Backward error and conditioning of Fiedler companion linearizations*

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Abstract

The standard way to solve polynomial eigenvalue problems is through linearizations. The family of Fiedler linearizations, which includes the classical Frobenius companion forms, presents many interesting properties from both the theoretical and the applied point of view. These properties make the Fiedler pencils a very attractive family of linearizations to be used in the solution of polynomial eigenvalue problems. However, their numerical features for general matrix polynomials had not been yet fully investigated. In this paper, we analyze the backward error of eigenpairs and the condition number of eigenvalues of Fiedler linearizations in the solution of polynomial eigenvalue problems. We get bounds for: (a) the ratio between the backward error of an eigenpair of the matrix polynomial and the backward error of the corresponding (computed) eigenpair of the linearization, and (b) the ratio between the condition number of an eigenvalue in the linearization and the condition number of the same eigenvalue in the matrix polynomial. A key quantity in these bounds is ρ, the ratio between the maximum norm of the coefficients of the polynomial and the minimum norm of the leading and trailing coefficient. If the matrix polynomial is well scaled (i. e., all its coefficients have a similar norm, which implies $\rho \approx 1$), then solving the polynomial eigenvalue problem with any Fiedler linearization will give a good performance from the point of view of backward error and conditioning. In the more general case of badly scaled matrix polynomials, dividing the coefficients of the polynomial by the maximum norm of its coefficients allows us to get better bounds. In particular, after this scaling, the ratio between the eigenvalue condition number in any two Fiedler linearizations is bounded by a quantity that depends only on the size and the degree of the polynomial. We also analyze the effect of parameter scaling in these linearizations, which improves significantly the backward error and conditioning in some cases where ρ is large. Several numerical experiments are provided to support our theoretical results.

Keywords: matrix polynomial, matrix pencil, eigenvalue, eigenvector, Polynomial Eigenvalue Problem, companion linearization, Fiedler pencil, conditioning, backward error, scaling

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1 Introduction

We consider matrix polynomials of degree ℓ in homogeneous coordinates $(\alpha, \beta) \in \mathbb{P}_1$ (the projective space of lines through the origin in $\mathbb{C}^2$), namely:

$$P(\alpha, \beta) = \sum_{i=0}^{\ell} \alpha^i \beta^{\ell-i} A_i, \tag{1}$$

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where $A_i \in \mathbb{C}^{n\times n}$, for $i = 0, 1, \ldots, \ell$, and $A_\ell \neq 0$. In the following, we assume that $P(\alpha, \beta)$ is a *regular* matrix polynomial, namely $\det P(\alpha, \beta)$ is not the (identically) zero polynomial (in other words, there is at least one pair (α, β) such that $\det P(\alpha, \beta) \neq 0$). The *Polynomial Eigenvalue Problem* (PEP) consists of finding *right* and *left eigenpairs* of P, denoted, respectively, by (x, α, β) and (y^*, α, β), such that

$$P(\alpha, \beta)x = 0, \qquad y^* P(\alpha, \beta) = 0^* \tag{2}$$

(where $(\cdot)^*$ denotes the conjugate transpose). In this situation, the pair (α, β) is an *eigenvalue* of $P(\alpha, \beta)$, and the (nonzero) vectors $x, y \in \mathbb{C}^n$ are, respectively, a *right* and a *left eigenvector* of P associated with the eigenvalue (α, β). Note that in the notation for left eigenpairs we are using the conjugate transpose y^* instead of y, to distinguish them from right eigenpairs.

The standard approach in the theory of matrix polynomials and the PEP (see, for instance, [3, 9, 11, 20, 30]) deals with a single variable λ, which forces to treat the *finite* and the *infinite eigenvalues* separately. Here we use homogeneous coordinates from the very beginning to deal uniformly with all eigenvalues (both finite and infinite). The infinite eigenvalue in our notation corresponds to the pair $(\alpha, 0)$, and the standard single-parameter notation can be recovered by taking $\lambda = \alpha/\beta$. We will also assume that the coordinates are normalized so that $|\alpha|, |\beta| \leq 1$.

The usual way to solve the PEP is by means of linearizations. These are introduced (in homogeneous coordinates) at the beginning of Section 2. For the moment, the only information we need to know is that a *linearization* of $P(\alpha, \beta)$ is an $\ell n \times \ell n$ matrix pencil $L(\alpha, \beta) = \alpha\mathcal{A} + \beta\mathcal{B}$ with the same eigenvalues as $P(\alpha, \beta)$, counting multiplicities. For the classical Frobenius companion linearizations there are also explicit simple formulas relating the right and left eigenvectors, x, y, of $P(\alpha, \beta)$ with the right and left eigenvectors, v, w, of the linearization. Hence, the PEP problem is classically solved as a *Generalized Eigenvalue Problem* (GEP):

$$L(\alpha, \beta)v = 0, \qquad w^* L(\alpha, \beta) = 0^*, \tag{3}$$

and then recovering the eigenvectors x, y from v, w. The PEP (2) and the corresponding GEP (3) are equivalent problems from the theoretical point of view (once we know the relationship between x, y and v, w). However, from the numerical point of view, they are different problems, and their differences can be quite relevant. In particular, the computed eigenpairs (v, α, β) and (w^*, α, β) may be close to exact eigenpairs of a nearby pencil to $L(\alpha, \beta)$, but the corresponding eigenpairs (x, α, β) and (y^*, α, β) may be far away from being the exact eigenpair of a nearby polynomial of $P(\alpha, \beta)$. Also, the eigenvalue (α, β) can be very sensitive to small changes of $L(\alpha, \beta)$, though not to small changes of $P(\alpha, \beta)$. Using the terminology of this paper, the backward errors $\eta_P(x, \alpha, \beta)$ and $\eta_P(y^*, \alpha, \beta)$ can be very different to the backward errors $\eta_L(v, \alpha, \beta)$ and $\eta_L(w^*, \alpha, \beta)$, and also the condition numbers $\kappa_P(\alpha, \beta)$ and $\kappa_L(\alpha, \beta)$ can be quite different (see Section 2 for the definition of backward error and condition number). A good linearization in this context would be a linearization $L(\alpha, \beta)$ such that the backward errors of the (computed) eigenpairs of $P(\alpha, \beta)$ are not too much larger than the backward errors of the corresponding eigenpairs of $L(\alpha, \beta)$, and such that the condition number $\kappa_L(\alpha, \beta)$ of the eigenvalue (α, β) in $L(\alpha, \beta)$ is not much larger than its condition number $\kappa_P(\alpha, \beta)$ in $P(\alpha, \beta)$. This would tell us, in particular, that the eigenpairs (x, α, β) and (y^*, α, β), corresponding to the computed eigenpairs (v, α, β) and (w^*, α, β), are the exact eigenpairs of a nearby matrix polynomial when (v, α, β) and (w^*, α, β) are the exact eigenpairs of a nearby pencil. This last condition is satisfied if the eigenpairs of $L(\alpha, \beta)$ are computed with a backward stable algorithm like, for instance, the standard QZ algorithm for the GEP [31].

The standard MATLAB code `polyeig` for solving PEPs, which uses the QZ algorithm on a Frobenius linearization, has a good performance for polynomials with coefficients whose norms are close to each other. This fact can be explained by our theoretical results since, in this case, the ratio between the condition number and the backward error of the linearization and those of the polynomial are moderate. However, when the polynomial is badly scaled, in the sense that it has some coefficients with very different norms, the results may be quite inaccurate (see, for instance, the `orr_sommerfeld` example in Section 8.1). An improvement of `polyeig` for quadratic PEPs has been presented in [21], by means of the `quadeig` MATLAB function. This improvement is based on an appropriate choice of the linearization "with favorable conditioning and stability properties" [21, 18:2], together with a deflation process of zero and infinite eigenvalues. This idea of looking for linearizations with good conditioning and stability properties for higher degree polynomials is the main motivation of the present work

In the last decade, several new families of linearizations have been introduced by different authors. Among them, two families have reached a prominent position. In the first place, the families $\mathbb{L}_1(P), \mathbb{L}_2(P)$, and $\mathbb{DL}(P)$ introduced in [28], and, in the second place, the family of Fiedler linearizations, introduced in [2] for regular matrix polynomials, in [9] for square singular matrix polynomials, and in [11] for rectangular matrix polynomials. The family of Fiedler linearizations presents many interesting features (see Section 3) that make them a very good candidate to be used in solving the PEP by linearization. Quite recently, a new very interesting family of linearizations has been introduced in [17], which extends, among others, the family of Fiedler linearizations considered in the present work. An even more general family of potential ("ansatz") linearizations can be found in the recent paper [19].

The backward error and conditioning of the families introduced in [28] have been analyzed, respectively, in [23] and [24]. In both papers, it has been shown that one of the Frobenius companion linearizations has best behavior within the family, depending on whether the eigenvalue that is to be computed belongs to the unit circle or not. The main goal of the present paper is to analyze the backward error of eigenpairs and the condition number of eigenvalues of all Fiedler linearizations. In particular, we aim to compare the backward error and condition number for the polynomial with the ones of the linearization.

Using explicit formulas that relate the left and right eigenvectors of a given Fiedler linearization with the corresponding left and right eigenvectors of the polynomial, we will obtain bounds for the ratios $\eta_P(x,\alpha,\beta)/\eta_F(v,\alpha,\beta), \eta_P(y^*,\alpha,\beta)/\eta_F(w^*,\alpha,\beta)$, and $\kappa_F(\alpha,\beta)/\kappa_P(\alpha,\beta)$, for any Fiedler linearization, $F(\alpha,\beta)$, of $P(\alpha,\beta)$. We want to emphasize that we get uniform bounds, which means that they are valid for all Fiedler linearizations, so that they do not depend on the specific linearization. These bounds allow us to conclude that when the matrix polynomial (1) is well scaled, namely when all coefficients A_i, for $i = 0, 1, \ldots, \ell$, have similar norm, then these ratios are moderate, so that any Fiedler linearization can be used to solve the PEP (2) with the same reliability as the classical Frobenius companion forms. Our results extend the ones recently obtained in [15] for the polynomial root-finding problem of monic scalar polynomials. The developments in Section 5 and 6 are also very close to those in the recent paper [6], valid for block symmetric linearizations of odd degree matrix polynomials.

If the matrix polynomial (1) has coefficients with very different magnitudes, then we can consider some scaling. In the first place, one can introduce a scaling of the coefficients consisting of dividing all them by the maximum norm of the coefficients. This strategy allows us to get a new matrix polynomial all whose coefficients have norm less than or equal to 1. We will see that, when this is the case, we get simpler and better bounds for all ratios mentioned in the previous paragraph. In particular, the ratio between the condition number of a given eigenvalue in any two Fiedler linearizations is bounded by $\ell^3\sqrt{n}$. This means that this simple scaling of the coefficients of $P(\alpha,\beta)$ allows us to bound the ratio between condition numbers of a given eigenvalue in any two Fiedler linearizations just in terms of the degree and the size of the polynomial. This, of course, does not guarantee that the condition number is small, but tells us that all Fiedler linearizations have a similar behavior (regarding the condition number).

We also consider a second kind of scaling, namely the *parameter scaling*, consisting of replacing the parameters (α,β) by some appropriate multiple $(s\alpha, s\beta)$, with $s \neq 0$. Our numerical experiments suggest that, if the parameter s is appropriately chosen, then this kind of scaling improves notably the backward error in some badly scaled polynomials.

We want to emphasize that, in this work, we are considering the backward error of single eigenpairs (and only for simple eigenvalues). Other notions of backward error have been analyzed in the literature, like the backward error of the polynomial, which considers the whole set of eigenpairs, instead of a single one. This backward error has been considered in [17, 32] for general matrix polynomials, in [16] for structured ones, and in [14, 27] for scalar polynomials.

The paper is organized as follows. In Section 2 we introduce the basic notation and recall the practical formulas for the backward error of eigenpairs and the condition number of eigenvalues of matrix polynomials (and, in particular, matrix pencils) that will be used in further sections. In Section 3 we recall the Fiedler pencils, together with the properties that we need throughout the paper. Section 4 is devoted to the formulas for the eigenvectors of Fiedler linearizations, which are used in Sections 5 and 6 to derive the bounds for the ratio between, respectively, backward errors of eigenpairs and condition numbers of eigenvalues of the Fiedler linearizations and the matrix polynomial. In Section 7 we analyze the effect of scaling in these bounds, and Section 8 contains some numerical experiments. Finally, in Section 9 we summarize the main contributions of the paper and indicate several lines of future research. Since the paper is quite technical, we have taken to an appendix (Appendices A and B) the most technical details.

2 Basic definitions and notation

We start by introducing the notion of unimodular matrix polynomials and linearization in homogeneous coordinates. These notions are all well known for a single coordinate. However, up to our knowledge, they have not been stated in the literature for homogeneous coordinates, so we formally introduce them here.

Definition 2.1. *An homogeneous matrix polynomial $U(\alpha,\beta)$ is said to be* unimodular in α *(respectively,* in β*) if* $\det U(\alpha,\beta)=c\beta^k$ *(resp.,* $c\alpha^k$*), for some* $c\neq 0$ *and some* $k\geq 0$.

Definition 2.2. *A* linearization in α *(resp.,* in β*) of a matrix polynomial* (1) *is a matrix pencil* $L(\alpha,\beta)=\alpha\mathcal{A}+\beta\mathcal{B}$, *with* $\mathcal{A},\mathcal{B}\in\mathbb{C}^{n\ell\times n\ell}$, *such that there are two matrix polynomials* $U(\alpha,\beta),V(\alpha,\beta)$, *unimodular in* α *(resp., in* β*), with*

$$U(\alpha,\beta)L(\alpha,\beta)V(\alpha,\beta)=\beta^k\begin{bmatrix} P(\alpha,\beta) & 0\\ 0 & \beta^\ell I_{(\ell-1)n}\end{bmatrix} \quad (resp., \alpha^k\begin{bmatrix} P(\alpha,\beta) & 0\\ 0 & \alpha^\ell I_{(\ell-1)n}\end{bmatrix}), \quad (4)$$

for some nonnegative integer k. *A matrix pencil* $L(\alpha,\beta)$ *is said to be a* strong linearization *of* $P(\alpha,\beta)$ *if* $L(\alpha,\beta)$ *is a linearization of* $P(\alpha,\beta)$ *in both* α *and* β.

We note that Definition 2.2 is the analogue, in homogeneous coordinates, of the standard definition of linearizations and strong linearizations in a single variable λ (see, for instance, [9, Def. 2.2]), which can be recovered by taking $\beta=1$ (linearization) and $\alpha=1$ (strong linearization).

Example 2.3. *The* second Frobenius companion pencil *for quadratic matrix polynomials* $P(\alpha,\beta)=\alpha^2A_2+\alpha\beta A_1+\beta^2A_0$,

$$C_2(\alpha,\beta)=\begin{bmatrix}\alpha A_2+\beta A_1 & -\beta I\\ \beta A_0 & \alpha I\end{bmatrix},$$

is a linearization in both α *and* β, *since*

$$\begin{bmatrix}\alpha I & \beta I\\ 0 & \alpha I\end{bmatrix} C_2(\alpha,\beta)\begin{bmatrix}\alpha I & 0\\ -\beta A_0 & \alpha I\end{bmatrix}=\alpha\begin{bmatrix}P(\alpha,\beta) & 0\\ 0 & \alpha^2 I\end{bmatrix},$$

and

$$\begin{bmatrix}\alpha I & \beta I\\ -\beta I & 0\end{bmatrix} C_2(\alpha,\beta)\begin{bmatrix}\beta I & 0\\ \beta A_1+\alpha A_2 & \beta I\end{bmatrix}=\beta\begin{bmatrix}P(\alpha,\beta) & 0\\ 0 & \beta^2 I\end{bmatrix}.$$

In the following, and for the sake of brevity, we will usually drop the dependence on the variables (α,β) when referring to matrix polynomials (or pencils).

The main advantage in dealing with linearizations for the PEP is that any strong linearization, L, of P has the same finite and infinite eigenvalues (with their corresponding partial multiplicities) as P. However, the eigenvectors are not preserved by linearization (note that the length of eigenvectors of L is $n\ell$, instead of n, which is the length of eigenvectors of P).

Nonetheless, we will see that, for any Fiedler pencil F, the eigenvectors of P can be easily recovered from the eigenvectors of F. Moreover, there are explicit formulas relating the eigenvectors of P and the eigenvectors of F (see Theorems 4.4 and 4.5).

The *(normwise) backward error* of a computed right eigenpair (x,α,β) of P is defined as

$$\eta_P(x,\alpha,\beta) := \min\{\, \epsilon : (P(\alpha,\beta)+\Delta P(\alpha,\beta))x = 0,\ \|\Delta A_i\|_2 \le \epsilon \|A_i\|_2,\ i=0,1,\ldots,\ell\,\},$$

(see [30]), where $\Delta P(\alpha,\beta) = \sum_{i=0}^{\ell} \alpha^i\beta^{\ell-i}\Delta A_i$. The backward error for a left eigenpair (y^*,α,β) is defined similarly as:

$$\eta_P(y^*,\alpha,\beta) := \min\{\, \epsilon : y^*(P(\alpha,\beta)+\Delta P(\alpha,\beta)) = 0,\ \|\Delta A_i\|_2 \le \epsilon \|A_i\|_2,\ i=0,1,\ldots,\ell\,\}.$$

There are explicit practical formulas for the backward error of right and left eigenpairs [30, Th. 1]:

$$\eta_P(x,\alpha,\beta) = \frac{\|P(\alpha,\beta)x\|_2}{\left(\sum_{j=0}^{\ell}|\alpha|^j|\beta|^{\ell-j}\|A_j\|_2\right)\|x\|_2}, \tag{5}$$

$$\eta_P(y^*,\alpha,\beta) = \frac{\|y^*P(\alpha,\beta)\|_2}{\left(\sum_{j=0}^{\ell}|\alpha|^j|\beta|^{\ell-j}\|A_j\|_2\right)\|y\|_2}. \tag{6}$$

Backward errors of (computed) eigenpairs measure the smallest distance from P to another matrix polynomial having this pair as an exact eigenpair. We note that the distance between matrix polynomials here is measured coefficient-wise. Other weaker measures have been considered, like the one that uses the norm of the whole polynomial (see, for instance, [32]).

Our aim is to compare $\eta_P(x,\alpha,\beta)$ and $\eta_P(y^*,\alpha,\beta)$ with, respectively,

$$\eta_L(v,\alpha,\beta) = \frac{\|L(\alpha,\beta)v\|_2}{(|\alpha|\|\mathcal{A}\|_2+|\beta|\|\mathcal{B}\|_2)\|v\|_2}, \quad \eta_L(w^*,\alpha,\beta) = \frac{\|w^*L(\alpha,\beta)\|_2}{(|\alpha|\|\mathcal{A}\|_2+|\beta|\|\mathcal{B}\|_2)\|w\|_2}, \tag{7}$$

which are obtained by applying (5) and (6) to a linearization $L(\alpha,\beta) = \alpha\mathcal{A}+\beta\mathcal{B}$.

The *normwise condition number,* $\kappa_P(\alpha,\beta)$, of a simple eigenvalue (α,β) of P is defined as

$$\kappa_P(\alpha,\beta) = \max_{\|\Delta A\|\le 1} \frac{\|K(\alpha,\beta)\Delta A\|_2}{\|[\alpha,\beta]\|_2},$$

where $K(\alpha,\beta) : (\mathbb{C}^{n\times n})^{\ell+1} \to T_{(\alpha,\beta)}\mathbb{P}_1$ is the differential of the map from $(A_0,A_1,\ldots,A_\ell)$ to (α,β) in the projective space, and $T_{(\alpha,\beta)}\mathbb{P}_1$ denotes the tangent space at (α,β) to $\mathbb{P}_1$. Here $\Delta A = (\Delta A_0,\Delta A_1,\ldots,\Delta A_\ell)$ and $\|\Delta A\| = \|[\omega_0^{-1}\Delta A_0,\omega_1^{-1}\Delta A_1,\ldots,\omega_\ell^{-1}\Delta A_\ell]\|_F$ with $\omega_i = \|A_i\|_2$. An extension of a result of Dedieu and Tisseur [8, Thm. 4.2], that treats the unweighted Frobenius norm, yields the explicit formula [24]:

$$\kappa_P(\alpha,\beta) = \left(\sum_{i=0}^{\ell}|\alpha|^{2i}|\beta|^{2(\ell-i)}\|A_i\|_2^2\right)^{1/2} \frac{\|x\|_2\|y\|_2}{\left|y^*(\bar\beta\mathcal{D}_\alpha P - \bar\alpha\mathcal{D}_\beta P)|_{(\alpha,\beta)}x\right|}, \tag{8}$$

where $\mathcal{D}_\alpha \equiv \frac{\partial}{\partial\alpha}$ and $\mathcal{D}_\beta \equiv \frac{\partial}{\partial\beta}$, and x,y are right and left eigenvectors of P associated with (α,β). The eigenvalue condition number $\kappa_L(\alpha,\beta)$ for the pencil $L(\alpha,\beta) = \beta\mathcal{A}+\alpha\mathcal{B}$ is defined in a similar way and an explicit formula is given by

$$\kappa_L(\alpha,\beta) = \sqrt{|\alpha|^2\|\mathcal{A}\|_2^2+|\beta|^2\|\mathcal{B}\|_2^2}\ \frac{\|v\|_2\|w\|_2}{\left|w^*(\bar\beta\mathcal{D}_\alpha L - \bar\alpha\mathcal{D}_\beta L)|_{(\alpha,\beta)}v\right|}, \tag{9}$$

where v,w are right and left eigenvectors of L associated with (α,β).

According to the notions introduced in [6, §2], we are considering the relative-relative backward errors and relative-relative condition numbers. This means that we are measuring relative changes in each coefficient of the matrix polynomial (where "relative" means compared to the size of each coefficient, and not to the norm of the whole polynomial, or to the coefficient with maximum norm). We also want to emphasize that other notions of "homogeneous" eigenvalue condition numbers have been considered in the literature (see the recent work [1] for a comparison between them).

Throughout the paper, we use x, y for right and left eigenvectors of the polynomial P, and v, w for right and left eigenvectors of the linearization L.

For $i \leq j$, we use MATLAB's notation $i : j$ to denote the set of integers from i to j (included).

For block partitioned matrices $A = [A_{ij}]_{\substack{i=1:m \\ j=1:n}}$, we use the notation $(\cdot)^{\mathcal{B}}$ to denote block transposition, that is, the (i, j) block of $A^{\mathcal{B}}$ is the (j, i) block of A.

In the following, the dth *Horner shift* of $P(\alpha, \beta)$, for $1 \leq d \leq \ell$, is the matrix polynomial in homogeneous coordinates

$$P_d(\alpha, \beta) := \alpha^d A_\ell + \alpha^{d-1}\beta A_{\ell-1} + \cdots + \alpha\beta^{d-1} A_{\ell-d+1} + \beta^d A_{\ell-d}, \qquad P_0(\alpha, \beta) := I_n, \tag{10}$$

and the *truncated degree−d* polynomial of $P(\alpha, \beta)$ is:

$$P_d^\sharp(\alpha, \beta) := \alpha^d A_d + \alpha^{d-1}\beta A_{d-1} + \cdots + \alpha\beta^{d-1} A_1 + \beta^d A_0.$$

We use the notation I_n for the $n \times n$ identity matrix (or just I, when there is no risk of confusion with the size).

3 Elementary properties of Fiedler linearizations

To introduce the family of Fiedler pencils we first need to define the following block-partitioned matrices associated with a given matrix polynomial (1):

$$M_0 = \operatorname{diag}(I_{n(\ell-1)}, A_0), \quad M_\ell = \operatorname{diag}(A_\ell, I_{(\ell-1)n}), \tag{11}$$

$$M_j = \begin{bmatrix} I_{n(\ell-j-1)} & & & \\ & A_j & -I_n & \\ & -I_n & 0 & \\ & & & I_{n(j-1)} \end{bmatrix}, \quad \text{for } j = 1, \ldots, \ell - 1. \tag{12}$$

The Fiedler pencils were first introduced in [2] for regular matrix polynomials, and later extended in [9] to square singular polynomials, and in [11] to rectangular polynomials. In all cases, they have been considered in a single variable λ. The notation we follow here is based on the one in [9], where the name "Fiedler pencils" was introduced.

Definition 3.1. *Let* $P(\alpha, \beta) = \sum_{j=0}^{\ell} a^j \beta^{\ell-j} A_j$*, with* $\ell \geq 2$*, and let* M_i*, for* $i = 0, 1, \ldots, \ell$*, be the matrices defined in* (11)–(12)*. Given any bijection* $\sigma : \{0, 1, \ldots, \ell - 1\} \to \{1, \ldots, \ell\}$*, the* Fiedler pencil of $P(\alpha, \beta)$ associated with σ *is the* $n\ell \times n\ell$ *matrix pencil*

$$F_\sigma(\alpha, \beta) := \alpha M_\ell + \beta M_{\sigma^{-1}(1)} \cdots M_{\sigma^{-1}(\ell)}. \tag{13}$$

In the definition of $F_\sigma(\alpha, \beta)$ we drop the dependence on the matrix polynomial P for the ease of notation. When necessary, we explicitly indicate it in the form $F_\sigma(P)$.

Note that $\sigma(i)$ in (13) denotes the position of the factor M_i in the product

$$M_\sigma := M_{\sigma^{-1}(1)} \cdots M_{\sigma^{-1}(\ell)}, \tag{14}$$

i.e., $\sigma(i) = j$ means that M_i is the jth factor in M_σ. Note also that the building factors (11)–(12) of the Fiedler pencil (13) depend on P (to be precise, they depend on its coefficients). We will drop this dependence for simplicity, except in some cases where there is more than one polynomial involved. In these cases, we use the notation $M_\sigma(P)$.

The following commutativity relations of matrices (11)–(12):

$$M_i M_j = M_j M_i \quad \text{for } |i - j| \neq 1 \tag{15}$$

indicate that there are less than $(\ell-1)!$ different Fiedler pencils associated with an arbitrary polynomial P. In particular, it is known [13, Cor. 2.7] that there are only $2^{\ell-1}$ generically different Fiedler pencils for matrix polynomials of degree ℓ. They depend on the notions of *consecutions* and *inversions*.

Definition 3.2. *Let $\sigma : \{0,1,\dots,\ell-1\} \to \{1,\dots,\ell\}$ be a bijection.*

(a) *For $i = 0,\dots,\ell-2$, we say that σ has a* consecution *at i if $\sigma(i) < \sigma(i+1)$, and that σ has an* inversion *at i if $\sigma(i) > \sigma(i+1)$.*

(b) *The* positional consecution-inversion sequence of σ*, denoted by* $\mathrm{PCIS}(\sigma)$*, is the $(\ell-1)$-tuple $(v_0(\sigma), v_1(\sigma),\dots,v_{\ell-2}(\sigma))$ such that $v_j(\sigma) = 1$ if σ has a consecution at j and $v_j(\sigma) = 0$ if σ has an inversion at j.*

(c) *Given $0 \le a, b \le n-2$, the* number of consecutions of σ from a to b*, denoted by $\mathfrak{c}_\sigma(a:b)$, is the nonnegative integer $\mathfrak{c}_\sigma(a:b) := \sum_{j=a}^{b} v_j(\sigma)$, and the* number of inversions from a to b*, denoted by $\mathfrak{i}_\sigma(a:b)$, is the nonnegative integer $\mathfrak{i}_\sigma(a:b) := \sum_{j=a}^{b}(1-v_j(\sigma))$, where if $a > b$ we set $\mathfrak{c}_\sigma(a:b) = \mathfrak{i}_\sigma(a:b) = 0$. The* number of inversions *(resp.* consecutions*) of σ is the number $\mathfrak{i}(\sigma) := \mathfrak{i}(0:\ell-2)$ (resp. $\mathfrak{c}(\sigma) := \mathfrak{c}(0:\ell-2)$).*

Note that, for $0 \le i \le n-2$, σ has a consecution at i if and only if M_i is to the left of M_{i+1} in the product (13) defining F_σ, and that σ has an inversion at i if M_i is to the right of M_{i+1}.

The first and second Frobenius companion forms, denoted by C_1 and C_2, respectively, are particular cases of Fiedler pencils. They are associated with bijections σ_1 and σ_2, respectively, such that $\mathrm{PCIS}(\sigma_1) = (0,0,\dots,0)$ (that is, all inversions), and $\mathrm{PCIS}(\sigma_2) = (1,\dots,1)$ (that is, all consecutions). In other words:

$$C_1(\alpha,\beta) = \alpha M_\ell + \beta M_{\ell-1}\cdots M_1 M_0 = \alpha \begin{bmatrix} A_\ell & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix} + \beta \begin{bmatrix} A_{\ell-1} & \dots & A_1 & A_0 \\ -I_n & & & \\ & \ddots & & \\ & & -I_n & \end{bmatrix}, \tag{16}$$

and

$$C_2(\alpha,\beta) = \alpha M_\ell + \beta M_0 M_1 \cdots M_{\ell-1} = \alpha \begin{bmatrix} A_\ell & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix} + \beta \begin{bmatrix} A_{\ell-1} & -I_n & & \\ A_{\ell-2} & & \ddots & \\ \vdots & & & -I_n \\ A_0 & & & \end{bmatrix}. \tag{17}$$

Note that $C_2(\alpha,\beta) = C_1(\alpha,\beta)^{\mathcal{B}}$.

From both the theoretical and the applied point of view, Fiedler pencils have several interesting features. Here we list some of them:

(P1) They are always strong linearizations [2, 9] (namely, linearizations in both α and β).

(P2) They are, moreover, *companion forms*, that is, they can be easily constructed, by means of a uniform template, without performing any arithmetic operation, from the coefficients of the polynomial [12, Def. 5.1].

(P3) The left and right eigenvectors of $P(\alpha,\beta)$ can be easily recovered for the left and right eigenvectors of any Fiedler linearization. Moreover, there are explicit formulas relating the left and right eigenvectors of a given Fiedler pencil with the left and right eigenvectors of $P(\alpha,\beta)$ [5, 9] (see also Theorem 4.5).

(P4) The leading matrix M_ℓ is in block upper triangular form, thereby reducing the computational cost of the Hessenberg-triangular reduction step of the QZ-algorithm [26].

(P5) If the matrix polynomial is well scaled (i. e., $\|A_i\| \approx 1$, for $i = 0,1,\dots,\ell$), then the Fiedler linearizations have good conditioning and backward stability properties.

Properties (P1)–(P4) are already known. (P5) is a consequence of the developments carried out in the present paper. The goal of this paper is to analyze the conditioning and backward error properties of Fiedler linearizations.

4 Formulas for the eigenvectors

The central ingredient of this section is the notion of *eigencolumn*. This notion will allow us to describe all left and right eigenvectors of Fiedler linearizations from left and right eigenvectors of the polynomial.

Definition 4.1. (Right and left eigencolumn). *Let P be a matrix polynomial, and L be a linearization of P. A matrix polynomial $\mathcal{R}$ (respectively, $\mathcal{L}$) of size $(n\ell)\times n$ is a* right eigencolumn *(resp.,* left eigencolumn*) of L if $L(\alpha,\beta)\mathcal{R}(\alpha,\beta)\equiv e_i\otimes P(\alpha,\beta)$, for some $1\le i\le \ell$ (resp., $L(\alpha,\beta)^*\mathcal{L}(\alpha,\beta)\equiv e_j\otimes P(\alpha,\beta)^*$, for some $1\le j\le\ell$).*

We note that right and left eigencolumns are particular cases of *one-sided factorizations*, in the specialized form (2.4a)–(2.4b) in [22].

In Theorem 4.2 we present explicit formulas for right and left eigencolumns of Fiedler linearizations. These formulas are already known for polynomials in a single variable λ (see [5, Th. 3.1] and [15, Th. 4.1] for scalar polynomials). However, the main (new) information in Theorem 4.2 are the identities (20)-(21) and (23)-(24), which are key to get the bounds for the ratio between backward errors and condition numbers in Sections 5 and 6, and that will allow us to relate the (computed) eigenvectors of F_σ with (approximate) eigenvectors of P. The proof of this result, which is quite technical, is taken to Appendix A. In the statement, $\mathfrak{i}_\sigma(\ell-1:\ell-2)$ is understood to be 0.

Theorem 4.2. (Formulas for eigencolumns of Fiedler linearizations). *Let P be an $n\times n$ matrix polynomial, and let P_d be its dth Horner shift, for $d=1,\ldots,\ell$. Let $\sigma:\{0,1,\ldots,n-1\}\to\{1,\ldots,n\}$ be a bijection, and let F_σ be the Fiedler pencil associated with σ. Then:*

(a) $\mathcal{R}_\sigma(\alpha,\beta)=\begin{bmatrix}[\mathcal{R}_\sigma(\alpha,\beta)]_1 & \cdots & [\mathcal{R}_\sigma(\alpha,\beta)]_\ell\end{bmatrix}^{\mathcal{B}}$*, where, for $i=1,\ldots,\ell$,*

$$[\mathcal{R}_\sigma(\alpha,\beta)]_i=\begin{cases}\alpha^{\mathfrak{i}_\sigma(0:\ell-i-1)}\beta^{\ell-i-\mathfrak{i}_\sigma(0:\ell-i-1)}P_{i-1}(\alpha,\beta), & \text{if } \sigma \text{ has a consecution at } \ell-i,\\ \alpha^{\mathfrak{i}_\sigma(0:\ell-i-1)}\beta^{\ell-1-\mathfrak{i}_\sigma(0:\ell-i-1)}I_n, & \text{if } \sigma \text{ has an inversion at } \ell-i \text{ or } i=1,\end{cases}\tag{18}$$

and

$$[\widetilde{\mathcal{R}}_\sigma(\alpha,\beta)]_i=\begin{cases}-\alpha^{i-2-\mathfrak{i}_\sigma(\ell-i:\ell-2)}\beta^{\mathfrak{i}_\sigma(\ell-i:\ell-2)+1}P^\sharp_{\ell-i}(\alpha,\beta), & \text{if } \sigma \text{ has a consecution at } \ell-i,\\ \alpha^{\ell-1-\mathfrak{i}_\sigma(\ell-i:\ell-2)}\beta^{\mathfrak{i}_\sigma(\ell-i:\ell-2)}I_n, & \text{if } \sigma \text{ has an inversion at } \ell-i \text{ or } i=1,\end{cases}\tag{19}$$

are right eigencolumns of $F_\sigma(\alpha,\beta)$. Moreover:

$$F_\sigma(\alpha,\beta)\mathcal{R}_\sigma(\alpha,\beta)=e_{\ell-i_0}\otimes P(\alpha,\beta),\tag{20}$$

with $i_0=\min\{i:\sigma \text{ has a consecution at } i\}$ (and $i_0=\ell-1$ if σ has no consecutions), and

$$F_\sigma(\alpha,\beta)\widetilde{\mathcal{R}}_\sigma(\alpha,\beta)=e_1\otimes P(\alpha,\beta),\tag{21}$$

(b) $\mathcal{L}_\sigma(\alpha,\beta)=\begin{bmatrix}[\mathcal{L}_\sigma(\alpha,\beta)]_1 & \cdots & [\mathcal{L}_\sigma(\alpha,\beta)]_\ell\end{bmatrix}^*$*, where, for $i=1,\ldots,\ell$,*

$$[\mathcal{L}_\sigma(\alpha,\beta)]_i=\begin{cases}\alpha^{\mathfrak{c}_\sigma(0:\ell-i-1)}\beta^{\ell-i-\mathfrak{c}_\sigma(0:\ell-i-1)}P_{i-1}(\alpha,\beta), & \text{if } \sigma \text{ has an inversion at } \ell-i,\\ \alpha^{\mathfrak{c}_\sigma(0:\ell-i-1)}\beta^{\ell-1-\mathfrak{c}_\sigma(0:\ell-i-1)}I_n, & \text{if } \sigma \text{ has a consecution at } \ell-i \text{ or } i=1,\end{cases}\tag{22}$$

and

$$[\widetilde{\mathcal{L}}_\sigma(\alpha,\beta)]_i=\begin{cases}-\alpha^{i-2-\mathfrak{c}_\sigma(\ell-i:\ell-2)}\beta^{\mathfrak{c}_\sigma(\ell-i:\ell-2)+1}P^\sharp_{\ell-i}(\alpha,\beta), & \text{if } \sigma \text{ has an inversion at } \ell-i,\\ \alpha^{\ell-1-\mathfrak{c}_\sigma(\ell-i:\ell-2)}\beta^{\mathfrak{c}_\sigma(\ell-i:\ell-2)}I_n, & \text{if } \sigma \text{ has a consecution at } \ell-i \text{ or } i=1,\end{cases}$$

are left eigencolumns of $F_\sigma(\alpha,\beta)$*. Moreover:*

$$F_\sigma(\alpha,\beta)^*\mathcal{L}_\sigma(\alpha,\beta) = e_{\ell-j_0}\otimes P(\alpha,\beta)^*, \tag{23}$$

with $j_0 = \min\{j : \sigma \text{ has an inversion at } j\}$ *(and* $j_0 = \ell-1$ *if* σ *has no inversions), and*

$$F_\sigma(\alpha,\beta)^*\widetilde{\mathcal{L}}_\sigma(\alpha,\beta) = e_1\otimes P(\alpha,\beta)^*, \tag{24}$$

In Lemma B.2 we will get a bound on the norm of the left and right eigencolumns $\mathcal{R}_\sigma(\alpha,\beta)$ and $\mathcal{L}_\sigma(\alpha,\beta)$ that will be used later.

Remark 4.3. *The left and right eigencolumns* $\mathcal{R}_\sigma(\alpha,\beta)$ *and* $\mathcal{L}_\sigma(\alpha,\beta)$ *in Theorem* 4.2 *satisfy the following properties:*

1. *The first block is equal to:*

$$[\mathcal{R}_\sigma(\alpha,\beta)]_1 = \alpha^{\mathfrak{i}(\sigma)}\beta^{\ell-1-\mathfrak{i}(\sigma)}I_n, \quad \text{and} \quad [\mathcal{L}_\sigma(\alpha,\beta)]_1 = \alpha^{\mathfrak{c}(\sigma)}\beta^{\ell-1-\mathfrak{c}(\sigma)}I_n.$$

2. *There is always one block of the form* $\beta^{\ell-1}I_n$ *in both* $\mathcal{R}_\sigma(\alpha,\beta)$ *and* $\mathcal{L}_\sigma(\alpha,\beta)$*, namely:*
   - (a) *If* σ *has consecutions at* $0,1,\dots,k-1$ *and an inversion at* k*, for some* $0\le k\le \ell-2$*, then* $[\mathcal{R}_\sigma(\alpha,\beta)]_{\ell-k} = \beta^{\ell-1}I_n$*, since* $\mathfrak{i}_\sigma(0:\ell-(\ell-k)-1) = \mathfrak{i}_\sigma(0:k-1) = 0$*. If* σ *has no inversions at all, then* $[\mathcal{R}_\sigma(\alpha,\beta)]_{\ell-1} = \beta^{\ell-1}I_n$*.*
   - (b) *If* σ *has inversions at* $0,1,\dots,k-1$ *and a consecution at* k*, for some* $0\le k\le\ell-2$*, then* $[\mathcal{L}_\sigma(\alpha,\beta)]_{\ell-k} = \beta^{\ell-1}I_n$*, since* $\mathfrak{c}_\sigma(0:\ell-(\ell-k)-1) = \mathfrak{c}_\sigma(0:k-1) = 0$*. If* σ *has no consecutions at all, then* $[\mathcal{L}_\sigma(\alpha,\beta)]_{\ell-1} = \beta^{\ell-1}I_n$*.*

The following results relate the left and right eigenvectors of a matrix polynomial P with those of any Fiedler linearization of P and viceversa.

Theorem 4.4. (Eigenvectors of F_σ from eigenvectors of P). *Let* P *be an* $n\times n$ *matrix polynomial, let* $\sigma : \{0,1,\dots,n-1\}\to\{1,\dots,n\}$ *be a bijection, and let* F_σ *be the Fiedler pencil associated with* σ*. Let* $\mathcal{R}_\sigma$ *and* $\mathcal{L}_\sigma$ *be as in Theorem* 4.2*, and let* (α,β) *be a simple eigenvalue of* P*. Then:*

- (a) *If* $\beta\neq 0$*, a vector* v *(respectively,* w*) is a right (resp., left) eigenvector of* F_σ *associated with* (α,β) *if and only if* $v = \mathcal{R}_\sigma(\alpha,\beta)x$ *(resp.,* $w = \mathcal{L}_\sigma(\alpha,\beta)y$*), with* x *(resp.,* y*) being a right (resp., left) eigenvector of* P *associated with* (α,β)*.*
- (b) *If* $\beta = 0$*, a vector* v *(respectively,* w*) is a right (resp., left) eigenvector of* F_σ *associated with* $(\alpha,0)$ *if and only if* $v = (e_1\otimes I_n)x$ *(resp.,* $w = (e_1\otimes I_n)y$*), with* x *(resp.,* y*) being a right (resp., left) eigenvector of* P *associated with* $(\alpha,0)$*.*

Proof. Claim (a) is a rewriting, using homogeneous coordinates, of Theorem 3.1 in [5]. The powers of α and β in (18) and (22) are the ones that make $\mathcal{R}_\sigma(\alpha,\beta)$ and $\mathcal{L}_\sigma(\alpha,\beta)$ to be homogeneous matrix polynomials of degree $\ell-1$.

Claim (b) is also trivial from (13) and the definition of M_ℓ (see also [9, Th. 7.2(a)]). □

Theorem 4.5. (Eigenvectors of P from eigenvectors of F_σ). *Let* P *be an* $n\times n$ *matrix polynomial, let* $\sigma : \{0,1,\dots,n-1\}\to\{1,\dots,n\}$ *be a bijection and let* F_σ *be the Fiedler pencil associated with* σ*. Let* (α,β) *be an eigenvalue of* F_σ *with associated right and left eigenvectors* v *and* w*, respectively. Then:*

- (a) $x = (e_{\ell-j_0}^T\otimes I_n)v$ *is a right eigenvector of* P *associated with* (α,β)*, where* $j_0 = \min\{j : \sigma \text{ has an inversion at } j\}$*, and* $j_0 = \ell-1$ *if* σ *has no inversions.*
- (b) $y = (e_{\ell-i_0}^T\otimes I_n)w$ *is a left eigenvector of* P *associated with* (α,β)*, where* $i_0 = \min\{i : \sigma \text{ has a consecution at } i\}$*, and* $i_0 = \ell-1$ *if* σ *has no consecutions.*
- (c) $x = (e_1^T\otimes I_n)v$ *and* $y = (e_1^T\otimes I_n)w$ *are, respectively, a right and a left eigenvector of* P *associated with* (α,β)*.*

Proof. Claims (a) and (b) are an immediate consequence of (23) and (20), since these identities imply: $0 = \mathcal{L}_\sigma(\alpha,\beta)^* F_\sigma(\alpha,\beta) v = \left(e_{\ell-j_0}^T \otimes P(\alpha,\beta)\right) v = P(\alpha,\beta)(e_{\ell-j_0}^T \otimes I_n)v$, and $0 = \mathcal{R}_\sigma(\alpha,\beta)^* F_\sigma(\alpha,\beta)^* w = \left(e_{\ell-i_0}^T \otimes P(\alpha,\beta)^*\right) w = P(\alpha,\beta)^*(e_{\ell-i_0}^T \otimes I_n)w$, respectively. As for claim (c), it is a consequence of (24) and (21), using the same arguments after replacing $\ell - j_0$ and $\ell - i_0$ by 1. □

5 Bounds for the ratio between backward errors

Armed with the results from previous sections and, in particular, with Theorem 4.5, we are in the position to state and prove one of the main results of this paper.

Theorem 5.1. *Let $\sigma : \{0,1,\ldots,\ell-1\} \to \{1,\ldots,\ell\}$ be a bijection and let F_σ be the Fiedler pencil associated with σ. Let v be an approximate right eigenvector of F_σ associated with the approximate simple eigenvalue (α,β). Set $j_0 := \min\{j : \sigma$ has an inversion at $j\}$ and $j_0 := \ell-1$ if σ has no inversions, and define*

$$x = \begin{cases} (e_{\ell-j_0}^T \otimes I_n)v, & \text{if } |\alpha| \le |\beta|, \\ (e_1^T \otimes I_n)v, & \text{if } |\alpha| > |\beta|. \end{cases}$$

Then x is an approximate right eigenvector of P associated with (α,β) and

$$\frac{\eta_P(x,\alpha,\beta)}{\eta_{F_\sigma}(v,\alpha,\beta)} \le \ell^{5/2} \frac{\max_i (1, \|A_i\|_2)^2}{\min(\|A_0\|_2, \|A_\ell\|_2)} \frac{\|v\|_2}{\|x\|_2}. \tag{25}$$

Proof. The fact that x is an approximate eigenvector of P follows from Theorem 4.5. Using (5) for P and (7) for F_σ we get

$$\begin{aligned}\frac{\eta_P(x,\alpha,\beta)}{\eta_{F_\sigma}(v,\alpha,\beta)} &= \frac{|\alpha|\|M_\ell\|_2 + |\beta|\|M_\sigma\|_2}{\sum_{i=0}^{\ell} |\alpha|^i |\beta|^{\ell-i} \|A_i\|_2} \frac{\|P(\alpha,\beta)x\|_2}{\|F_\sigma(\alpha,\beta)v\|_2} \frac{\|v\|_2}{\|x\|_2} \\ &\le \frac{|\alpha|\|M_\ell\|_2 + |\beta|\|M_\sigma\|_2}{\sum_{i=0}^{\ell} |\alpha|^i |\beta|^{\ell-i} \|A_i\|_2} \cdot \|\mathcal{L}_\sigma(\alpha,\beta)\|_2 \cdot \frac{\|v\|_2}{\|x\|_2},\end{aligned}$$

where for the last inequality we have used (23). For the numerator of the first factor in the last expression we consider the identity $\|M_\ell\|_2 = \max(1, \|A_\ell\|_2)$, together with the inequality $\|M_\sigma\|_2 \le \ell \max_i(1, \|A_i\|_2)$, which is a direct consequence of Lemma 3.5 in [23]. The denominator can be bounded as: $\sum_{i=0}^{\ell} |\alpha|^i |\beta|^{\ell-i} \|A_i\|_2 \ge (|\alpha|^\ell + |\beta|^\ell) \min(\|A_0\|_2, \|A_\ell\|_2)$. The second factor in this expression can be bounded as follows, using again Lemma 3.5 in [23] in the first inequality:

$$\begin{aligned}\|\mathcal{L}_\sigma(\alpha,\beta)\|_2 &\le \sqrt{\ell} \max_{i,j} \left\{ |\alpha|^{j-1} |\beta|^{\ell-j}, |\alpha|^{\mathfrak{c}_\sigma(0:\ell-i-1)-1} |\beta|^{\ell-\mathfrak{c}_\sigma(0:\ell-i-1)-i} \|P_i(\alpha,\beta)\|_2 \right\} \\ &\le \sqrt{\ell} \max_{i,j} \left\{ |\alpha|^{j-1}|\beta|^{\ell-j}, \left(\sum_{k=1}^{i+1} |\alpha|^{\mathfrak{c}_\sigma(0:\ell-i-1)+i-k} |\beta|^{\ell-\mathfrak{c}_\sigma(0:\ell-i-1)-i+k-1} \right) \max_{s=\ell-i:\ell} \|A_s\|_2 \right\} \\ &\le \sqrt{\ell} \left(\sum_{i=1}^{\ell} |\alpha|^{i-1} |\beta|^{\ell-i} \right) \max_i (1, \|A_i\|_2).\end{aligned}$$

All this together gives:

$$\frac{\eta_P(x,\alpha,\beta)}{\eta_{F_\sigma}(v,\alpha,\beta)} \le \ell^{3/2} \frac{(|\alpha|+|\beta|)\left(\sum_{i=1}^{\ell} |\alpha|^{i-1}|\beta|^{\ell-i}\right)}{|\alpha|^\ell + |\beta|^\ell} \frac{\max_i(1,\|A_i\|_2)^2}{\min(\|A_0\|_2, \|A_\ell\|_2)} \frac{\|v\|_2}{\|x\|_2}.$$

Finally, (25) follows from the inequality

$$\frac{(|\alpha|+|\beta|)\left(\sum_{i=1}^{\ell}|\alpha|^{i-1}|\beta|^{\ell-i}\right)}{|\alpha|^{\ell}+|\beta|^{\ell}}\leq \ell,$$

which is an immediate consequence of the bound for $f_1(x)$ in [24, Lemma A.1] (taking $x^2 = |\alpha|/|\beta|$). □

The counterpart of Theorem 5.1 for left eigenvectors is Theorem 5.2.

Theorem 5.2. *Let $\sigma : \{0, 1, \ldots, \ell-1\} \to \{1, \ldots, \ell\}$ be a bijection and F_σ be the Fiedler pencil associated with σ. Let w be an approximate left eigenvector of F_σ associated with the approximate simple eigenvalue (α, β). Set $i_0 := \min\{i : \sigma$ has a consecution at $i\}$ and $i_0 := \ell - 1$ if σ has no consecutions, and define*

$$y = \begin{cases} (e_{\ell-i_0}^T \otimes I_n)w, & \text{if } |\alpha| \leq |\beta|, \\ (e_1^T \otimes I_n)w, & \text{if } |\alpha| > |\beta|. \end{cases}$$

Then y is an approximate left eigenvector of P associated with (α, β) and

$$\frac{\eta_P(y^*,\alpha,\beta)}{\eta_{F_\sigma}(w^*,\alpha,\beta)} \leq \ell^{5/2}\frac{\max_i\left(1,\|A_i\|_2\right)^2}{\min\left(\|A_0\|_2,\|A_\ell\|_2\right)}\frac{\|w\|_2}{\|y\|_2}. \tag{26}$$

Proof. The proof of this result is completely analogous to the proof of Theorem 5.1, but using (6) and (20) instead of (5) and (23). □

The bounds (25) and (26) are not expected to be sharp. Note, for instance, that to get the power 5/2 of ℓ we have added an extra factor ℓ in the bound for $\|M_\sigma\|_2$. This factor is, for most Fiedler pencils F_σ, a number $2 \leq s \leq \ell$, with s being the largest number of nonzero blocks in all block rows of F_σ, and can be obtained using Lemma A.1. Also, in the bound for $\|\mathcal{L}_\sigma(\alpha,\beta)\|_2$ we have considered all subindices i, j, instead of those $i \in \{i : \sigma \text{ has an inversion at } \ell-i-1\}$ and those $j = \mathfrak{c}_\sigma(0 : \ell-k-1)$, where σ has a consecution at $\ell-k-1$. As a consequence, we could replace the factor $\max_i \|A_i\|_2$ by $\max_{i\geq \ell-i_0-1}\|A_i\|_2$, with $i_0 = \max\{i : \sigma \text{ has a consecution at } i\}$. We will show in Theorem 7.1 that the exponent 2 of $\max_i \|A_i\|_2$ in both (25) and (26) can be reduced to 1 after scaling the coefficients of P.

6 Bounds for the ratio between condition numbers

This section is the counterpart of Section 5 for the condition number of eigenvalues. More precisely, we get bounds for the ratio $\kappa_{F_\sigma}(\alpha,\beta)/\kappa_P(\alpha,\beta)$, for any Fiedler linearization F_σ of P.

Theorem 6.1. *Let F_σ be a Fiedler linearization of P, and let (α, β) be a simple eigenvalue of P. Then*

$$\frac{1}{\sqrt{2\ell}}\frac{\max(1,\max_{i=0:\ell-1}\|A_i\|_2)}{\max_i\|A_i\|_2} \leq \frac{\kappa_{F_\sigma}(\alpha,\beta)}{\kappa_P(\alpha,\beta)} \leq \sqrt{2}\,\ell^4\frac{\max_i(1,\|A_i\|_2)^3}{\min(\|A_0\|_2,\|A_\ell\|_2)}. \tag{27}$$

Proof. Let us start proving the upper bound, and consider first the case $\beta \neq 0$. In this case, and following Theorem 4.4, if x, y denote, respectively, a right and a left eigenvector of P associated with (α, β), then $v := \mathcal{R}_\sigma(\alpha,\beta)x$ and $w := \mathcal{L}_\sigma(\alpha,\beta)y$ are a right and a left eigenvector of F_σ, respectively. From Proposition B.1 we get:

$$\frac{|y^*(\overline{\beta}(D_\alpha P)|_{(\alpha,\beta)} - \overline{\alpha}(D_\beta P)|_{(\alpha,\beta)})x|}{|w^*(\overline{\beta}(D_\alpha F_\sigma)|_{(\alpha,\beta)} - \overline{\alpha}(D_\beta F_\sigma)|_{(\alpha,\beta)})v|} = |\beta|^{1-\ell}, \tag{28}$$

so that replacing this into (8)–(9) we achieve:

$$\frac{\kappa_{F_\sigma}(\alpha,\beta)}{\kappa_P(\alpha,\beta)} = |\beta|^{1-\ell}\frac{\left(|\alpha|^2\|M_\ell\|_2^2+|\beta|^2\|M_\sigma\|_2^2\right)^{1/2}}{\left(\displaystyle\sum_{i=0}^{\ell}|\alpha|^{2i}|\beta|^{2(\ell-i)}\|A_i\|_2^2\right)^{1/2}}\frac{\|\mathcal{R}_\sigma(\alpha,\beta)x\|_2\|\mathcal{L}_\sigma(\alpha,\beta)y\|_2}{\|x\|_2\|y\|_2}. \tag{29}$$

Using Lemma 3.5 in [23], we can bound:

$$\begin{aligned}\left(|\alpha|^2\|M_\ell\|_2^2+|\beta|^2\|M_\sigma\|_2^2\right)^{1/2} &\leq \left(|\alpha|^2\max(1,\|A_\ell\|_2^2)+|\beta|^2\ell^2\max(1,\max_{i=0:\ell-1}\|A_i\|_2^2)\right)^{1/2}\\ &\leq \ell(|\alpha|^2+|\beta|^2)^{1/2}\max_i(1,\|A_i\|_2).\end{aligned} \tag{30}$$

Also:

$$\left(\sum_{i=0}^{\ell}|\alpha|^{2i}|\beta|^{2(\ell-i)}\|A_i\|_2^2\right)^{1/2} \geq \left(|\alpha|^{2\ell}+|\beta|^{2\ell}\right)^{1/2}\min(\|A_0\|_2,\|A_\ell\|_2). \tag{31}$$

To bound the rightmost factor in (29) or, more precisely, the ratios $\|\mathcal{R}_\sigma(\alpha,\beta)x\|_2/\|x\|_2$ and $\|\mathcal{L}_\sigma(\alpha,\beta)y\|_2/\|y\|_2$, we consider separately the cases:

(i) $|\alpha|\leq|\beta|$: In this case, we first consider the inequality:

$$\frac{\|\mathcal{R}_\sigma(\alpha,\beta)x\|_2\|\mathcal{L}_\sigma(\alpha,\beta)y\|_2}{\|x\|_2\|y\|_2}\leq\|\mathcal{R}_\sigma(\alpha,\beta)\|_2\|\mathcal{L}_\sigma(\alpha,\beta)\|_2. \tag{32}$$

Replacing the bounds (50) in (29), together with (30)–(32), we achieve:

$$\begin{aligned}\frac{\kappa_{F_\sigma}(\alpha,\beta)}{\kappa_P(\alpha,\beta)} &\leq \ell^4|\beta|^{1-\ell}\frac{(|\alpha|^2+|\beta|^2)^{1/2}\max_i(1,\|A_i\|_2)}{(|\alpha|^{2\ell}+|\beta|^{2\ell})^{1/2}\min(\|A_0\|_2,\|A_\ell\|_2)}|\beta|^{2\ell-2}\max_i(1,\|A_i\|_2)^2\\ &\leq \ell^4|\beta|^{\ell-1}\frac{(2|\beta|^2)^{1/2}}{|\beta|^\ell}\frac{\max_i(1,\|A_i\|_2)^3}{\min(\|A_0\|_2,\|A_\ell\|_2)}\leq\sqrt{2}\,\ell^4\frac{\max_i(1,\|A_i\|_2)^3}{\min(\|A_0\|_2,\|A_\ell\|_2)},\end{aligned}$$

where in the second inequality we have used that $0\leq|\alpha|\leq|\beta|$.

(ii) $|\alpha|>|\beta|$: Replacing the inequalities (51) in (29), together with (30)–(31), we achieve (27) proceeding exactly as in case (i) (just interchanging the roles of $|\alpha|$ and $|\beta|$).

Now, let us consider the case $\beta=0$. In this case, and as a consequence of Theorem 4.4, if x,y denote, respectively, a right and a left eigenvector of P associated with (α,β), then a right and a left eigenvector of F_σ are $(e_1\otimes I_n)x$ and $(e_1\otimes I_n)y$, respectively. Now, using Proposition B.1, together with (30)–(31), and the identity:

$$\frac{\|(e_1\otimes I_n)x\|_2\|(e_1\otimes I_n)y\|_2}{\|x\|_2\|y\|_2}=1,$$

we get the identities:

$$\frac{\kappa_{F_\sigma}(\alpha,\beta)}{\kappa_P(\alpha,\beta)}=|\alpha|^{\ell-1}\frac{|\alpha|\max(1,\|A_\ell\|_2)}{|\alpha|^\ell\|A_\ell\|_2}\frac{\|(e_1\otimes I_n)x\|_2\|(e_1\otimes I_n)y\|_2}{\|x\|_2\|y\|_2}=\frac{\max(1,\|A_\ell\|_2)}{\|A_\ell\|_2}, \tag{33}$$

which clearly satisfies the statement.

For the lower bound, we also start with the case $\beta\neq0$. First notice that, from Remark 4.3,

$$\|\mathcal{R}_\sigma(\alpha,\beta)x\|_2\|\mathcal{L}_\sigma(\alpha,\beta)y\|_2\geq|\beta|^{\ell-1}\max(|\alpha|,|\beta|)^{\ell-1}\|x\|_2\|y\|_2,$$

so, from (28) and (8)–(9):

$$\begin{aligned}
\frac{\kappa_{F_\sigma}(\alpha,\beta)}{\kappa_P(\alpha,\beta)} &\geq \max(|\alpha|,|\beta|)^{\ell-1}\frac{(|\alpha|^2\|M_\ell\|_2^2+|\beta|^2\|M_\sigma\|_2^2)^{1/2}}{\left(\sum_{i=0}^{\ell}|\alpha|^{2i}|\beta|^{2(\ell-i)}\|A_i\|_2^2\right)^{1/2}} \\
&\geq \max(|\alpha|,|\beta|)^{\ell-1}\frac{1}{\sqrt{2}}\frac{|\alpha|\|A_\ell\|_2+|\beta|\max(1,\max_{i=0:\ell-1}\|A_i\|_2)}{\left(|\alpha|^{2\ell}\|A_\ell\|_2^2+\ell|\beta|^2\max(|\alpha|,|\beta|)^{2\ell-2}\max\limits_{i=0:\ell}\|A_i\|_2^2\right)^{1/2}} \\
&\geq \max(|\alpha|,|\beta|)^{\ell-1}\frac{1}{\sqrt{2}}\frac{|\alpha|\|A_\ell\|_2+|\beta|\max(1,\max_{i=0:\ell-1}\|A_i\|_2)}{|\alpha|^{\ell}\|A_\ell\|_2+\sqrt{\ell}|\beta|\max(|\alpha|,|\beta|)^{\ell-1}\max\limits_{i=0:\ell}\|A_i\|_2} \\
&\geq \max(|\alpha|,|\beta|)^{\ell-1}\frac{1}{\sqrt{2\ell}}\frac{|\alpha|\|A_\ell\|_2+|\beta|\max(1,\max_{i=0:\ell-1}\|A_i\|_2)}{|\alpha|^{\ell}\|A_\ell\|_2+|\beta|\max(|\alpha|,|\beta|)^{\ell-1}\max\limits_{i=0:\ell}\|A_i\|_2} \\
&\geq \frac{1}{\sqrt{2\ell}}\frac{\max(1,\max_{i=0:\ell-1}\|A_i\|_2)}{\max_i\|A_i\|_2}.
\end{aligned}$$

The result for the case $\beta = 0$ is an immediate consequence of (33). □

Remark 6.2. *Note that if $\|A_\ell\|_2 \geq 1$ then $\frac{\kappa_{F_\sigma}(\alpha,0)}{\kappa_P(\alpha,0)} = 1$. The reason for that is the following: The leading term of F_σ, which is the only relevant one for computing the eigenvalue $(\alpha, 0)$, is of the form $\operatorname{diag}(A_\ell, I)$, and the corresponding eigenvectors are the first block of the eigenvectors of P. Therefore, computing the eigenvalue $(\alpha, 0)$ through F_σ is essentially the same problem as computing it in the polynomial P.*

As for the case of backward errors in Theorem 5.1, the upper bound in (27) is not expected to be sharp. We will see in Theorem 7.4 that the exponent 3 in the factor $\max_i \|A_i\|_2$ is in some sense "fake", since after dividing the coefficients of P by the norm of the largest coefficient this exponent decreases to 1.

7 Scaling

7.1 Scaling the coefficients

Replacing P by sP in (1), with $s \in \mathbb{C}$, gives a new matrix polynomial whose coefficients are sA_i, for $i = 0, 1, \ldots, \ell$, and which has the same eigenpairs as P. We refer to this replacement as *coefficient scaling*. From (5)–(6), we see that the backward error of left and right eigenpairs is invariant under this scaling (provided that the computed eigenpairs obtained using both P and sP coincide). Similarly, (8) tells us that the condition number of eigenvalues is invariant as well. However, when solving the PEP (2) by linearization, the scaling $P \mapsto sP$ does not correspond to a coefficient scaling in the linearization, so that the backward error of eigenpairs, as well as the condition number of eigenvalues in the linearization, may change. In Theorems 7.1 and 7.4, we see that this scaling significantly reduces the bounds obtained in (25)–(27).

In the following, we set

$$\rho := \frac{\max_i \|A_i\|_2}{\min(\|A_0\|_2, \|A_\ell\|_2)}, \tag{34}$$

since this is a key quantity in the bounds that we get in Theorems 7.1 and 7.4, and allows us to measure the ratio between the backward error of eigenpairs and the condition number of eigenvalues in the matrix polynomial and the Fiedler linearization.

Theorem 7.1. *Let F_σ be a Fiedler linearization of P. Let $s := \max_i \|A_i\|_2$, let ρ be as in (34), and set $\widehat{P} := s^{-1}P$. Let (v, α, β) and (w^*, α, β) be, respectively, right and left (approximate) eigenpairs of $\widehat{F}_\sigma := F_\sigma(\widehat{P})$. Set*

$$j_0 := \begin{cases} \min\{j : \sigma \text{ has an inversion at } j\} \\ \ell - 1, \text{ if } \sigma \text{ has no inversions,} \end{cases} \quad \text{and} \quad i_0 := \begin{cases} \min\{i : \sigma \text{ has a consecution at } i\} \\ \ell - 1, \text{ if } \sigma \text{ has no consecutions,} \end{cases}$$

and set

$$x := \begin{cases} (e_{\ell-j_0}^T \otimes I_n)v, & \text{if } |\alpha| \le |\beta| \\ (e_1^T \otimes I_n)v, & \text{if } |\alpha| > |\beta| \end{cases}, \qquad y := \begin{cases} (e_{\ell-i_0}^T \otimes I_n)w, & \text{if } |\alpha| \le |\beta| \\ (e_1^T \otimes I_n)w, & \text{if } |\alpha| > |\beta| \end{cases}.$$

Then (x, α, β) *and* (y^*, α, β) *are, respectively, right and left (approximate) eigenpairs of* P*. If* (α, β) *is an (approximate) simple eigenvalue, then*

$$\frac{\eta_P(x,\alpha,\beta)}{\eta_{\widehat{F}_\sigma}(v,\alpha,\beta)} \le \ell^{5/2}\rho\,\frac{\|v\|_2}{\|x\|_2}, \qquad \frac{\eta_P(y^*,\alpha,\beta)}{\eta_{\widehat{F}_\sigma}(w^*,\alpha,\beta)} \le \ell^{5/2}\rho\,\frac{\|w\|_2}{\|y\|_2}. \tag{35}$$

Proof. The result is a direct consequence of Theorems 5.1 and 5.2, since $\max_i \|s^{-1}A_i\|_2 = 1$, and $\min(\|s^{-1}A_0\|_2, \|s^{-1}A_\ell\|_2) = s^{-1}\min(\|A_0\|_2, \|A_\ell\|_2)$. □

The bounds in (35) depend on two factors: the first one, $\ell^{5/2}\rho$ is given explicitly in terms of the norm of the coefficients of the polynomial $P(\alpha, \beta)$, so it is possible to get an idea on how large it can be once $P(\alpha, \beta)$ is given. The second factor is a ratio between the norm of the eigenvector of F_σ and the corresponding eigenvector of P. The next result provides a bound on this ratio for the unscaled case (namely, the one considered in Theorems 5.1 and 5.2), provided that the eigenvectors are the exact ones. However, we want to emphasize that the computed eigenvectors may differ very much from the exact ones.

Lemma 7.2. *Let* v *and* w *be, respectively, a right and a left (exact) eigenvector of* $F_\sigma(\alpha, \beta)$ *associated with a simple eigenvalue* (α, β)*, and let* x *and* y *be as in Theorems* 5.1 *and* 5.2*. Then*

$$\frac{\|v\|_2}{\|x\|_2}, \ \frac{\|w\|_2}{\|y\|_2} \le \ell^{3/2} \max_{0 \le i \le \ell}(1, \|A_i\|_2).$$

Proof. We only prove the bound for the right eigenvectors, since the proof for the left eigenvectors follows similar arguments.

Let us first consider the case $|\alpha| \le |\beta|$. By Theorem 4.4, the right eigenvector, v, of F_σ associated with (α, β) is of the form $v = \mathcal{R}_\sigma(\alpha, \beta)z$, for some right eigenvector, z, of P. By (18), we have $x = (e_{\ell-j_0}^T \otimes I_n)\mathcal{R}_\sigma(\alpha, \beta)z = \beta^{\ell-1}z$, where the last identity follows from Remark 4.3 2(a). Therefore

$$\begin{aligned} \frac{\|v\|_2}{\|x\|_2} &= \frac{\|\mathcal{R}_\sigma(\alpha,\beta)z\|_2}{\|x\|_2} = \frac{\|\mathcal{R}_\sigma(\alpha,\beta)z\|_2}{|\beta|^{\ell-1}\|z\|_2} \le \frac{\|\mathcal{R}_\sigma(\alpha,\beta)\|_2}{|\beta|^{\ell-1}} \le \frac{\ell^{3/2}|\beta|^{\ell-1}\max_i(1,\|A_i\|_2)}{|\beta|^{\ell-1}} \\ &= \ell^{3/2}\max(1, \max_i \|A\|_2), \end{aligned}$$

where in the last inequality we have used (50).

Now, let us address the case $|\alpha| > |\beta|$ and $\beta \neq 0$. Again, by Theorem 4.4, $v = \mathcal{R}_\sigma(\alpha, \beta)z$, for some right eigenvector, z, of P. Now, by definition of x and using (18), we get

$$x = (e_1^T \otimes I_n)v = (e_1^T \otimes I_n)\mathcal{R}_\sigma(\alpha,\beta)z = \alpha^{\mathrm{i}(\sigma)}\beta^{\ell-1-\mathrm{i}(\sigma)}z.$$

Now, from (54) (which is valid for any right eigenvector, x, of P), we get

$$\frac{\|v\|_2}{\|x\|_2} = \frac{\|\mathcal{R}_\sigma(\alpha,\beta)z\|_2}{|\alpha|^{\mathrm{i}(\sigma)}|\beta|^{\ell-1-\mathrm{i}(\sigma)}\|z\|_2} \le \ell^{3/2}\max_{0\le i\le \ell}(1, \|A_i\|_2),$$

as wanted.

If $\beta = 0$, by Theorem 4.4(b), $v = (e_1 \otimes I_n)x$, so $\|v\|_2/\|x\|_2 = 1$.

For left eigenvectors we proceed in a similar way using $\mathcal{L}_\sigma(\alpha, \beta)$ instead of $\mathcal{R}_\sigma(\alpha, \beta)$. □

As an immediate consequence of Lemma 7.2 we have the following corollary, which gives a bound on the ratios $\|v\|_2/\|x\|_2$ and $\|w\|_2/\|y\|_2$ in Theorem 7.1, provided that v, w, x, y are the exact eigenvectors.

Corollary 7.3. *Let v and w be, respectively, a right and a left (exact) eigenvector of $\widehat{F}_\sigma(\alpha,\beta)$ associated with a simple eigenvalue (α,β) of P, and let x and y be as in Theorem 7.1. Then*

$$\frac{\|v\|_2}{\|x\|_2}\,,\;\frac{\|w\|_2}{\|y\|_2} \le \ell^{3/2}.$$

The proof of Lemma 7.2 shows the reason for taking a different piece of the eigenvector v (respectively, w) of F_σ in Theorem 5.1 (resp., Theorem 5.2), or of $\widehat{F}_\sigma$ in Theorem 7.1, to get an eigenvector, x (resp., y), of P, depending on whether $|\alpha| \le |\beta|$ or $|\alpha| > |\beta|$. Since, by Theorem 4.5, both $x = (e_{\ell-j_0}^T \otimes I_n)v$ and $x = (e_1^T \otimes I_n)v$ are right eigenvectors of P, we could take any of them to recover an eigenvector of P from v. However, there may be a relevant difference in norm between these two vectors, so it is advisable to get the one with largest norm, in order to minimize the ratio $\|v\|_2/\|x\|_2$. Corollary 7.3 guarantees that the piece taken in Theorem 7.1 provides an eigenvector x whose norm is close to the one of the whole vector v (up to $\ell^{3/2}$). The same happens with the left eigenvectors.

Regarding the condition number, the counterpart of Theorem 6.1 in the scaled case is the following result.

Theorem 7.4. *Let F_σ be a Fiedler linearization of P, let $s := \max_i \|A_i\|_2$, let ρ be as in (34), and set $\widehat{P} := s^{-1}P$, $\widehat{F}_\sigma := F_\sigma(\widehat{P})$. If (α,β) is a simple eigenvalue of P, then*

$$\frac{\kappa_{\widehat{F}_\sigma}(\alpha,\beta)}{\kappa_P(\alpha,\beta)} \le \sqrt{2}\ell^4\rho. \tag{36}$$

Proof. As for the backward errors, the result is a direct consequence of Theorem 6.1, since $\max_i \|s^{-1}A_i\|_2 = 1$, and $\min(\|s^{-1}A_0\|_2, \|s^{-1}A_\ell\|_2) = s^{-1}\min(\|A_0\|_2, \|A_\ell\|_2)$. ∎

Theorems 7.1 and 7.4 show that scaling the coefficients of the polynomial reduces significantly the bound for the ratio between both backward errors of eigenpairs and condition numbers of eigenvalues in the linearization and the polynomial, for any Fiedler linearization. Besides this, we will see in Theorem 7.5 that the ratio between the condition number of a given eigenvalue in any two Fiedler linearizations can be bounded by some quantity which depends only on the size and the degree of the polynomial P, and which is by other means independent on the coefficients of P. Theorem 7.5 relies on Lemma B.3 which, because of its technical nature, is taken to Appendix B.

Theorem 7.5. *Let P be a matrix polynomial, set $s := \max_i \|A_i\|_2$, and $\widehat{P} := s^{-1}P$. Let F_{σ_1} and F_{σ_2} be two Fiedler linearizations of P, and set $\widehat{F}_{\sigma_1} := F_{\sigma_1}(\widehat{P})$, $\widehat{F}_{\sigma_2} := F_{\sigma_2}(\widehat{P})$. Let (α,β) be a simple eigenvalue of P. Then*

$$\frac{1}{\ell^3\sqrt{n}} \le \frac{\kappa_{\widehat{F}_{\sigma_1}}(\alpha,\beta)}{\kappa_{\widehat{F}_{\sigma_2}}(\alpha,\beta)} \le \ell^3\sqrt{n}\,. \tag{37}$$

Proof. Along the proof, let us denote by $\widehat{x},\widehat{y}$ the right and left eigenvectors of $\widehat{P}$ associated with (α,β), and by $\widehat{v}_\sigma,\widehat{w}_\sigma$ the corresponding right and left eigenvectors of $\widehat{F}_\sigma := \alpha\widehat{M}_\ell + \beta\widehat{M}_\sigma$ as described in the statement of Proposition B.1. Let us first notice that, as a consequence of Proposition B.1,

$$\widehat{w}_{\sigma_1}^*(\overline{\beta}(D_\alpha\widehat{F}_{\sigma_1})|_{(\alpha,\beta)} - \overline{\alpha}(D_\beta\widehat{F}_{\sigma_1})|_{(\alpha,\beta)})\widehat{v}_{\sigma_1} = \widehat{w}_{\sigma_2}^*(\overline{\beta}(D_\alpha\widehat{F}_{\sigma_2})|_{(\alpha,\beta)} - \overline{\alpha}(D_\beta\widehat{F}_{\sigma_2})|_{(\alpha,\beta)})\widehat{v}_{\sigma_2},$$

and, replacing this in (8):

$$\frac{\kappa_{\widehat{F}_{\sigma_1}}(\alpha,\beta)}{\kappa_{\widehat{F}_{\sigma_2}}(\alpha,\beta)} = \frac{(|\alpha|^2\|\widehat{M}_\ell\|_2^2 + |\beta|^2\|\widehat{M}_{\sigma_1}\|_2^2)^{1/2}}{(|\alpha|^2\|\widehat{M}_\ell\|_2^2 + |\beta|^2\|\widehat{M}_{\sigma_2}\|_2^2)^{1/2}}\,\frac{\|\widehat{v}_{\sigma_1}\|_2\|\widehat{w}_{\sigma_1}\|_2}{\|\widehat{v}_{\sigma_2}\|_2\|\widehat{w}_{\sigma_2}\|_2}. \tag{38}$$

Now, from the inequalities $(1/\sqrt{n})\|A\|_F \leq \|A\|_2 \leq \|A\|_F$, valid for every matrix A [25, p. 314], with $\|\cdot\|_F$ being the Frobenius norm, and the identity $\|\widehat{M}_{\sigma_1}\|_F = \|\widehat{M}_{\sigma_2}\|_F$, we get:

$$\frac{1}{\sqrt{n}} \frac{\|\widehat{v}_{\sigma_1}\|_2 \|\widehat{w}_{\sigma_1}\|_2}{\|\widehat{v}_{\sigma_2}\|_2 \|\widehat{w}_{\sigma_2}\|_2} \leq \frac{\kappa_{\widehat{F}_{\sigma_1}}(\alpha,\beta)}{\kappa_{\widehat{F}_{\sigma_2}}(\alpha,\beta)} \leq \sqrt{n} \frac{\|\widehat{v}_{\sigma_1}\|_2 \|\widehat{w}_{\sigma_1}\|_2}{\|\widehat{v}_{\sigma_2}\|_2 \|\widehat{w}_{\sigma_2}\|_2}. \tag{39}$$

From Lemma B.3,

$$\frac{1}{\ell^3} \leq \frac{\|\widehat{v}_{\sigma_1}\|_2 \|\widehat{w}_{\sigma_1}\|_2}{\|\widehat{v}_{\sigma_2}\|_2 \|\widehat{w}_{\sigma_2}\|_2} \leq \ell^3, \tag{40}$$

and replacing (39) and (40) in (38), the result follows. □

Note that Theorem 7.5 does not guarantee that the condition number ratio is small after scaling the coefficients of P. However, it says that after this elementary scaling, the conditioning of any eigenvalue of P does not differ too much from one Fiedler linearization to another. Moreover, our numerical experiments indicate that the condition number ratio is indeed small after this scaling.

7.2 Parameter scaling

Another kind of scaling that has been considered in the PEP with a single parameter λ, namely $P(\lambda)x = 0$, with $P(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$, is the *parameter scaling* [3, 18, 23, 24, 29]. This scaling consists of replacing λ by $\gamma\lambda$, with $\gamma > 0$, so that the polynomial $P(\lambda)$ is replaced by $Q(\lambda) := P(\gamma\lambda) = \sum_{j=0}^{\ell} \lambda^j(\gamma^j A_j)$, whose jth coefficient is $\gamma^j A_j$. Note that λ_0 is a simple eigenvalue of $Q(\lambda)$ with associated right and left eigenvectors x and y, respectively, if and only if $\gamma\lambda_0$ is a simple eigenvalue of $P(\lambda)$ with the same associated eigenvectors. In the light of equations (35) and (36), the goal is to find a value of γ such that

$$\rho(\gamma) = \frac{\max_i \gamma^i \|A_i\|_2}{\min(\|A_0\|_2, \gamma^\ell \|A_\ell\|_2)} \tag{41}$$

is small. It has been proved in [3] that the (unique) value which minimizes (41) is:

$$\gamma_{\text{opt}} = \left(\frac{\|A_0\|_2}{\|A_\ell\|_2}\right)^{1/\ell}. \tag{42}$$

This is the value that we have considered in some of the numerical experiments included in Section 8.

8 Numerical experiments

In this section, for simplicity, we follow the non-homogeneous notation for eigenvalues, that is $\lambda = (\alpha, 1)$. All numerical examples have been run on MATLAB-R2018b, with unit roundoff 2^{-53}. For the backward error, we have focused on the right eigenvectors, for brevity.

8.1 NLEVP collection

Here we report our tests on a couple of examples with degree ≥ 3 within the NLEVP collection [4], more precisely:

1. `orr_sommerfeld`: This is a 64×64 quartic PEP with $\|A_0\|_2 = 1$, $\|A_1\|_2 = 5.8 \cdot 10^3$, $\|A_2\|_2 = 1.7 \cdot 10^6$, $\|A_3\|_2 = 2.4 \cdot 10^7$, and $\|A_4\|_2 \approx 2 \cdot 10^{12}$. We have computed all eigenvalues using the standard MATLAB code `polyeig`[1] and the Fiedler linearization F_σ such that PCIS(σ)=$(1, 0, 1)$. In Figure 1 (a) we plot the ratio between the backward error of all computed eigenpairs in the

[1] The linearization used by `polyeig` is the reversal of the first Frobenius companion form associated with the reversal polynomial, where the reversal of a polynomial $P(\lambda)$ is rev $P(\lambda) := \lambda^\ell P(1/\lambda)$

polynomial and in the linearization. As it can be seen, in the unscaled case the ratio is quite large. The coefficient scaling improves significantly these results, though there is still a large ratio for small eigenvalues. However, the coefficient+parameter scaling has an excellent performance (ratios close to 1). Note that the results are in accordance with (35). For the coefficient scaling, the bound is quite pessimistic, whereas for the coefficient+parameter scaling it is more accurate. We did not plot the bound (25) for the unscaled case, since it is quite large, due to the fact that $\max_i \|A_i\|_2 \approx 2 \cdot 10^{12}$ is very large. The condition number ratio in Figure 1 (b) has a similar behavior. Again, the ratio is in accordance with the theoretical bound (36), though for the largest eigenvalues is quite pessimistic. Note, in particular, that, again, adding a parameter scaling we get ratios close to 1 for all eigenvalues. This difference in the ratios can be explained after computing the values of ρ for both scalings. For the coefficient scaling we get $\rho \approx 3.5 \cdot 10^{11}$, whereas using $\gamma_{\rm opt}$ as in (42), we get $\rho(\gamma_{\rm opt}) \approx 4.86$ for the parameter scaling. The maximum ratio $\|v\|_2/\|x\|_2$, with v and x as in Theorem 7.1 is around 1.4 for both scalings, which is in accordance with Corollary 7.3 (even though v and x are computed eigenvectors), since $\ell^{3/2} = 8$.

In Figure 2 we have plotted the backward error for the computed eigenpairs, including the computation carried out with `polyeig`. For small eigenvalues (those with modulus less than 0.1), both `polyeig` and the unscaled Fiedler linearization give poor results. Even scaling the coefficients gives poor results for eigenvalues with modulus less than, say, 10^{-2}. However, parameter scaling provides good results for all eigenvalues (backward error of order 10^{-15}).

2. `plasma_drift`: This is a 128×128 cubic PEP having coefficients with moderate norm (more precisely, $\|A_0\|_2 \approx 128$, $\|A_1\|_2 \approx 1.2 \cdot 10^3$, $\|A_2\|_2 \approx 5.3$, and $\|A_3\|_2 \approx 12.7$). In Figure 3 we plot the backward error ratio of the right eigenpairs computed with all four Fiedler linearizations. As it can be seen, there is no much difference between the results obtained for all linearizations. In the case of coefficient scaling, the backward error ratio is less than 10 for all eigenpairs. Coefficient scaling just slightly improves these results. In both cases, the results are between two and three orders of magnitude better than the ones expected by the bound (35). Here $\rho \approx 97.1$ and $\rho(\gamma_{\rm opt}) \approx 21$, with $\gamma_{\rm opt}$ being as in (42), and the maximum ratio $\|v\|_2/\|x\|_2$, with v and x as in Theorem 7.1, is less than 2 for both scalings, which is again in accordance with Corollary 7.3 (up to the fact that v is the computed eigenvector), since $\ell^{3/2} = \sqrt{27} \approx 5.2$.

In Figure 4 we compare the ratio between conditioning of eigenvalues in the linearization and the polynomial, that is, $\kappa_{F_\sigma}(\lambda_i)/\kappa_P(\lambda_i)$ (without scaling), and $\kappa_{\widehat{F}_\sigma}(\lambda_i)/\kappa_P(\lambda_i)$ (with coefficient and parameter scaling) for all four Fiedler linearizations F_σ. As it can be seen, both the coefficient and the coefficient+parameter scaling have a very good performance for all Fiedler linearizations (the ratio is between 1 and 10 in most cases, and only for a few eigenvalues it is around 10^2 in the coefficient+parameter scaling), and the differences between any two linearizations are negligible. Note that the bound (36) is satisfied, though it is about four orders of magnitude larger than the real ratio (in the coefficient scaling case) and between two and three orders of magnitude larger (in the coefficient+parameter scaling). We did not plot the bound (27) in the unscaled case, since it is of order 10^{10}, which is quite large.

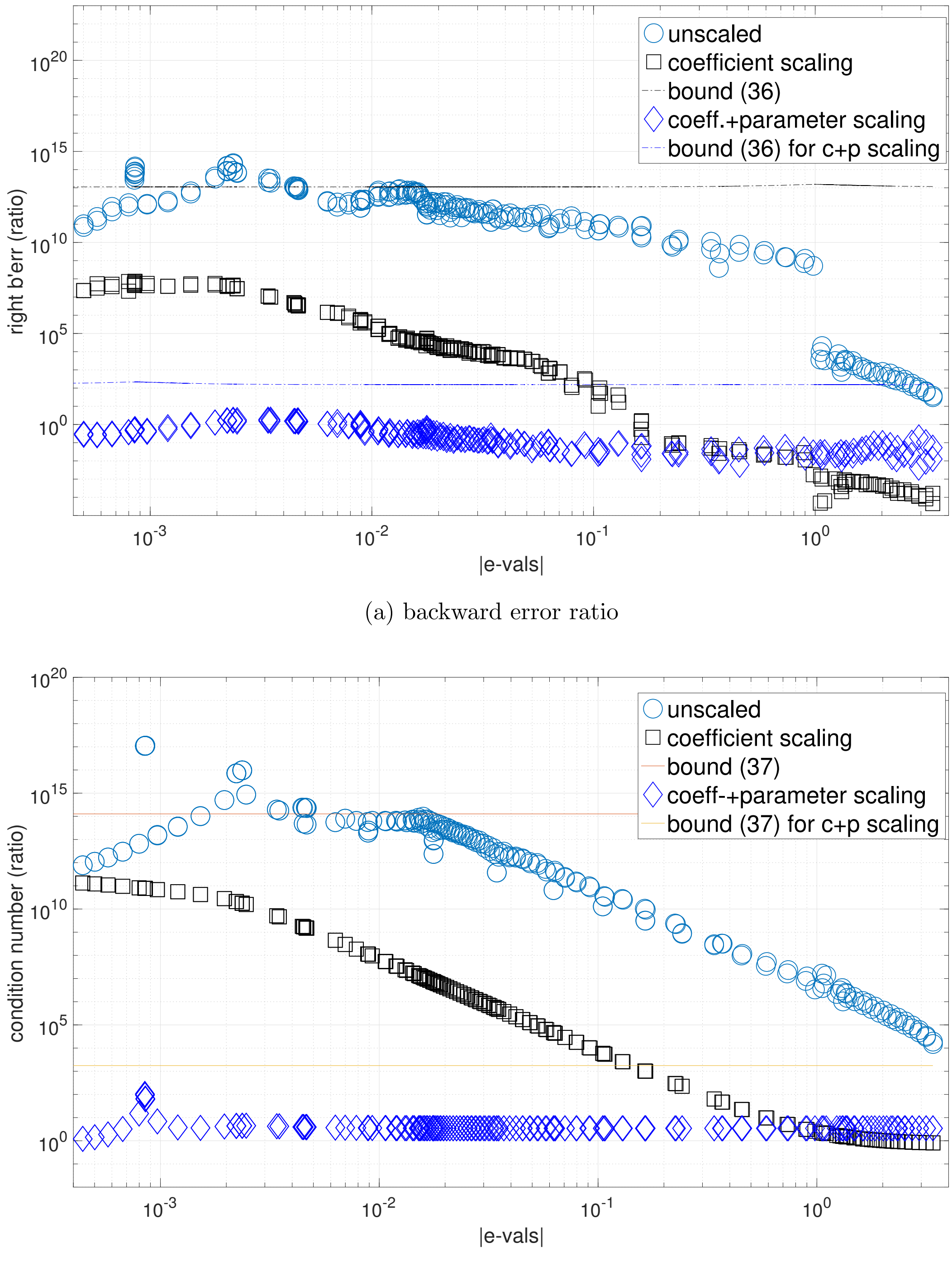
The maximum ratio of the conditioning for any eigenvalue computed with any two different linearizations is:

$$\max_k \left(\max_{1 \le i < j \le 4} \frac{\kappa_{\widehat{F}_{\sigma_i}}(\lambda_k)}{\kappa_{\widehat{F}_{\sigma_j}}(\lambda_k)} \right) = 2.87,$$

which is in accordance with (37), since in this case the upper bound is $\ell^3\sqrt{n} = 27\sqrt{128} \approx 305.47$.

8.2 Random matrix polynomials

Test 1. Our first example is a random 50×50 cubic matrix polynomial $P(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 + \lambda^3 A_3$, with $A_0 = 10^6 \times \texttt{rand}(50), A_1 = 10^2 \times \texttt{rand}(50), A_2 = 10^{-2} \times \texttt{rand}(50), A_3 = 10^{-6} \times \texttt{rand}(50)$. This polynomial has coefficients with quite different norms, and the value of ρ in (34) is large (around 10^{12}). We have solved the PEP with all Fiedler linearizations for cubic matrix polynomials (unscaled, with diagonal scaling, and with both diagonal and parameter scaling, respectively). In Figure 5 we plot the backward error ratio for all right eigenpairs, and



(a) backward error ratio

(b) condition number ratio

Figure 1: `orr_sommerfeld`. Backward error ratio for right eigenpairs and condition number ratio for eigenvalues (represented by the absolute values of the eigenvalues) computed with the Fiedler linearization F_σ with PCIS(σ)=$(1, 0, 1)$.

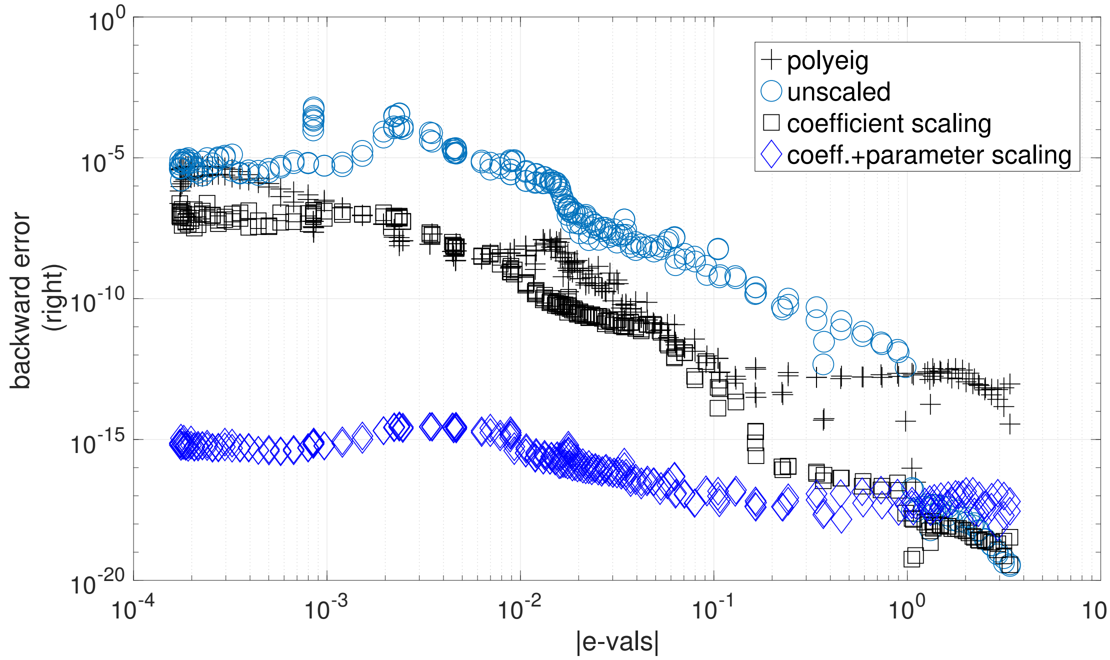
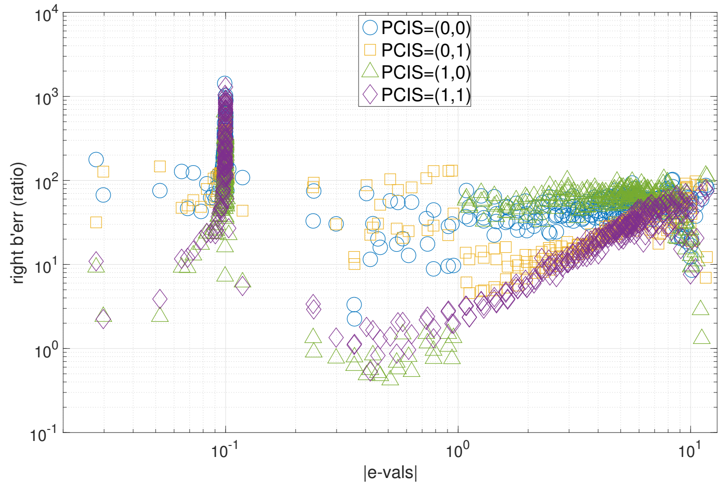


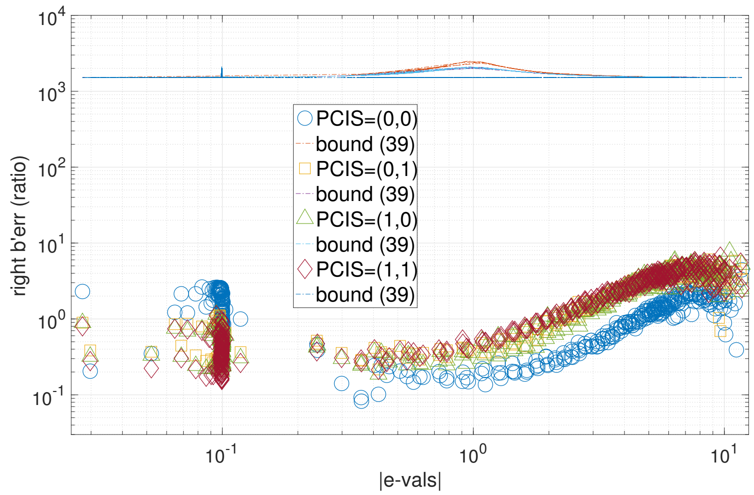
Figure 2: `orr_sommerfeld`. Backward error for right eigenpairs (represented by the absolute values of the eigenvalues) computed with `polyeig` and the Fiedler linearization F_σ with PCIS(σ)=$(1,0,1)$.

in Figure 6 we plot the condition number ratio for all eigenvalues λ_i. As it can be seen, the coefficient scaling slightly improves the backward error, but not the condition number. This is in accordance with Theorems 7.1 and 7.4, because $\rho \approx 10^{12}$ is large. By contrast, a further parameter scaling significantly improves both the backward error and (even more) the condition number. In particular, coefficient+parameter scaling gives an excellent performance, with ratios close to 1 in both the backward error and condition number, which means that the backward error and conditioning of the PEP is essentially the one of the linearization. This is explained by the bounds (35) and (36), since $\rho(\gamma_{\rm opt}) \approx 1$. Note that the bound (36) for the coefficient scaling case is pessimistic in between two and five orders of magnitude. For the coefficient+parameter scaling this bound is around 10^2 (we did not plot it), so it is sharper, but still pessimistic. As for the backward error, the bound (35) in the coefficient scaling case is of order 10^{10} (we did not plot it), so it is very pessimistic. We did not plot the bounds (26) and (27), again, because they are very large and completely inaccurate. Note also that, in the unscaled case, there are differences of about 10^3 in the condition number ratio using different linearizations. However, in the scaled cases these differences are negligible, which is in accordance with Theorem 7.5.

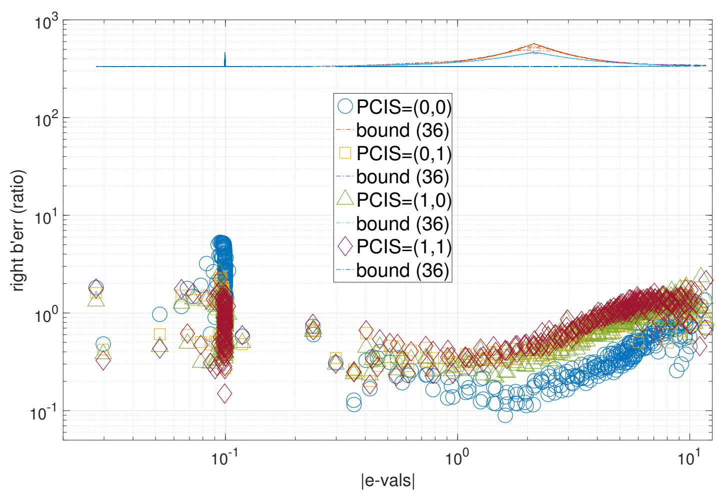
Test 2. The previous experiments suggest that coefficient scaling may be beneficial when the polynomial has some coefficients with large norm but ρ is small. In order to confirm this, we have solved the PEP for a matrix polynomial of degree 4, with $A_0 = 10^6 \times \texttt{rand}(20), A_1 = 10^7 \times \texttt{rand}(20), A_2 = 10^{-3} \times \texttt{rand}(20), A_3 = 10 \times \texttt{rand}(20), A_4 = 10^6 \times \texttt{rand}(20)$ using the Fiedler linearization with PCIS $= (1,0,1)$. Note that $\rho \approx 10$ in this case. In Figure 7(a) and (b) we show the backward error ratio for all right eigenpairs and the computed condition number ratio for the polynomial, respectively. Note that in all cases coefficient scaling gives much better results than the unscaled linearization. In particular, the coefficient scaling provides backward stable results (backward error ratio close to 1). The combined action of coefficient and parameter scaling gives similar results. This is in accordance with our theoretical results, since $\rho(\gamma_{\rm opt}) \approx \rho$. The bound (25) for the unscaled case is around 10^{17}, which is quite pessimistic, as it is the bound (27), which is of order 10^{19}.



(a) Unscaled

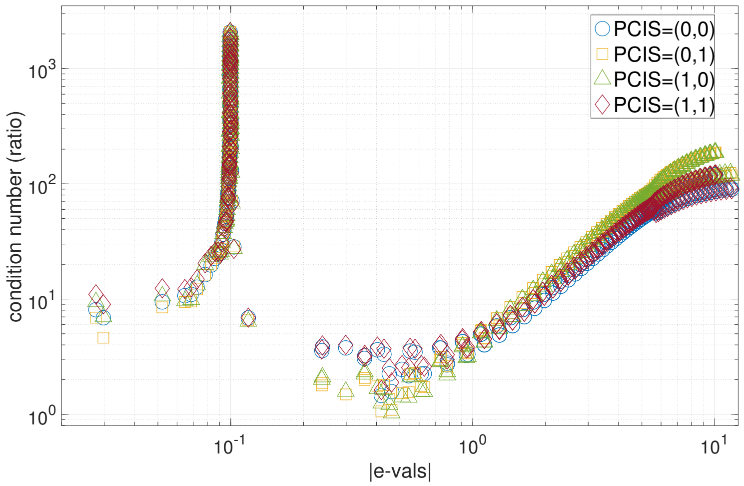


(b) Coefficient scaling

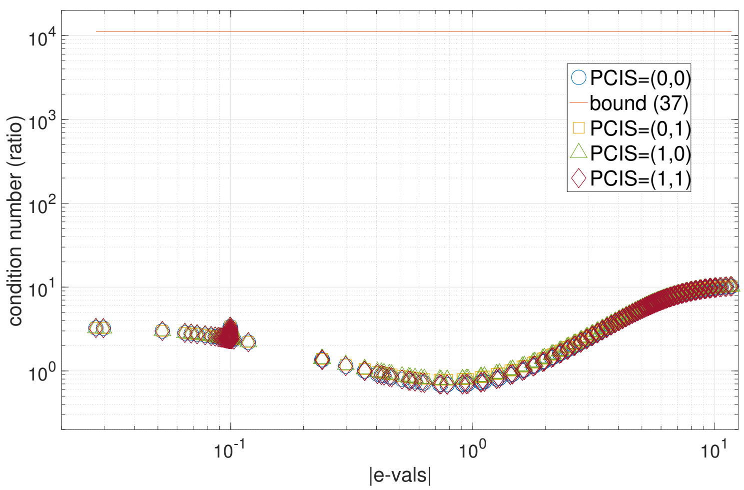


(c) Coefficient+parameter scaling

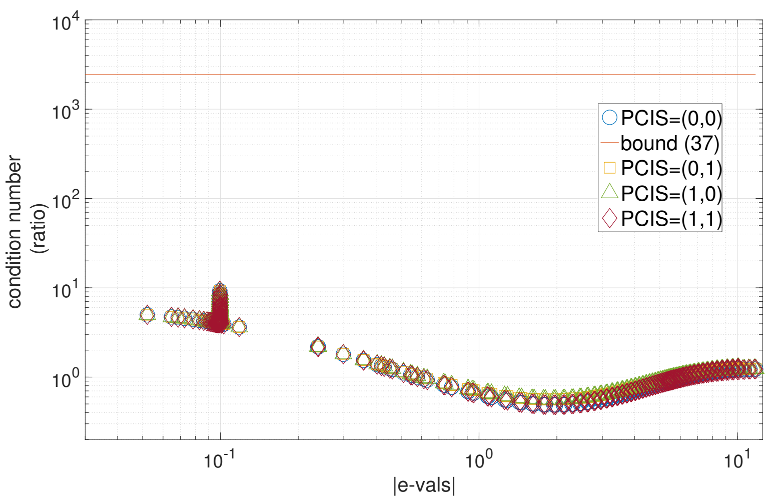
Figure 3: `plasma_drift`. Comparison of backward error ratio for right eigenpairs (represented by the absolute values of the eigenvalues) computed with all four Fiedler linearizations.



(a) Unscaled

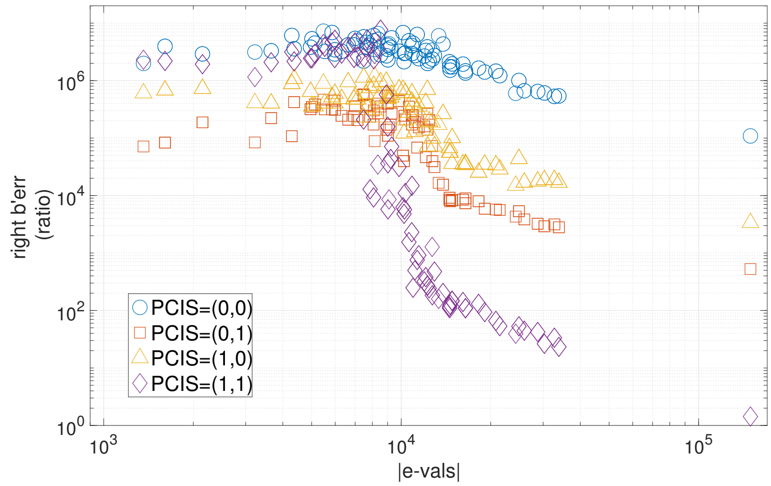


(b) Coefficient scaling

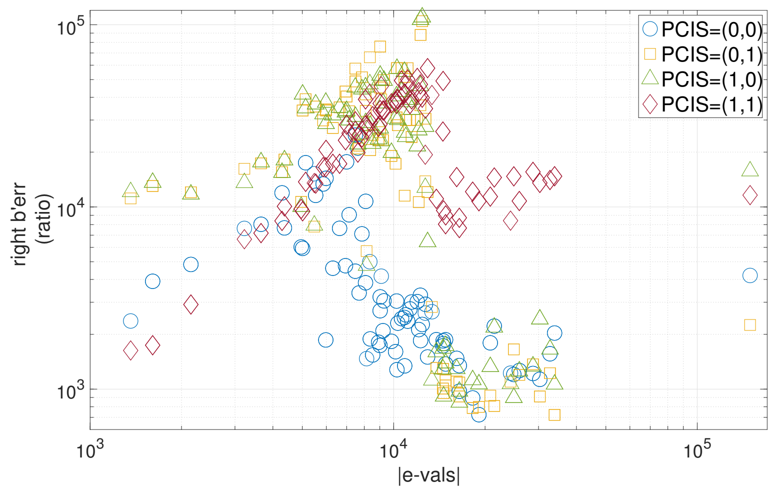


(c) Coefficient+parameter scaling

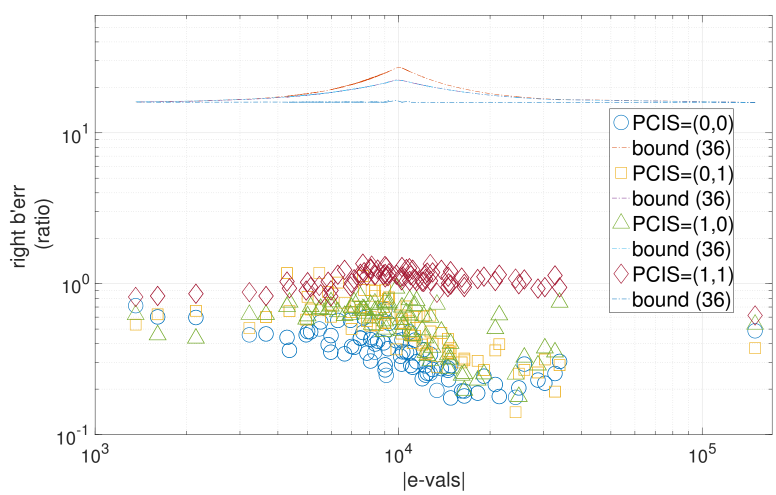
Figure 4: `plasma_drift`. Ratio between conditioning for the linearization and the polynomial for all eigenvalues (represented by their absolute values) and all Fiedler linearizations.



(a) Unscaled

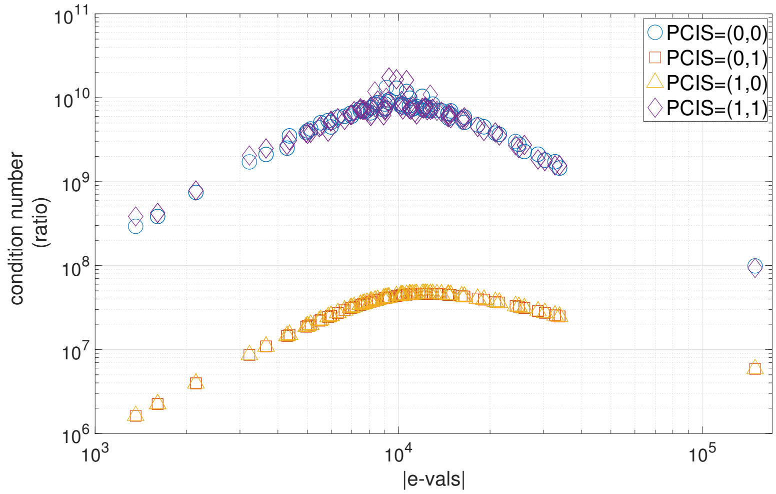


(b) Coefficient scaling

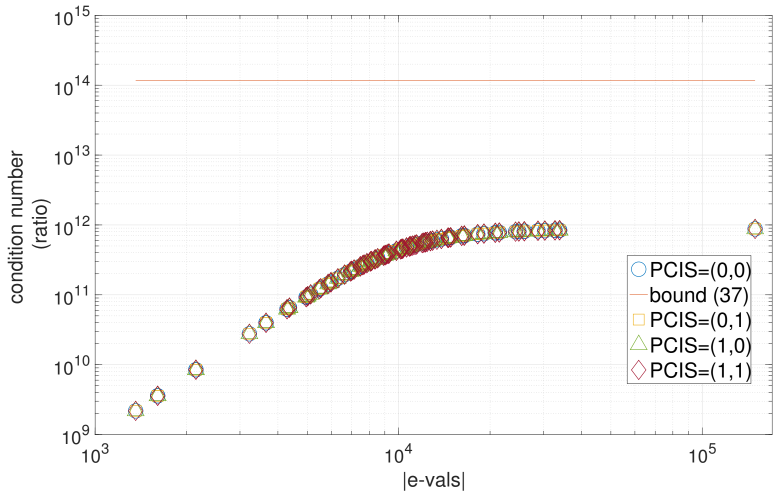


(c) Coefficient+parameter scaling

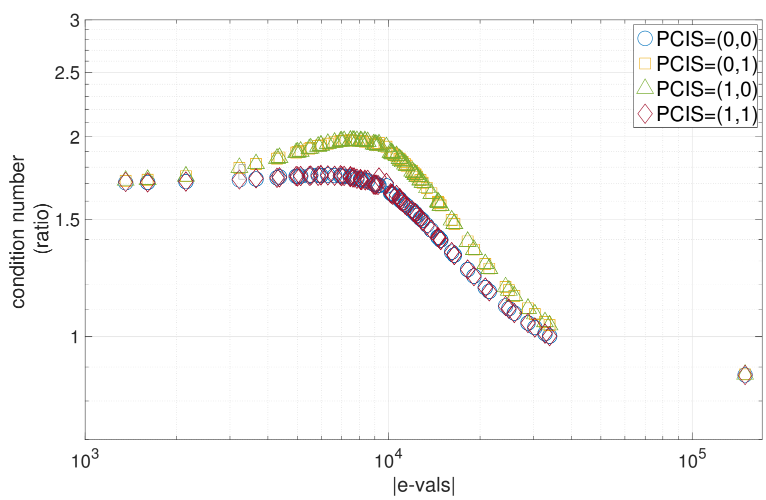
Figure 5: Backward error ratio for right eigenpairs (x, λ) for a random cubic 50×50 polynomial whose coefficients have orders of magnitude $(10^6, 10^2, 10^{-2}, 10^{-6})$.



(a) Unscaled

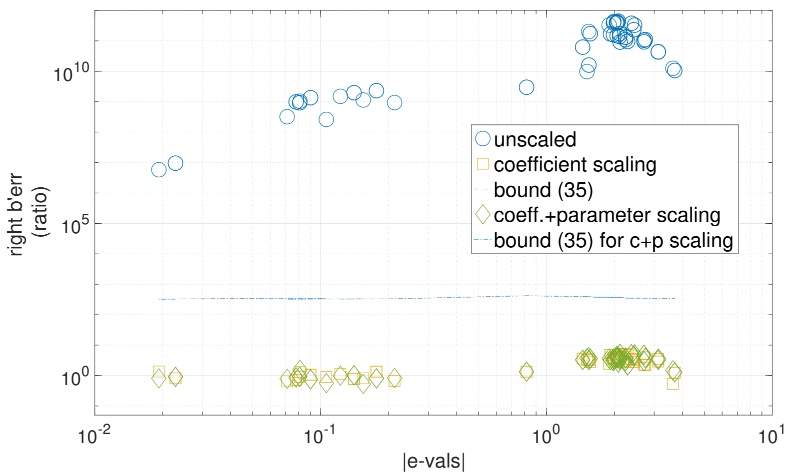


(b) Coefficient scaling

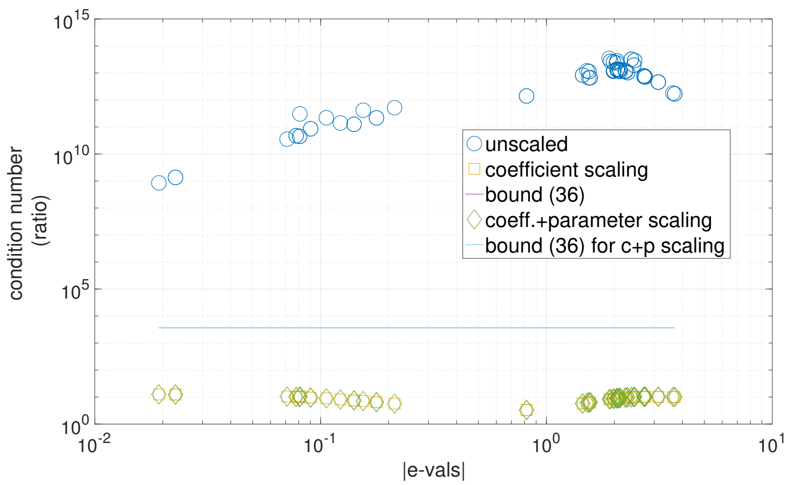


(c) Coefficient+parameter scaling

Figure 6: Condition number ratio for right eigenpairs (x, λ) for a random cubic 50×50 polynomial whose coefficients have orders of magnitude $(10^6, 10^2, 10^{-2}, 10^{-6})$.



(a) Backward error



(b) Condition number

Figure 7: Backward error ratio for right and left eigenpairs and condition number ratio of eigenvalues for a random 20×20 quartic polynomial whose coefficients have orders of magnitude $(10^6, 10^7, 10^{-3}, 10, 10^6)$ computed with a Fiedler linearization whose PCIS $= (1, 0, 1)$

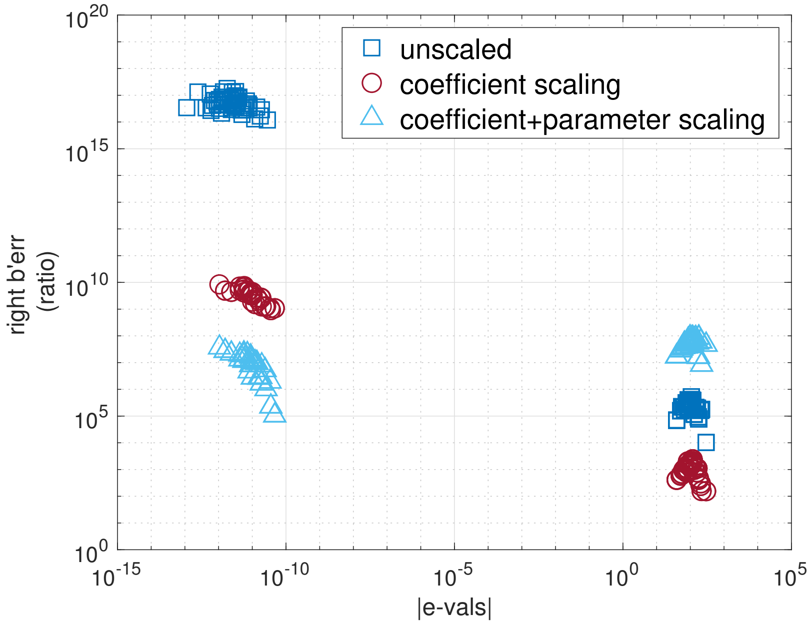
Test 3. Unlike coefficient scaling, which seems to be an advisable scaling by default in all cases, the parameter scaling described in Section 7.2 is not so useful in general. In this numerical experiment, we consider, again, a 50×50 cubic matrix polynomial $P(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 + \lambda^3 A_3$, with $A_0 = 10^{-5} \times \texttt{rand}(50), A_1 = 10^6 \times \texttt{rand}(50), A_2 = 10^{-3} \times \texttt{rand}(50), A_3 = 10^2 \times \texttt{rand}(50)$. In this case, $\rho \approx 10^{11}$ and $\rho(\gamma_{\text{opt}}) \approx 4.7 \cdot 10^8$. We have solved the PEP using the Fiedler linearization with PCIS $= (0, 1)$. Figures 8 shows the right backward error and the condition number ratio for all eigenvalues. As can be seen, the coefficient scaling always improves the results with respect the unscaled case. However, the parameter scaling only improves the backward error and condition number for small eigenvalues. In this example, the eigenvalues split in two well separated sets, so a tropical scaling [29] could improve these results.

Test 4 (A polynomial with high degree). Even though matrix polynomials that typically arise in applications are of low degree (for instance, the maximum degree of a polynomial in the NLEVP collection [4] is 4), it may be interesting to check the main theoretical results of this paper in higher degree polynomials. This example consists of a degree-50 matrix polynomial with coefficients $A_i = \texttt{rand(3,3)}$, for $i = 0, \ldots, 50$. We have solved the PEP using a Fiedler linearization with a random PCIS and with coefficient scaling. The results are in accordance with the theoretical results, even though the theoretical results are quite pessimistic in this case. More precisely, the upper bound in (36) is, approximately, 10^7, and the maximum value for the bound in (35) is almost 10^6. However, the maximum condition number ratio is 258, and the maximum right backward error ratio is 69.7. Figure 9 shows the condition number ratio and the right backward error ratio for all eigenvalues.

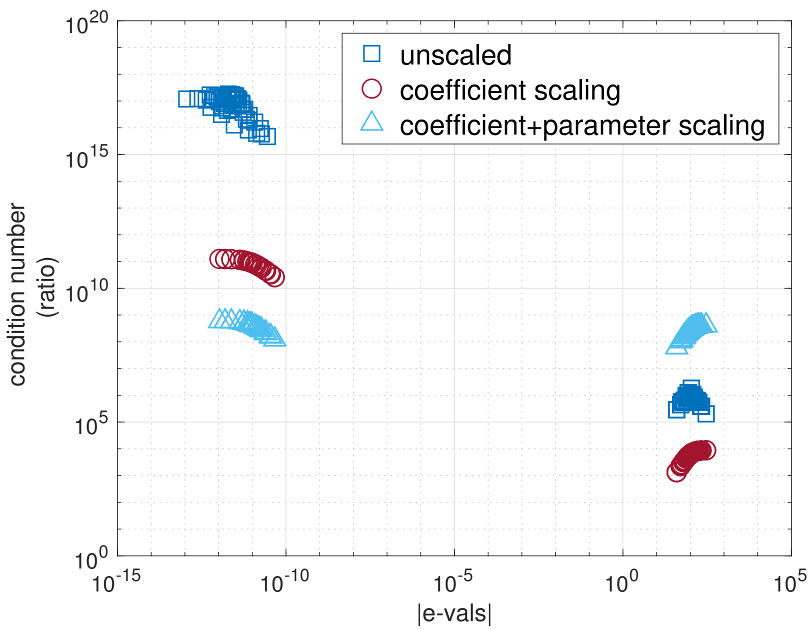
9 Conclusions and future work

We have analyzed the numerical features (concerning backward error and conditioning) of the PEP solved with Fiedler linearizations. In particular, we have compared the backward errors, $\eta_{F_\sigma}(v, \alpha, \beta)$, $\eta_{F_\sigma}(w^*, \alpha, \beta)$, of right and left eigenpairs, (v, α, β) and (w^*, α, β), and the condition number of eigenvalues, $\kappa_{F_\sigma}(\alpha, \beta)$, in any Fiedler linearization F_σ with, respectively, the backward errors, $\eta_P(x, \alpha, \beta)$, $\eta_P(y^*, \alpha, \beta)$, of the corresponding right and left eigenpairs, (x, α, β) and (y, α, β), and the condition number, $\kappa_P(\alpha, \beta)$, in the polynomial P. For this, we have introduced the notion of *right* and *left eigencolumns* of linearizations, and we have derived explicit expressions for the eigencolumns of an arbitrary F_σ. From these expressions, we have obtained explicit formulas for the right and left eigenvectors of F_σ in terms of the corresponding right and left eigenvectors of P. These formulas were already known [5], but we have presented them in a slightly simpler form and in homogeneous variables, which allow us to address all eigenvalues (finite and infinite) in a unified way. Using these formulas we have obtained bounds for the ratios $\eta_P(x, \alpha, \beta)/\eta_{F_\sigma}(v, \alpha, \beta), \eta_P(y^*, \alpha, \beta)/\eta_{F_\sigma}(w^*, \alpha, \beta)$, and $\kappa_{F_\sigma}(\alpha, \beta)/\kappa_P(\alpha, \beta)$. Our bounds depend on the size, n, and the degree, ℓ, of P, and on the norm of the coefficients of P, $A_0, \ldots, A_\ell$. These bounds show that, if P is well scaled (that is, $\|A_i\|_2 \approx 1$, for all $i = 0, 1, \ldots, \ell$), then solving the PEP with any Fiedler linearization (including the classical Frobenius companion linearizations) can be considered as a backward stable procedure, as long as we solve the associated GEP with a backward stable algorithm. However, when some $\|A_i\|_2$ is large or $\min(\|A_0\|_2, \|A_\ell\|_2)$ is small, then the ratios between the corresponding backward errors and the corresponding condition numbers in the polynomial and the linearization can be large. In this case, two different Fiedler linearizations could have quite different backward error and/or conditioning. Large norms in the coefficients can be avoided by just dividing the polynomial by the norm of the largest coefficient, something we refer to as "coefficient scaling".

We have analyzed the effect of scaling. We have seen that multiplying all coefficients of P by $(\max_i \|A_i\|)^{-1}$ allows us to get better (smaller) bounds for the ratios between backward errors of eigenpairs and condition numbers of eigenvalues in the Fiedler linearization and the polynomial. Moreover, after this scaling, the condition number of a given eigenvalue differs at most by a factor $\ell^3\sqrt{n}$ from one Fiedler linearization to another, a quantity that is completely independent of the coefficients of P. We have also considered experimentally the (optimal)



(a) Backward error



(b) Condition number

Figure 8: Backward error and condition number ratio for a PEP of a cubic 50×50 matrix polynomial whose coefficients have order of magnitude $(10^{-5}, 10^{6}, 10^{-3}, 10^{2})$, solved with a Fiedler linearization with PCIS $= (0, 1)$

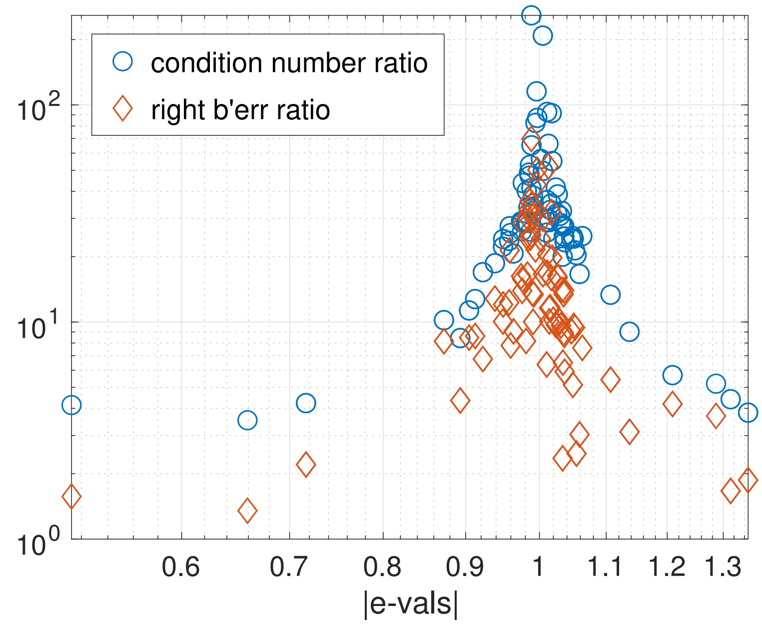


Figure 9: A degree-50 PEP for a matrix polynomial with 3×3 random coefficients solved with a Fiedler linearization with random PCIS(σ) (with coefficient scaling)

parameter scaling obtained in [3].

Our numerical experiments show that all Fiedler linearizations have a similar performance in solving the PEP (after coefficient scaling), even in some badly scaled polynomials. They also show that parameter scaling (as described in Section 7.2) seems to provide much better results in some cases where ρ in (34) is large, and allows to get a backward error and condition number in the PEP of the same order as the one for the linearization, even for polynomials having coefficients with large norm or with large norm ratio. This behavior is in accordance with bounds (25)–(26), (27), and (35)–(36), even though these bounds seem to be quite pessimistic.

By contrast with the situation for the families of linearizations $\mathbb{L}_1(P), \mathbb{L}_2(P)$, and $\mathbb{DL}(P)$ [23, 24], there is no Fiedler linearization with best backward error and conditioning properties that can be identified in advance. Hence, the main open problem regarding the practical application of Fiedler linearizations in the PEP is to be able to know in advance which is the best Fiedler linearization to be used for a given matrix polynomial. For this, a future research would be to carry out a detailed comparison of the backward error of eigenpairs and conditioning of eigenvalues in different Fiedler linearizations.

Another line of research is to extend the analysis carried out in this paper to the more general class of Fiedler-like linearizations [5, 33], including the structure-preserving families in [7, 10].

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A Proof of Theorem 4.2

The main goal of this Appendix is to provide a proof of Theorem 4.2. This theorem presents the formulas for the eigencolumns of Fiedler pencils, which are the ones for eigenvectors in [5, Th. 3.1] (and in [15, Th. 4.1] for scalar polynomials). It also contains the key identities (20)–(21) and (23)–(24), and the main effort is devoted to prove these identities. For completeness, we are going to provide a complete proof of Theorem 4.2, including the first part (the formulas for the eigencolumns). We first need to recall some structural properties of Fiedler pencils.

It is shown in [11, Th. 3.10] that the matrix M_σ is a block partitioned matrix with $\ell \times \ell$ blocks of size $n \times n$. The only nonzero blocks are the coefficient matrices $A_0, A_1, \ldots, A_{\ell-1}$, together with $n-1$ identities, I_n. This can be easily seen through the recursive construction of the matrix M_σ in (14) given by `Algorithm 1` in [11]. For completeness and further references, we include the algorithm here. In the following, we use MATLAB's notation for block partitioned matrices, namely $W(i_1 : i_2, j_1 : j_2)$ is the submatrix of W consisting of block rows from i_1 to i_2 and block columns from j_1 to j_2.

`Algorithm 1.` Given $P(\alpha, \beta) = \sum_{j=0}^{\ell} \alpha^j \beta^{\ell-j} A_j$ and a bijection σ, the following algorithm constructs M_σ.

if σ has a consecution at 0 then

$$W_0 = \begin{bmatrix} A_1 & -I_n \\ A_0 & 0 \end{bmatrix}$$

else

$$W_0 = \begin{bmatrix} A_1 & A_0 \\ -I_n & 0 \end{bmatrix}$$

endif

for $i = 1 : \ell - 2$

if σ has a consecution at i then

$$W_i = \begin{bmatrix} A_{i+1} & -I_n & 0 \\ W_{i-1}(:,1) & 0 & W_{i-1}(:, 2 : i+1) \end{bmatrix}$$

else

$$W_i = \begin{bmatrix} A_{i+1} & W_{i-1}(1,:) \\ -I_n & 0 \\ 0 & W_{i-1}(2 : i+1, :) \end{bmatrix}$$

endif
endfor
$M_\sigma = W_{\ell-2}$.

Moreover, `Algorithm 1` allows us to describe the ith block row of M_σ.

Lemma A.1. *Let $\sigma : \{0,1,\ldots,\ell-1\} \to \{1,\ldots,\ell\}$ be a bijection and $M_\sigma = M_{\sigma^{-1}(1)} \cdots M_{\sigma^{-1}(\ell)}$, where M_j, for $j = 1,\ldots,\ell-1$, are the matrices defined in (11)–(12). Then M_σ is a $(n\ell)\times(n\ell)$ block partitioned matrix with $\ell\times\ell$ blocks of size $n\times n$, whose ith block row is:*

(i) *For $i = 1$:*

$$[M_\sigma]_1 = \begin{bmatrix} A_{\ell-1} & A_{\ell-2} & \ldots & A_{\ell-k+1} & -I_n & 0 & \ldots & 0 \end{bmatrix}, \tag{43}$$

where σ has inversions at $\ell-2,\ldots,\ell-k+1$, and a consecution at $\ell-k$, for $k \geq 1$ (so that, if σ has a consecution at $\ell-2$, only the coefficient $A_{\ell-1}$ appears). If σ has inversions at $0,1,\ldots,\ell-2$, then $-I_n$ is replaced by A_0 and no zero blocks appear.

(ii) *If σ has an inversion at $\ell-i$, for $i \geq 2$:*

$$[M_\sigma]_i = \begin{bmatrix} 0 & \ldots & 0 & -I_n & 0 & \ldots & 0 \end{bmatrix}, \tag{44}$$

where the $-I_n$ block appears at the j_0th position, with $j_0 = \max\{j < i : \sigma \text{ has an inversion at } \ell-j\}$. If σ has consecutions at $\ell-i+1,\ldots,\ell-2$, so that no such a j_0 exists, then the $-I_n$ block is at the first position (that is, $j_0 := 1$).

(iii) *If σ has a consecution at $\ell-i$, for $i \geq 2$:*

$$[M_\sigma]_i = \begin{bmatrix} 0 & \ldots & 0 & A_{\ell-i} & 0 & \ldots & 0 & A_{\ell-i-1} & \ldots & A_{\ell-i-k} & -I_n & 0\ldots & 0 \end{bmatrix}, \tag{45}$$

where the block $A_{\ell-i}$ appears at the $(\ell-j_0)$th position, and the blocks $A_{\ell-i-1},\ldots,A_{\ell-i-k},-I_n$ appear at positions $i+1,\ldots,i+k,i+k+1$, respectively, with $\ell-i \leq j_0 \leq \ell-2$ and $k \geq 0$ defined as:

- *$j_0 = \min\{j \geq \ell-i : \sigma \text{ has an inversion at } j\}$;*
- *σ has inversions at $\ell-i-1,\ldots,\ell-i-k$ and a consecution at $\ell-i-k-1$.*

If σ has a consecution at j, for all $j \geq \ell-i$, then the block $A_{\ell-i}$ appears at the first position.

If σ has inversions at $\ell-i-1,\ldots,1,0$, so that no such k as above exists, then:

$$[M_\sigma]_i = \begin{bmatrix} 0 & \ldots & 0 & A_{\ell-i} & 0 & \ldots & 0 & A_{\ell-i-1} & \ldots & A_0 \end{bmatrix}. \tag{46}$$

Proof. Claim (i) is an immediate consequence of `Algorithm 1`.
For claim (ii), if σ has an inversion at $\ell-i$, then $W_{\ell-i}$ in `Algorithm 1` is:

$$W_{\ell-i} = \begin{bmatrix} A_{\ell-i+1} & W_{\ell-i-1}(1,:) \\ -I_n & 0 \\ 0 & W_{\ell-i-1}(2:\ell-i+1,:) \end{bmatrix}.$$

Now, at each step of `Algorithm 1` we add one block row above the second block row of $W_{\ell-i}$, so that the second block row of $W_{\ell-i}$ goes to the ith position in M_σ. Note that, up to the first inversion within the indices $\ell-i+1,\ldots,\ell-2$, the block $-I_n$ in the second block row of $W_{\ell-i}$ remains at the first block column. After this step, this block moves one position to the right at each step. From this, claim (ii) follows.

For claim (iii), let us first assume that σ has inversions at $\ell-i-1,\ldots,\ell-i-k$ and a consecution at $\ell-i-k-1$, for some $k \geq 0$, and that j_0 is as in the statement. From `Algorithm 1`, if σ has a consecution at $\ell-i$:

$$W_{\ell-i} = \begin{bmatrix} A_{\ell-i+1} & -I_n & 0 \\ W_{\ell-i-1}(:,1) & 0 & W_{\ell-i-1}(:,2:\ell-i+1) \end{bmatrix}.$$

Now, since σ has inversions at $\ell-i-1,\dots,\ell-i-k$ and a consecution at $\ell-i-k-1$, by `Algorithm 1` again, the first block row of $W_{\ell-i-1}$ is of the form

$$W_{\ell-i-1}(1,:) = \left[\begin{array}{cccccccc} A_{\ell-i} & A_{\ell-i-1} & \dots & A_{\ell-i-k} & -I_n & 0 & \dots & 0 \end{array}\right],$$

where the number of block columns is $\ell-i+1$. Note that, if σ has inversions at $\ell-i-1,\dots,1,0$, then the block A_0 appears instead of the block $-I_n$, and there are no zero blocks at the end.

Since σ has consecutions at $\ell-i+1,\dots,j_0-1$, at each of these steps, `Algorithm 1` adds one zero block to the right of $A_{\ell-i}$, so that after all these steps there are $j_0+i-\ell-1$ zero blocks between $A_{\ell-i}$ and $A_{\ell-i-1}$. At steps $j_0, j_0+1,\dots,\ell-2$, `Algorithm 1` adds one zero block to the left of $A_{\ell-i}$, so that this block ends up at the $(\ell-j_0)$th block column of M_σ. Since σ has a consecution at $\ell-i$, the first block row of $W_{\ell-i-1}$ goes to the ith block row of M_σ, and this proves claim (iii) in the case where there is some $r<\ell-i$ such that σ has a consecution at r.

If σ has inversions at $0,1,\dots,\ell-i-1$, then

$$W_{\ell-i-1}(1,:) = \left[\begin{array}{cccc} A_{\ell-i} & A_{\ell-i-1} & \dots & A_0 \end{array}\right],$$

and the result follows with similar arguments to the ones for the previous case. □

Proof of Theorem 4.2. Let us first consider part (a) and $\mathcal{R}_\sigma(\alpha,\beta)$. We first show that $[F_\sigma(\alpha,\beta)\mathcal{R}_\sigma(\alpha,\beta)]_i \equiv 0$ when σ has an inversion at $\ell-i$. From (13) and (44), the ith block row of $F_\sigma(\alpha,\beta)$ is equal to

$$[F_\sigma(\alpha,\beta)]_i = \left[\begin{array}{ccccccccc} 0 & \dots & -\beta I_n & 0 & \dots & 0 & \alpha I_n & \dots & 0 \end{array}\right], \tag{47}$$

where the block αI_n is at the ith position, and the block $-\beta I_n$ is at the j_0th position, with $j_0=\max\{j<i : \sigma \text{ has an inversion at } \ell-j\}$. Now, by (18), $[\mathcal{R}_\sigma(\alpha,\beta)]_{j_0} = \alpha^{\mathfrak{i}_\sigma(0:\ell-j_0-1)}\beta^{\ell-1-\mathfrak{i}_\sigma(0:\ell-j_0-1)}\, I_n$, and $[\mathcal{R}_\sigma(\alpha,\beta)]_i = \alpha^{\mathfrak{i}_\sigma(0:\ell-i-1)}\beta^{\ell-1-\mathfrak{i}_\sigma(0:\ell-i-1)}\, I_n$. But, by definition of j_0, $\mathfrak{i}_\sigma(0:\ell-j_0-1) = \mathfrak{i}_\sigma(0:\ell-i-1)+1$, so

$$[F_\sigma(\alpha,\beta)\mathcal{R}_\sigma(\alpha,\beta)]_i = (-\beta)\cdot\alpha^{\mathfrak{i}_\sigma(0:\ell-j_0-1)}\beta^{\ell-1-\mathfrak{i}_\sigma(0:\ell-j_0-1)}I_n + \alpha\cdot\alpha^{\mathfrak{i}_\sigma(0:\ell-i-1)}\beta^{\ell-1-\mathfrak{i}_\sigma(0:\ell-i-1)}I_n \equiv 0.$$

If σ has consections at $\ell-j$, for all $j<i$, so that no such a j_0 exists, then the previous arguments also work by setting $j_0=1$.

Now, let us consider the case where σ has a consecution at $\ell-i$. Let us first assume that there is some $j\geq \ell-i$ such that σ has an inversion at j, and let j_0 and k be as in the statement of Lemma A.1 (iii). Then

$$\begin{array}{lcl}
[\mathcal{R}_\sigma(\alpha,\beta)]_{\ell-j_0} & = & \alpha^{\mathfrak{i}_\sigma(0:j_0-1)}\beta^{\ell-1-\mathfrak{i}_\sigma(0:j_0-1)}I_n = \alpha^{\mathfrak{i}_\sigma(0:\ell-i-1)}\beta^{\ell-1-\mathfrak{i}_\sigma(0:\ell-i-1)}\, I_n \\
[\mathcal{R}_\sigma(\alpha,\beta)]_i & = & \alpha^{\mathfrak{i}_\sigma(0:\ell-i-1)}\beta^{\ell-i-\mathfrak{i}_\sigma(0:\ell-i-1)}\, P_{i-1}(\alpha,\beta) \\
[\mathcal{R}_\sigma(\alpha,\beta)]_{i+1} & = & \alpha^{\mathfrak{i}_\sigma(0:\ell-i-1)-1}\beta^{\ell-\mathfrak{i}_\sigma(0:\ell-i-1)}\, I_n \\
 & \vdots & \\
[\mathcal{R}_\sigma(\alpha,\beta)]_{i+k} & = & \alpha^{\mathfrak{i}_\sigma(0:\ell-i-1)-k}\beta^{\ell-k-1-\mathfrak{i}_\sigma(0:\ell-i-1)}\, I_n \\
[\mathcal{R}_\sigma(\alpha,\beta)]_{i+k+1} & = & \alpha^{\mathfrak{i}_\sigma(0:\ell-i-1)-k}\beta^{\ell-i-1-\mathfrak{i}_\sigma(0:\ell-i-1)}\, P_{i+k}(\alpha,\beta),
\end{array}$$

so that, from (45),

$$\begin{array}{rl}
[F_\sigma(\alpha,\beta)\,\mathcal{R}_\sigma(\alpha,\beta)]_i = & \alpha^{\mathfrak{i}_\sigma(0:\ell-i-1)}\beta^{\ell-\mathfrak{i}_\sigma(0:\ell-i-1)}A_{\ell-i} + \alpha^{\mathfrak{i}_\sigma(0:\ell-i-1)+1}\beta^{\ell-i-\mathfrak{i}_\sigma(0:\ell-i-1)}\, P_{i-1}(\alpha,\beta) \\
 & +\alpha^{\mathfrak{i}_\sigma(0:\ell-i-1)-1}\beta^{\ell-\mathfrak{i}_\sigma(0:\ell-i-1)+1}A_{\ell-i-1} + \cdots \\
 & +\alpha^{\mathfrak{i}_\sigma(0:\ell-i-1)-k}\beta^{\ell-k-\mathfrak{i}_\sigma(0:\ell-i-1)}A_{\ell-i-k} \\
 & -\alpha^{\mathfrak{i}_\sigma(0:\ell-i-1)-k}\beta^{\ell-i-\mathfrak{i}_\sigma(0:\ell-i-1)}\, P_{i+k}(\alpha,\beta) \equiv 0.
\end{array}$$

Now, let us assume that σ has inversions at $\ell-i-1,\dots,1,0$, so that $[M_\sigma]_i$ is as in (46). In this case, $\mathfrak{i}(0:\ell-i-1)=\ell-i$, so

$$[F_\sigma(\alpha,\beta)\,\mathcal{R}_\sigma(\alpha,\beta)]_i = \alpha^{\ell-i}\beta^iA_{\ell-i} + \alpha^{\ell-i+1}\, P_{i-1}(\alpha,\beta) + \alpha^{\ell-i-1}\beta^{i+1}A_{\ell-i-1} + \cdots + \beta^\ell A_0 \equiv P(\alpha,\beta).$$

Note that $\ell - i$ is the minimum index where σ has as consecution, which proves also the last claim of part (a) for $i > 1$.

For $i = 1$, we first consider the case where $i_0 < \ell - 1$, so that σ has at least one consecution. In this case, let us assume that σ has inversions at $\ell - 2, \ldots, \ell - k + 1$, and a consecution at $\ell - k$. Then

$$\mathcal{R}_\sigma(\alpha,\beta) = \begin{bmatrix} \alpha^{\mathfrak{i}(\sigma)}\beta^{\ell-1-\mathfrak{i}(\sigma)} I_n \\ \alpha^{\mathfrak{i}(\sigma)-1}\beta^{\ell-\mathfrak{i}(\sigma)} I_n \\ \vdots \\ \alpha^{\mathfrak{i}(\sigma)-k+2}\beta^{\ell+k-3-\mathfrak{i}(\sigma)} I_n \\ \alpha^{\mathfrak{i}(\sigma)-k+2}\beta^{\ell-\mathfrak{i}(\sigma)-2} P_{k-1}(\alpha,\beta) \\ \star \end{bmatrix},$$

where the blocks $\star$ have no relevance in our arguments. Now, from (43),

$$\begin{aligned} [F_\sigma(\alpha,\beta)\mathcal{R}_\sigma(\alpha,\beta)]_1 &= \alpha^{\mathfrak{i}(\sigma)}\beta^{\ell-1-\mathfrak{i}(\sigma)}(\alpha A_\ell + \beta A_{\ell-1}) + \alpha^{\mathfrak{i}(\sigma)-1}\beta^{\ell-\mathfrak{i}(\sigma)+1}A_{\ell-2} + \cdots \\ &\quad + \alpha^{\mathfrak{i}(\sigma)-k+2}\beta^{\ell+k-2-\mathfrak{i}(\sigma)}A_{\ell-k+1} - \alpha^{\mathfrak{i}(\sigma)-k+2}\beta^{\ell-\mathfrak{i}(\sigma)-1}P_{k-1}(\alpha,\beta) \\ &\equiv 0. \end{aligned}$$

However, if σ has no consecutions at all, then

$$\mathcal{R}_\sigma(\alpha,\beta) = \begin{bmatrix} \alpha^{\ell-1} I_n \\ \alpha^{\ell-2}\beta\, I_n \\ \vdots \\ \beta^{\ell-1} I_n \end{bmatrix},$$

and from (46) we get $[F_\sigma(\alpha,\beta)\mathcal{R}_\sigma(\alpha,\beta)]_1 = \alpha^{\ell-1}(\alpha A_\ell + \beta A_{\ell-1}) + \alpha^{\ell-2}\beta^2 A_{\ell-2} + \cdots + \beta^\ell A_0 = P(\alpha,\beta)$.

Now, let us consider $\widetilde{\mathcal{R}}_\sigma(\alpha,\beta)$. Let us first assume that σ has inversions at $\ell-2,\ldots,\ell-k+1$, and a consecution at $\ell - k$, for some $k \geq 2$ (which includes the case $k = 2$, namely when σ has a consecution at $\ell - 2$). Then, by (19) and (43), we get

$$\begin{aligned} \left[F_\sigma(\alpha,\beta)\widetilde{\mathcal{R}}_\sigma(\alpha,\beta)\right]_1 &= (\alpha A_\ell + \beta A_{\ell-1})\cdot(\alpha^{\ell-1}I_n) + \beta A_{\ell-2}\cdot(\alpha^{\ell-2}\beta I_n) + \cdots + \\ &\quad \beta A_{\ell-k+1}\cdot(\alpha^{\ell-k+1}\beta^{k-2}I_n) + \beta I_n \cdot (\beta^{k-1}P^\sharp_{\ell-k}(\alpha,\beta)) \\ &= \alpha^\ell A_\ell + \alpha^{\ell-1}\beta A_{\ell-1} + \cdots + \alpha^{\ell-k+1}\beta^{k-1}A_{\ell-k} + \\ &\quad \beta^k \cdot (\alpha^{\ell-k}A_{\ell-k} + \alpha^{\ell-k-1}\beta A_{\ell-k-1} + \cdots + \beta^{\ell-k}A_0) \\ &= P(\alpha,\beta). \end{aligned}$$

Now, assume that σ has an inversion at $\ell - i$. Then, the ith block row of F_σ is as in (47). By definition of j_0, the permutation σ has consecutions at $\ell-i+1,\ldots,\ell-j_0-1$, and inversion at $\ell - j_0$ (including the case $j_0 = 1$ if σ has inversions at $\ell-i+1,\ldots,\ell-2$). Then, by (19),

$$\begin{aligned} \left[F_\sigma(\alpha,\beta)\widetilde{\mathcal{R}}_\sigma(\alpha,\beta)\right]_i &= -\beta I_n \cdot (\alpha^{\ell-1-\mathfrak{i}_\sigma(\ell-j_0:\ell-2)}\beta^{\mathfrak{i}_\sigma(\ell-j_0:\ell-2)}I_n) + \\ &\quad \alpha I_n \cdot (\alpha^{\ell-1-\mathfrak{i}_\sigma(\ell-i:\ell-2)}\beta^{\mathfrak{i}_\sigma(\ell-i:\ell-2)}I_n) \equiv 0, \end{aligned}$$

where the last identity follows from the identity $\mathfrak{i}_\sigma(\ell-i:\ell-2) = \mathfrak{i}_\sigma(\ell-j_0:\ell-2)+1$.

Now, let us assume that σ has a consecution at $\ell - i$. In this case,

$$\begin{aligned} \left[\widetilde{\mathcal{R}}_\sigma(\alpha,\beta)\right]_{\ell-j_0} &= \alpha^{\ell-1-\mathfrak{i}_\sigma(j_0:\ell-2)}\beta^{\mathfrak{i}_\sigma(j_0:\ell-2)}I_n, \\ \left[\widetilde{\mathcal{R}}_\sigma(\alpha,\beta)\right]_{i} &= -\alpha^{i-2-\mathfrak{i}_\sigma(\ell-i:\ell-2)}\beta^{\mathfrak{i}_\sigma(\ell-i:\ell-2)}\, P^\sharp_{\ell-i}(\alpha,\beta) \\ \left[\widetilde{\mathcal{R}}_\sigma(\alpha,\beta)\right]_{i+1} &= \alpha^{\ell-1-\mathfrak{i}_\sigma(\ell-i-1:\ell-2)}\beta^{\mathfrak{i}_\sigma(\ell-i-1:\ell-2)}\, I_n \\ &\vdots \\ \left[\widetilde{\mathcal{R}}_\sigma(\alpha,\beta)\right]_{i+k} &= \alpha^{\ell-1-\mathfrak{i}_\sigma(\ell-i-k:\ell-2)}\beta^{\mathfrak{i}_\sigma(\ell-i-k:\ell-2)}\, I_n \\ \left[\widetilde{\mathcal{R}}_\sigma(\alpha,\beta)\right]_{i+k+1} &= -\alpha^{i+k-1-\mathfrak{i}_\sigma(\ell-i-k:\ell-2)}\beta^{\mathfrak{i}_\sigma(\ell-i-k:\ell-2)+1}\, P^\sharp_{\ell-i-k-1}(\alpha,\beta), \end{aligned}$$

so that, from (45) again,

$$\begin{aligned}\Big[F_\sigma(\alpha,\beta)\widetilde{\mathcal{R}}_\sigma(\alpha,\beta)\Big]_i \quad = \quad & \alpha^{\ell-1-\mathfrak{i}_\sigma(j_0:\ell-2)}\beta^{\mathfrak{i}_\sigma(j_0:\ell-2)+1}A_{\ell-i}-\alpha^{i-1-\mathfrak{i}_\sigma(\ell-i:\ell-2)}\beta^{\mathfrak{i}_\sigma(\ell-i:\ell-2)+1}\,P^\sharp_{\ell-i}(\alpha,\beta)+\\ & \alpha^{\mathfrak{i}_\sigma(\ell-1-\mathfrak{i}_\sigma(\ell-i-1:\ell-2)}\beta^{\mathfrak{i}_\sigma(\ell-i-1:\ell-2)}\,A_{\ell-i-1}\,+\cdots+\\ & \alpha^{\ell-1-\mathfrak{i}_\sigma(\ell-i-k:\ell-2)}\beta^{\mathfrak{i}_\sigma(\ell-i-k:\ell-2)}\,A_{\ell-i-k}+\\ & \alpha^{i+k-1-\mathfrak{i}_\sigma(\ell-i-k:\ell-2)}\beta^{\mathfrak{i}_\sigma(\ell-i-k:\ell-2)+2}\,P^\sharp_{\ell-i-k-1}(\alpha,\beta)\equiv 0,\end{aligned} \tag{48}$$

where, for the last identity, we have used that σ has a consecution at $\ell-i-k-1$ and at $\ell-i,\ell-i+1,\dots,j_0-1$, and inversions at $\ell-i-k,\dots,\ell-i-1$, and at j_0, and that

$$P^\sharp_{\ell-i}(\alpha,\beta)=\alpha^{\ell-i}A_{\ell-i}+\alpha^{\ell-i-1}\beta A_{\ell-i-1}+\cdots+\alpha^{\ell-i-k}\beta^k A_{\ell-i-k}+\beta^{k+1}P^\sharp_{\ell-i-k-1}(\alpha,\beta).$$

Note that this includes the case where σ has consecutions at $\ell-i,\ell-i+1,\dots,\ell-2$, namely when $j_0=\ell-1$, as well as the case $k=0$. If σ has consecutions at $\ell-i-1,\dots,1,0$, so that k does not exist, then from (46) and (19) we just have to remove the last summand in the right-hand side of (48) and replace $\ell-i-k$ by 0, so that the whole sum again vanishes.

For part (b), note that $F_\sigma(P)^*=F_{\operatorname{rev}\sigma}(P^*)$, where $\operatorname{rev}\sigma:\{0,1,\dots,\ell-1\}\to\{1,\dots,\ell\}$ has a consecution at i if and only if σ has an inversion at i, for $i=0,1,\dots,\ell-2$. Hence, it suffices to show that $\mathcal{L}_\sigma(P)=\mathcal{R}_{\operatorname{rev}\sigma}(P^*)$ and that $\widetilde{\mathcal{L}}_\sigma(P)=\widetilde{\mathcal{R}}_{\operatorname{rev}\sigma}(P^*)$. But this becomes clear after comparing the definitions of both $\mathcal{L}_\sigma(\alpha,\beta)$ and $\mathcal{R}_\sigma(\alpha,\beta)$ and both $\widetilde{\mathcal{L}}_\sigma(\alpha,\beta)$ and $\widetilde{\mathcal{R}}_\sigma(\alpha,\beta)$: note that $\mathcal{L}_\sigma(\alpha,\beta)$ can be obtained from $\mathcal{R}_\sigma(\alpha,\beta)$ just replacing inversions by consecutions and vice versa, and also replacing the Horner shifts of $P(\alpha,\beta)$ by the Horner shifts of $P(\alpha,\beta)^*$, and the same happens with $\widetilde{\mathcal{L}}_\sigma(\alpha,\beta)$ and $\widetilde{\mathcal{R}}_\sigma(\alpha,\beta)$. □

B Some technical identities and bounds

The following technical result allows us to relate the denominators in (8) with the ones in (9) for $L=F_\sigma$.

Proposition B.1. *Let x,y be, respectively, a right and a left eigenvector of P associated with an eigenvalue (α,β). Let $\mathcal{R}_\sigma$ and $\mathcal{L}_\sigma$ be, respectively, the right and left eigencolumns of F_σ in* (18) *and* (22)*, and set*

$$v=\begin{cases}\mathcal{R}_\sigma(\alpha,\beta)x, & \text{if } |\alpha|\le|\beta|,\\ (e_1\otimes I_n)x, & \text{if } |\alpha|>|\beta|,\end{cases}\qquad \text{and}\qquad w=\begin{cases}\mathcal{L}_\sigma(\alpha,\beta)y, & \text{if } |\alpha|\le|\beta|,\\ (e_1\otimes I_n)y, & \text{if } |\alpha|>|\beta|.\end{cases}$$

Then

$$w^*(\overline{\beta}(D_\alpha F_\sigma)|_{(\alpha,\beta)}-\overline{\alpha}(D_\beta F_\sigma)|_{(\alpha,\beta)})v=\begin{cases}\beta^{\ell-1}y^*(\overline{\beta}(D_\alpha P)|_{(\alpha,\beta)}-\overline{\alpha}(D_\beta P)|_{(\alpha,\beta)})x, & \text{if } \beta\neq 0,\\ \alpha^{1-\ell}y^*(\overline{\beta}(D_\alpha P)|_{(\alpha,\beta)}-\overline{\alpha}(D_\beta P)|_{(\alpha,\beta)})x, & \text{if } \beta=0.\end{cases}$$

Proof. Let us first consider the case $|\alpha|\le|\beta|$. We first note that, if $\mathcal{L}_\sigma$ and $\mathcal{R}_\sigma$ are as in the statement, then

$$\mathcal{L}_\sigma(\alpha,\beta)^*F_\sigma(\alpha,\beta)\mathcal{R}_\sigma(\alpha,\beta)=\beta^{\ell-1}P(\alpha,\beta), \tag{49}$$

for all (α,β). To see this, first recall, from (20), that $F_\sigma\mathcal{R}_\sigma=e_{\ell-i_0}\otimes P$, and then notice that the $(\ell-i_0)$th block of $\mathcal{L}_\sigma$ is of the form $\alpha^{\mathfrak{c}_\sigma(0:i_0-1)}\beta^{\ell-1-\mathfrak{c}_\sigma(0:i_0-1)}\,I_n=\beta^{\ell-1}I_n$. Now, differentiating in (49), we get:

$$\begin{aligned}&(D_\alpha\mathcal{L}_\sigma^*)|_{(\alpha,\beta)}F_\sigma\mathcal{R}_\sigma+\mathcal{L}_\sigma^*(D_\alpha F_\sigma)|_{(\alpha,\beta)}\mathcal{R}_\sigma+\mathcal{L}_\sigma^*F_\sigma(D_\alpha\mathcal{R}_\sigma)|_{(\alpha,\beta)}=\beta^{\ell-1}(D_\alpha P)|_{(\alpha,\beta)}, \quad \text{and}\\ &(D_\beta\mathcal{L}_\sigma^*)|_{(\alpha,\beta)}F_\sigma\mathcal{R}_\sigma+\mathcal{L}_\sigma^*(D_\beta F_\sigma)|_{(\alpha,\beta)}\mathcal{R}_\sigma+\mathcal{L}_\sigma^*F_\sigma(D_\beta\mathcal{R}_\sigma)|_{(\alpha,\beta)}=(\ell-1)\beta^{\ell-2}P+\beta^{\ell-1}(D_\beta P)|_{(\alpha,\beta)}.\end{aligned}$$

Multiplying on the left by y^* and on the right by x, and using Theorem 4.4, we get:

$$y^*\mathcal{L}_\sigma^*(D_\alpha F_\sigma)|_{(\alpha,\beta)}\mathcal{R}_\sigma x=\beta^{\ell-1}y^*(D_\alpha P)|_{(\alpha,\beta)}x,\quad\text{and}\quad y^*\mathcal{L}_\sigma^*(D_\beta F_\sigma)|_{(\alpha,\beta)}\mathcal{R}_\sigma x=\beta^{\ell-1}y^*(D_\beta P)|_{(\alpha,\beta)}x,$$

from which the result follows just multiplying the first identity by $\overline{\beta}$, the second identity by $\overline{\alpha}$, and subtracting.

If $|\alpha| > |\beta|$, it suffices to prove that $y^*(e_1^T \otimes I_n)(D_\beta F_\sigma)|_{(\alpha,\beta)}(e_1 \otimes I_n)x = \alpha^{1-\ell} y^*(D_\beta P)|_{(\alpha,\beta)} x$. The left-hand side is equal to $y^*[(D_\beta F_\sigma)|_{(\alpha,\beta)}]_{11} x = y^* A_{\ell-1} x$. This is in turn equal to $\alpha^{1-\ell} y^*(D_\beta P)|_{(\alpha,\beta)} x$, so this concludes the proof. □

The next two results provide bounds which are used in the proof of Theorems 6.1 and 7.5.

Lemma B.2. *Let $\mathcal{R}_\sigma(\alpha,\beta)$ and $\mathcal{L}_\sigma(\alpha,\beta)$ be as in Theorem 4.2, and let x, y be, respectively, a right and a left eigenvector of P associated with a simple eigenvalue (α,β).*

(i) *If $|\alpha| \le |\beta|$, then*

$$\|\mathcal{R}_\sigma(\alpha,\beta)\|_2\,,\ \|\mathcal{L}_\sigma(\alpha,\beta)\|_2 \le \ell^{3/2}|\beta|^{\ell-1} \max_{0\le i\le \ell}(1, \|A_i\|_2). \tag{50}$$

(ii) *If $|\alpha| > |\beta|$, then*

$$\frac{\|\mathcal{R}_\sigma(\alpha,\beta)x\|_2}{\|x\|_2}, \frac{\|\mathcal{L}_\sigma(\alpha,\beta)^* y\|_2}{\|y\|_2} \le \ell^{3/2}|\alpha|^{\ell-1} \max_i(1, \|A_i\|_2). \tag{51}$$

Proof. (i) The norm of the ith block of the right eigencolumn is equal to:

$$\|[\mathcal{R}_\sigma(\alpha,\beta)]_i\|_2 = \begin{cases} |\alpha|^{\mathrm{i}_\sigma(0:\ell-i-1)}|\beta|^{\ell-i-\mathrm{i}_\sigma(0:\ell-i-1)}\|P_{i-1}(\alpha,\beta)\|_2, & \text{if } \sigma \text{ has a consecution at } \ell-i,\\ |\alpha|^{\mathrm{i}(0:\ell-i-1)}|\beta|^{\ell-1-\mathrm{i}_\sigma(0:\ell-i-1)}, & \text{otherwise.}\end{cases}$$

Now, since by definition of $P_{i-1}(\alpha,\beta)$:

$$\|P_{i-1}(\alpha,\beta)\|_2 \le \left(\sum_{k=1}^{i} |\alpha|^{i-k}|\beta|^{k-1}\right) \max_{j=\ell-i+1:\ell} \|A_j\|_2,$$

we get:

$$\begin{aligned}\|[\mathcal{R}_\sigma(\alpha,\beta)]_i\|_2 \le \quad & \max\big(|\alpha|^{\mathrm{i}_\sigma(0:\ell-i-1)}|\beta|^{\ell-1-\mathrm{i}_\sigma(0:\ell-i-1)}\ ,\\ & \left(\sum_{k=1}^{i} |\alpha|^{\mathrm{i}_\sigma(0:\ell-i-1)+i-k}|\beta|^{\ell-i-\mathrm{i}_\sigma(0:\ell-i-1)+k-1}\right) \max_{j=\ell-i+1:\ell}\|A_j\|_2).\end{aligned} \tag{52}$$

Now, we bound:

$$|\alpha|^{\mathrm{i}_\sigma(0:\ell-i-1)}|\beta|^{\ell-1-\mathrm{i}_\sigma(0:\ell-i-1)} \le \sum_{k=0}^{\ell-1}|\alpha|^k|\beta|^{\ell-k-1},$$

and also:

$$\sum_{k=1}^{i} |\alpha|^{\mathrm{i}_\sigma(0:\ell-i-1)+i-k}|\beta|^{\ell-i-\mathrm{i}_\sigma(0:\ell-i-1)+k-1} \le \sum_{k=0}^{\ell-1}|\alpha|^k|\beta|^{\ell-k-1}.$$

Replacing this in (52), we get:

$$\|[\mathcal{R}_\sigma(\alpha,\beta)]_i\|_2 \le \left(\sum_{j=0}^{\ell-1}|\alpha|^j|\beta|^{\ell-j-1}\right)\max_i(1,\|A_i\|_2) \le \ell|\beta|^{\ell-1}\max_i(1,\|A_i\|_2),$$

where in the last inequality we have used that $|\beta| \ge |\alpha|$. Then, using Lemma 3.5 in [23]:

$$\|\mathcal{R}_\sigma(\alpha,\beta)\|_2 \le \ell^{3/2}|\beta|^{\ell-1}\max_{0\le i\le\ell}(1,\|A_i\|_2).$$

Proceeding similarly with the ith block of the left eigencolumn, we can get exactly the same bound for the norm of $\mathcal{L}_\sigma(\alpha,\beta)$.

(ii) From the identity:

$$P_{i-1}(\alpha,\beta)x = -\frac{1}{\alpha^{\ell-i+1}}(\alpha^{\ell-i}\beta^i A_{\ell-i} + \alpha^{\ell-i-1}\beta^{i+1}A_{\ell-i-1} + \cdots + \beta^\ell A_0)x, \tag{53}$$

valid for $i = 1, \ldots, \ell$, we obtain, for each i such that σ has a consecution at $\ell - i$:

$$\begin{aligned} \|[\mathcal{R}_\sigma(\alpha,\beta)]_i x\|_2 &= \tfrac{|\alpha|^{\mathrm{i}_\sigma(0:\ell-i-1)}|\beta|^{\ell-i-\mathrm{i}_\sigma(0:\ell-i-1)}}{|\alpha|^{\ell-i+1}}\|(\alpha^{\ell-i}\beta^i A_{\ell-i} + \alpha^{\ell-i-1}\beta^{i+1}A_{\ell-i-1} + \cdots + \beta^\ell A_0)x\|_2 \\ &= \alpha|^{\mathrm{i}_\sigma(0:\ell-i-1)}|\beta|^{\ell-1-\mathrm{i}_\sigma(0:\ell-i-1)}\|\left(\tfrac{\beta}{\alpha}\right)A_{\ell-i} + \left(\tfrac{\beta}{\alpha}\right)^2 A_{\ell-i+1} + \cdots + \left(\tfrac{\beta}{\alpha}\right)^{\ell-i+1} A_0 x\|_2 \\ &\le \ell|\alpha|^{\mathrm{i}(\sigma)-1}|\beta|^{\ell-1-\mathrm{i}(\sigma)} \max_{j=0:\ell-i}\|A_j\|_2\|x\|_2, \end{aligned}$$

where we have used that $|\beta| \le |\alpha|$. For those i such that σ has an inversion at $\ell - i$, or $i = 1$, we also have, by (18),

$$\|[\mathcal{R}_\sigma(\alpha,\beta)]_i\|_2 \le |\alpha|^{\mathrm{i}_\sigma(0:\ell-i-1)}\,|\beta|^{\ell-\mathrm{i}_\sigma(0:\ell-i-1)} \le |\alpha|^{\mathrm{i}(\sigma)}|\beta|^{\ell-1-\mathrm{i}(\sigma)}.$$

Hence, from the definition of $\mathcal{R}_\sigma(\alpha,\beta)$ and Lemma 3.5 in [23], we get

$$\|\mathcal{R}_\sigma(\alpha,\beta)x\|_2 \le \ell^{3/2}|\alpha|^{\mathrm{i}(\sigma)}|\beta|^{\ell-1-\mathrm{i}(\sigma)} \max_i(1, \|A_i\|_2)\|x\|_2. \tag{54}$$

Now (51) follows from (54) and the fact that $|\alpha| > |\beta|$. Proceeding similarly with the left eigencolumns, and starting with the identity

$$y^* P_{i-1}(\alpha,\beta) = -\frac{1}{\alpha^{\ell-i+1}}y^*(\alpha^{\ell-i}\beta^i A_{\ell-i} + \alpha^{\ell-i-1}\beta^{i+1}A_{\ell-i-1} + \cdots + \beta^\ell A_0),$$

we get the bound $\|\mathcal{L}_\sigma(\alpha,\beta)^* y\|_2 \le \ell^{3/2}|\alpha|^{\ell-1}\max_i(1, \|A_i\|_2)\|y\|_2$, and the rest of the proof follows similar steps as the ones for $\mathcal{R}_\sigma$. □

Lemma B.3. *Let $\widehat{P} = \sum_{i=0}^{\ell}\lambda^i\widehat{A}_i$ be a matrix polynomial with $\|\widehat{A}_i\|_2 \le 1$, for $i = 0, 1, \ldots, \ell$. Let $\widehat{x}, \widehat{y}$ be, respectively, right and left eigenvectors associated with the simple eigenvalue (α, β) of $\widehat{P}$, and let $\widehat{v}, \widehat{w}$ be the corresponding right and left eigenvectors of the Fiedler linearization $\widehat{F}_\sigma := F_\sigma(\widehat{P})$, as described in the statement of Proposition* B.1. *Then:*

(i) $|\alpha|^{\ell-1}|\beta|^{\ell-1}\|\widehat{x}\|_2\|\widehat{y}\|_2 \le \|\widehat{v}\|_2\|\widehat{w}\|_2 \le \ell^3|\alpha|^{\ell-1}|\beta|^{\ell-1}\|\widehat{x}\|_2\|\widehat{y}\|_2$ *, if* $|\alpha| \ge |\beta|$, *and*

(ii) $|\beta|^{2\ell-2}\|\widehat{x}\|_2\|\widehat{y}\|_2 \le \|\widehat{v}\|_2\|\widehat{w}\|_2 \le \ell^3|\beta|^{2\ell-2}\|\widehat{x}\|_2\|\widehat{y}\|_2$, *if* $|\beta| \ge |\alpha|$.

Proof. Let us first prove that, in the conditions of the statement,

$$\|\widehat{P}_{i-1}(\alpha,\beta)\widehat{x}\|_2 \le i\max(|\alpha|^{i-1}, |\beta|^{i-1})\|\widehat{x}\|_2, \qquad \text{and} \tag{55}$$

$$\|\widehat{P}_{i-1}(\alpha,\beta)^*\widehat{y}\|_2 \le i\max(|\alpha|^{i-1}, |\beta|^{i-1})\|\widehat{y}\|_2, \tag{56}$$

for $i = 1, \ldots, \ell$. We only prove the first inequality, since the second one is similar. For this, let us consider the sequence of inequalities:

$$\begin{aligned} \|\widehat{P}_{i-1}(\alpha,\beta)\widehat{x}\|_2 &= \|\alpha^{i-1}\widehat{A}_\ell\widehat{x} + \alpha^{i-2}\beta\widehat{A}_{\ell-1}\widehat{x} + \cdots + \beta^{i-1}\widehat{A}_{\ell-i+1}\widehat{x}\|_2 \\ &\le \left(|\alpha|^{i-1}\|\widehat{A}_\ell\|_2 + |\alpha|^{i-2}|\beta|\|\widehat{A}_{\ell-1}\|_2 + \cdots + |\beta|^{i-1}\|\widehat{A}_{\ell-i+1}\|_2\right)\|\widehat{x}\|_2 \\ &\le \left(|\alpha|^{i-1} + |\alpha|^{i-2}|\beta| + \cdots + |\beta|^{i-1}\right)\|\widehat{x}\|_2 \le i\max(|\alpha|^{i-1}, |\beta|^{i-1})\|\widehat{x}\|_2. \end{aligned}$$

Now, let us first prove (i), so we assume $|\beta| \le |\alpha|$, and then the upper bounds in (55) and (56) are, respectively, $i|\alpha|^{i-1}\|\widehat{x}\|_2$ and $i|\alpha|^{i-1}\|\widehat{y}\|_2$.

If $\mathcal{R}_\sigma$ is a right eigencolumn of F_σ then, as a consequence of (55), we get:

$$\|[\mathcal{R}_\sigma(\alpha,\beta)]_i\widehat{x}\|_2 \le \begin{cases} i|\alpha|^{\mathrm{i}_\sigma(0:\ell-i-1)+i-1}|\beta|^{\ell-i-\mathrm{i}_\sigma(0:\ell-i-1)}\|\widehat{x}\|_2, & \text{if } \sigma \text{ has a consecution at } \ell-i, \\ |\alpha|^{\mathrm{i}_\sigma(0:\ell-i-1)}|\beta|^{\ell-1-\mathrm{i}_\sigma(0:\ell-i-1)}\|\widehat{x}\|_2, & \text{if } \sigma \text{ has an inversion at } \ell-i, \text{ or } i=1. \end{cases} \tag{57}$$

Similarly,

$$\|[\mathcal{L}_\sigma(\alpha,\beta)]_i\widehat{y}\|_2 \le \begin{cases} i|\alpha|^{\mathfrak{c}_\sigma(0:\ell-i-1)+i-1}|\beta|^{\ell-i-\mathfrak{c}_\sigma(0:\ell-i-1)}\|\widehat{y}\|_2, & \text{if } \sigma \text{ has an inversion at } \ell-i, \\ |\alpha|^{\mathfrak{c}_\sigma(0:\ell-i-1)}|\beta|^{\ell-1-\mathfrak{c}_\sigma(0:\ell-i-1)}\|\widehat{y}\|_2, & \text{if } \sigma \text{ has a consecution at } \ell-i, \text{ or } i=1. \end{cases} \tag{58}$$

Note that, for $i = 1$, the above inequalities are equalities, so

$$\begin{aligned} \|\widehat{v}\|_2 &\ge \|[\mathcal{R}_\sigma(\alpha,\beta)]_1\widehat{x}\|_2 = |\alpha|^{\mathfrak{i}(\sigma)}|\beta|^{\ell-1-\mathfrak{i}(\sigma)}\|\widehat{x}\|_2, \quad \text{and} \\ \|\widehat{w}\|_2 &\ge \|[\mathcal{L}_\sigma(\alpha,\beta)]_1\widehat{y}\|_2 = |\alpha|^{\mathfrak{c}(\sigma)}|\beta|^{\ell-1-\mathfrak{c}(\sigma)}\|\widehat{y}\|_2. \end{aligned}$$

Hence, $\|\widehat{v}\|_2\|\widehat{w}\|_2 \ge |\alpha|^{\ell-1}|\beta|^{\ell-1}\|\widehat{x}\|_2\|\widehat{y}\|_2$, and the first inequality in part (i) follows.

To prove the second inequality in part (i), we first notice that, for all $1 \le i \le \ell$:

$$\|[\mathcal{R}_\sigma(\alpha,\beta)]_i\widehat{x}\|_2\|[\mathcal{L}_\sigma(\alpha,\beta)]_i\widehat{y}\|_2 \le \ell^2|\alpha|^{\ell-1}|\beta|^{\ell-1}\|\widehat{x}\|_2\|\widehat{y}\|_2. \tag{59}$$

This is an immediate consequence of the inequalities (57)–(58). In particular:

- If $i > 1$ and σ has a consecution at $\ell - i$, then (59) is an immediate consequence of the upper inequality in (57) and the lower inequality in (58).
- If $i > 1$ and σ has an inversion at $\ell - i$, then (59) follows from lower inequality in (57) and the upper inequality in (58).
- If $i = 1$, then (59) follows from the lower inequalities in both (57) and (58).

Now, from Lemma 3.5 in [23],

$$\|\widehat{v}\|_2 \le \ell^{1/2}\max_i \|[\mathcal{R}(\alpha,\beta)]_i\widehat{x}\|_2\,, \quad \text{and} \quad \|\widehat{w}\|_2 \le \ell^{1/2}\max_i \|[\mathcal{L}(\alpha,\beta)]_i\widehat{y}\|_2,$$

and this, together with (59), gives $\|\widehat{v}\|_2\|\widehat{w}\|_2 \le \ell^3|\alpha|^{\ell-1}|\beta|^{\ell-1}\|\widehat{x}\|_2\|\widehat{y}\|_2$, as wanted.

Let us now prove (ii), so we assume $|\beta| \ge |\alpha|$. The left-hand side inequality in (ii) follows from claim 2 in Remark 4.3.

To prove the right-hand side inequality, we first use (55)–(56) to get the analogous inequalities to (57)–(58), namely:

$$\|[\mathcal{R}_\sigma(\alpha,\beta)]_i\widehat{x}\|_2 \le \begin{cases} i|\alpha|^{\mathfrak{i}_\sigma(0:\ell-i-1)}|\beta|^{\ell-1-\mathfrak{i}_\sigma(0:\ell-i-1)}\|\widehat{x}\|_2, & \text{if } \sigma \text{ has a consecution at } \ell-i, \\ |\alpha|^{\mathfrak{i}_\sigma(0:\ell-i-1)}|\beta|^{\ell-1-\mathfrak{i}_\sigma(0:\ell-i-1)}\|\widehat{x}\|_2, & \text{if } \sigma \text{ has an inversion at } \ell-i, \text{ or } i=1, \end{cases} \tag{60}$$

and

$$\|[\mathcal{L}_\sigma(\alpha,\beta)]_i\widehat{y}\|_2 \le \begin{cases} i|\alpha|^{\mathfrak{c}_\sigma(0:\ell-i-1)}|\beta|^{\ell-1-\mathfrak{c}_\sigma(0:\ell-i-1)}\|\widehat{y}\|_2, & \text{if } \sigma \text{ has an inversion at } \ell-i, \\ |\alpha|^{\mathfrak{c}_\sigma(0:\ell-i-1)}|\beta|^{\ell-1-\mathfrak{c}_\sigma(0:\ell-i-1)}\|\widehat{y}\|_2, & \text{if } \sigma \text{ has a consecution at } \ell-i, \text{ or } i=1. \end{cases} \tag{61}$$

Now, from (60) and (61), and using $|\alpha| \le |\beta|$ we get, instead of (59), the inequality

$$\|[\mathcal{R}_\sigma(\alpha,\beta)]_i\widehat{x}\|_2\|[\mathcal{L}_\sigma(\alpha,\beta)]_i\widehat{y}\|_2 \le \ell^2|\alpha|^{\ell-i}|\beta|^{i-1}||\beta|^{\ell-1}\|\widehat{x}\|_2\|\widehat{y}\|_2 \le \ell^2|\beta|^{2\ell-2}\|\widehat{x}\|_2\|\widehat{y}\|_2.$$

Hence, from Lemma 3.5 in [23], we arrive at $\|\widehat{v}\|_2\|\widehat{w}\|_2 \le \ell^3|\beta|^{2\ell-2}\|\widehat{x}\|_2\|\widehat{y}\|_2$, as wanted. □