ of $P(\lambda)$ are strictly equivalent if and only if $\mathfrak{c}(\sigma_1) = \mathfrak{c}(\sigma_2)$ (or equivalently, if $\mathfrak{i}(\sigma_1) = \mathfrak{i}(\sigma_2)$).*

Proof. It is well known [19, Chapter XII, Thm. 5] that two pencils are strictly equivalent if and only if they have the same finite and infinite elementary divisors and the same left and right minimal indices. Since every Fiedler pencil is a strong linearization of $P(\lambda)$ by Theorem 4.6, $F_{\sigma_1}(\lambda)$ and $F_{\sigma_2}(\lambda)$ always have the same finite and infinite elementary divisors (see Lemma 2.3). By Corollary 5.8 (resp., Corollary 5.11) the right (resp., left) minimal indices of $F_{\sigma_1}(\lambda)$ and $F_{\sigma_2}(\lambda)$ are equal if and only if $\mathfrak{i}(\sigma_1) = \mathfrak{i}(\sigma_2)$ $\big($resp., $\mathfrak{c}(\sigma_1) = \mathfrak{c}(\sigma_2)\big)$. But $\mathfrak{c}(\sigma_1) + \mathfrak{i}(\sigma_1) = k - 1 = \mathfrak{c}(\sigma_2) + \mathfrak{i}(\sigma_2)$, so $\mathfrak{c}(\sigma_1) = \mathfrak{c}(\sigma_2)$ is a necessary and sufficient condition for the equality of *both* the left and the right minimal indices of $F_{\sigma_1}(\lambda)$ and $F_{\sigma_2}(\lambda)$, and hence also for the strict equivalence of $F_{\sigma_1}(\lambda)$ and $F_{\sigma_2}(\lambda)$. ☐

For regular polynomials $P(\lambda)$, it is well known that any two strong linearizations of $P(\lambda)$ are strictly equivalent, since they are regular pencils with the same finite and infinite elementary divisors as $P(\lambda)$ (see [19, Chapter XII, Thm. 2] or [20, Prop. 1.2]). This is in stark contrast with the situation for square singular $P(\lambda)$, where from Theorem 6.1 it is now clear that there are always Fiedler pencils of $P(\lambda)$ that are *not* strictly equivalent. Indeed, among all Fiedler pencils of $P(\lambda)$ the companion forms always stand in strict equivalence classes of their own, as we show in the next result.

COROLLARY 6.2. *For a singular square matrix polynomial $P(\lambda)$ of degree $k \geq 2$, the first companion form is never strictly equivalent to any other Fiedler pencil of $P(\lambda)$. The same holds for the second companion form. In particular, the first and second companion forms of $P(\lambda)$ are never strictly equivalent to each other.*

Proof. We consider only the first companion form $C_1(\lambda)$. Recall from Section 3 that $C_1(\lambda) = F_{\sigma_1}(\lambda)$ with bijection $\sigma_1 = (k, k-1, \ldots, 2, 1)$; note that $\mathfrak{c}(\sigma_1) = 0$. It is not hard to see inductively that σ_1 is the *only* bijection σ with $\mathfrak{c}(\sigma) = 0$. Thus by Theorem 6.1 there is no other Fiedler pencil that is strictly equivalent to $C_1(\lambda)$. ☐

REMARK 6.3. In [4, Thm. 2.3] it was shown that every Fiedler pencil $F_\sigma(\lambda)$ of a *regular* polynomial $P(\lambda)$ is a strong linearization by proving that $F_\sigma(\lambda)$ is always strictly equivalent to $C_1(\lambda)$. Corollary 6.2 shows that it is impossible to extend this proof strategy to singular polynomials. Note that a different proof that the first and second companion forms of square singular matrix polynomials are never strictly equivalent was presented in [11, Cor. 5.11].

7. Recovery of eigenvectors of regular matrix polynomials. A polynomial $P(\lambda)$ that is *regular* has no minimal indices or minimal bases, of course, but recovery of the eigenvectors of $P(\lambda)$ from those of any of its linearizations is still an important task. In this section we show how to recover the eigenvectors of a regular $P(\lambda)$ from those of its Fiedler pencils. Since these recovery procedures follow easily from the results in Sections 4 and 5, we only sketch the main ideas and state the results; the reader can easily fill in the details. Recall that a *finite eigenvalue* of a regular $P(\lambda)$ is a number $\lambda_0 \in \mathbb{F}$ such that $\det P(\lambda_0) = 0$, equivalently a $\lambda_0 \in \mathbb{F}$ such that there exist nonzero vectors $x_0 \in \mathbb{F}^{n\times 1}$ and $y_0^T \in \mathbb{F}^{1\times n}$ satisfying $P(\lambda_0)x_0 = 0$ and $y_0^T P(\lambda_0) = 0$; x_0 is a right eigenvector and y_0^T a left eigenvector of $P(\lambda)$ corresponding to λ_0, i.e., elements of the right and left nullspaces $\mathcal{N}_r\big(P(\lambda_0)\big)$ and $\mathcal{N}_\ell\big(P(\lambda_0)\big)$, respectively.

The first key idea for recovering eigenvectors of $P(\lambda)$ from *any* of its linearizations, not just from a Fiedler pencil, starts from the defining equation (2.3) for linearizations, then evaluates λ at the finite eigenvalue λ_0 of interest to give

$$U(\lambda_0)L(\lambda_0)V(\lambda_0) = \operatorname{diag}\big[\, I_{(k-1)n}\,,\, P(\lambda_0)\,\big]\,,$$

where $U(\lambda_0)$ and $V(\lambda_0)$ are nonsingular constant matrices. Letting U_0^L and V_0^R denote the last block-row of $U(\lambda_0)$ and the last block-column of $V(\lambda_0)$, respectively, then the same kind of argument as used in Lemma 5.1 shows that

$$\begin{array}{rccc} \mathcal{L}_0 : & \mathcal{N}_\ell\big(P(\lambda_0)\big) & \longrightarrow & \mathcal{N}_\ell\big(L(\lambda_0)\big) \\ & w^T & \longmapsto & w^T \cdot U_0^L \end{array} \quad \text{and} \quad \begin{array}{rccc} \mathcal{R}_0 : & \mathcal{N}_r\big(P(\lambda_0)\big) & \longrightarrow & \mathcal{N}_r\big(L(\lambda_0)\big) \\ & v & \longmapsto & V_0^R \cdot v \end{array}$$

are isomorphisms of $\mathbb{F}$-vector spaces.

This can now be applied to recovering eigenvectors specifically from the Fiedler pencils, by taking as V_0^R the last block-column $\Lambda_\sigma^R(P)$ of $V(\lambda)$ found in Lemma 5.3, evaluated at $\lambda = \lambda_0$. Similarly we can use as U_0^L the matrix polynomial $\Lambda_\sigma^L(P)$ found in Theorem 5.9, evaluated at $\lambda = \lambda_0$. Since the unique I_n block in each of $\Lambda_\sigma^R(P)$ and $\Lambda_\sigma^L(P)$ will still be present (in the same positions) after evaluating at λ_0, we are led to the following eigenvector recovery procedures, which are direct analogs of the minimal basis recovery procedures in Corollaries 5.8 and 5.11.

COROLLARY 7.1 (**Eigenvector recovery from Fiedler pencils**).
Let $P(\lambda)$ be an $n \times n$ regular matrix polynomial with degree $k \geq 2$, let $F_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ associated with a bijection σ having $\operatorname{CISS}(\sigma) = (c_1, i_1, \ldots, c_\ell, i_\ell)$, *and suppose that λ_0 is a* finite *eigenvalue of $P(\lambda)$.*

Right eigenvectors: (*partition $nk \times 1$ vectors as $k \times 1$ block vectors with $n \times 1$ blocks*)

(a) *If $z \in \mathbb{F}^{nk\times 1}$ is a right eigenvector of $F_\sigma(\lambda)$ with finite eigenvalue $\lambda_0 \in \mathbb{F}$, and x is the $(k - c_1)^{\text{th}}$ block of z, then x is a right eigenvector of $P(\lambda)$ with finite eigenvalue λ_0.*

(b) *If $\{z_1, \ldots, z_p\}$ is a basis of $\mathcal{N}_r\big(F_\sigma(\lambda_0)\big)$, and x_j is the $(k - c_1)^{\text{th}}$ block of z_j for $j = 1, \ldots, p$, then $\{x_1, \ldots, x_p\}$ is a basis of $\mathcal{N}_r\big(P(\lambda_0)\big)$.*

Left eigenvectors: (*partition $1 \times nk$ vectors as $1 \times k$ block vectors with $1 \times n$ blocks*)

(c) *If $w^T \in \mathbb{F}^{1\times nk}$ is a left eigenvector of $F_\sigma(\lambda)$ with finite eigenvalue $\lambda_0 \in \mathbb{F}$, and*

$$y^T \text{ is the } \begin{cases} k^{\text{th}} \text{ block of } w^T & \text{if } c_1 > 0 \\ (k - i_1)^{\text{th}} \text{ block of } w^T & \text{if } c_1 = 0\,, \end{cases}$$

then y^T is a left eigenvector of $P(\lambda)$ with finite eigenvalue λ_0.

(d) *If $\{w_1^T, \ldots, w_p^T\}$ is a basis of $\mathcal{N}_\ell\big(F_\sigma(\lambda_0)\big)$, and*

$$y_j^T \text{ is the } \begin{cases} k^{\text{th}} \text{ block of } w_j^T & \text{if } c_1 > 0 \\ (k - i_1)^{\text{th}} \text{ block of } w_j^T & \text{if } c_1 = 0 \end{cases}$$

for $j = 1, \ldots, p$, then $\{y_1^T, \ldots, y_p^T\}$ is a basis of $\mathcal{N}_\ell\big(P(\lambda_0)\big)$.

Note that these results hold for the first companion form of $P(\lambda)$ by taking $(c_1, i_1) = (0, k - 1)$, and for the second companion form taking $(c_1, i_1) = (k - 1, 0)$.

Finally we consider the recovery of left and right eigenvectors corresponding to the eigenvalue ∞, which for Fiedler pencils turns out to be very simple. Recall that a regular matrix polynomial $P(\lambda)$ has an *infinite eigenvalue* if and only if $\operatorname{rev} P(\lambda)$ has eigenvalue zero, and the corresponding left and right eigenvectors of $P(\lambda)$ at the eigenvalue ∞ are just the left and right null vectors of $(\operatorname{rev} P)(0) = A_k$, the leading coefficient of $P(\lambda)$. Since the leading coefficient of every Fiedler pencil for $P(\lambda)$ is $M_k = \operatorname{diag}\big[\, A_k, I_{(k-1)n}\,\big]$, we see immediately that there is a very simple relationship between the left and right eigenvectors of $P(\lambda)$ at ∞ and those of any of its Fiedler pencils, as expressed in the following theorem.

THEOREM 7.2 (**Eigenvector recovery at ∞ from Fiedler pencils**). *Suppose $P(\lambda)$ is an $n \times n$ regular matrix polynomial with degree $k \geq 2$, and let $F_\sigma(\lambda)$ be any Fiedler pencil of $P(\lambda)$.*

Right eigenvectors at ∞:

(a) *$z \in \mathbb{F}^{nk\times 1}$ is a right eigenvector of $F_\sigma(\lambda)$ for the eigenvalue ∞ if and only if $z = [\, x^T \;\; 0^T_{(k-1)n\times 1}\,]^T$, where x is a right eigenvector of $P(\lambda)$ at ∞.*

(b) *$\{z_1, \ldots, z_p\} \subset \mathbb{F}^{nk\times 1}$ is a basis of the right eigenspace of $F_\sigma(\lambda)$ at ∞ if and only if $z_j = [\, x_j^T \;\; 0^T_{(k-1)n\times 1}\,]^T$ for $j = 1, \ldots, p$, where $\{x_1, \ldots, x_p\}$ is a basis of the right eigenspace of $P(\lambda)$ at ∞.*

Left eigenvectors at ∞:

(a) *$w^T \in \mathbb{F}^{1\times nk}$ is a left eigenvector of $F_\sigma(\lambda)$ for the eigenvalue ∞ if and only if $w^T = [\, y^T \;\; 0_{1\times(k-1)n}\,]$, where y^T is a left eigenvector of $P(\lambda)$ at ∞.*

(b) *$\{w_1^T, \ldots, w_p^T\} \subset \mathbb{F}^{1\times nk}$ is a basis of the left eigenspace of $F_\sigma(\lambda)$ at ∞ if and only if $w_j^T = [\, y_j^T \;\; 0_{1\times(k-1)n}\,]$ for $j = 1, \ldots, p$, where $\{y_1^T, \ldots, y_p^T\}$ is a basis of the left eigenspace of $P(\lambda)$ at ∞.*

8. Conclusions and future work. We have proved that every Fiedler pencil of a given square matrix polynomial $P(\lambda)$ is always a strong linearization of $P(\lambda)$, even in the case that $P(\lambda)$ is singular. In addition, we have derived an extremely simple procedure to recover the minimal indices and bases of a singular square matrix polynomial from the minimal indices and bases of any of its Fiedler pencils, at no computational cost. This simple procedure has been further extended to the recovery of the eigenvectors of a regular matrix polynomial from the eigenvectors of its Fiedler pencils. These results now make it possible to use well-established numerical algorithms on any Fiedler pencil [6, 8, 9, 15, 36] to obtain the complete eigenstructure of a square matrix polynomial (regular or singular).

This paper continues the work initiated by the authors in [11], with the aim of creating for *singular* matrix polynomials a wider arena of linearizations that allow the easy recovery of the complete eigenstructure of the polynomial. Note that the mere definition of linearization does not guarantee that the minimal indices and bases of $P(\lambda)$ can be easily recovered, or even that they have any simple relationship at all to those of a given linearization. Consequently each family of linearizations requires a separate study in order to establish convenient recovery procedures.

Another goal of our continuing work on linearizations for singular polynomials is to create more possibilities for finding linearizations that preserve any structure that a matrix polynomial might possess. The results of [11] show that for many types of structure, this cannot be achieved using any of the linearizations defined in [33]. Thus the next steps in our investigation are to modify Fiedler pencils with the aim of finding structured linearizations for singular structured matrix polynomials [12, 13], and also to extend Fiedler pencils to deal with the very important case of (non-square) rectangular matrix polynomials [14].

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