RECOVERY OF EIGENVECTORS AND MINIMAL BASES OF MATRIX POLYNOMIALS FROM GENERALIZED FIEDLER LINEARIZATIONS *

MARÍA I. BUENO†, FERNANDO DE TERÁN ‡, AND FROILÁN M. DOPICO §

Abstract. A standard way to solve polynomial eigenvalue problems $P(\lambda)x = 0$ is to convert the matrix polynomial $P(\lambda)$ into a matrix pencil that preserves its elementary divisors and, therefore, its eigenvalues. This process is known as linearization and is not unique, since there are infinitely many linearizations with widely varying properties associated with $P(\lambda)$. This freedom has motivated the recent development and analysis of new classes of linearizations that generalize the classical first and second Frobenius companion forms, with the goals of finding linearizations that retain whatever structures that $P(\lambda)$ might possess and/or of improving numerical properties, as conditioning or backward errors, with respect the companion forms. In this context, an important new class of linearizations is what we name generalized Fiedler linearizations, introduced in 2004 by Antoniou and Vologiannidis as an extension of certain linearizations introduced previously by Fiedler for scalar polynomials. On the other hand, the mere definition of linearization does not imply the existence of simple relationships between the eigenvectors, minimal indices, and minimal bases of $P(\lambda)$ and those of the linearization. So, given a class of linearizations, to provide easy recovery procedures for eigenvectors, minimal indices, and minimal bases of $P(\lambda)$ from those of the linearizations is essential for the usefulness of this class. In this paper we develop such recovery procedures for generalized Fiedler linearizations and pay special attention to structure preserving linearizations inside this class.

Key words. eigenvector, Fiedler pencils, linearizations, matrix polynomials, minimal bases, minimal indices, palindromic polynomials, symmetric polynomials

AMS subject classifications. 65F15, 15A18, 15A22.

1. Introduction.

Throughout this work we consider $n \times n$ matrix polynomials with degree $k \geq 2$ of the form

$$P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i \,, \qquad A_0, \ldots, A_k \in \mathbb{F}^{n\times n}, \quad A_k \neq 0 \,, \tag{1.1}$$

where $\mathbb{F}$ is an arbitrary field and λ is a scalar variable in $\mathbb{F}$. An $n \times n$ polynomial $P(\lambda)$ is said to be *singular* if $\det P(\lambda)$ is identically zero, i.e., if all its coefficients are zero, otherwise it is *regular*. For regular matrix polynomials, the Polynomial Eigenvalue Problem (PEP) consists of finding scalars $\lambda_0 \in \mathbb{F}$ and nonzero vectors x and y in $\mathbb{F}^n$ satisfying $P(\lambda_0)x = 0$ and $y^T P(\lambda_0) = 0$. The values λ_0 are known as the *eigenvalues* of $P(\lambda)$ and the associated nonzero vectors x and y^T are known as *right* and *left* eigenvectors of $P(\lambda)$, respectively. Along this paper we follow the convention of writing right eigenvectors as column vectors and left eigenvectors as row vectors. A matrix polynomial $P(\lambda)$ may have *infinite eigenvalues*. These are the zero eigenvalues of the *reversal* polynomial $\operatorname{rev} P(\lambda) = \lambda^k P(1/\lambda) = \sum_{i=0}^{k} \lambda^i A_{k-i}$. The PEP problem arises in many applications [16, 29] and attracts nowadays the attention of many researchers.

In the case of singular matrix polynomials, the above definition of eigenvalue makes no sense (otherwise, all numbers in $\mathbb{F}$ would be eigenvalues of $P(\lambda)$), and one has to be more careful to define eigenvalues [8, Section 2]. In addition, other magnitudes that do not exist for

*This work was partially supported by the Ministerio de Ciencia e Innovación of Spain through grant MTM-2009-09281

†Department of Mathematics, University of California, Santa Barbara, CA 93106, USA (mbueno@math.ucsb.edu).

‡Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. Universidad 30, 28911 Leganés, Spain (fteran@math.uc3m.es). Part of this work was done while this author was visiting the Department of Mathematics at UCSB, whose hospitality is gratefully acknowledged.

§Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM and Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. Universidad 30, 28911 Leganés, Spain (dopico@math.uc3m.es).

regular polynomials are of interest: the *minimal bases* and *minimal indices* of $P(\lambda)$, which are relevant in many control problems [13, 25]. A short summary on these concepts can be found in [8, Section 2].

A standard way to solve the PEP is by using linearizations. A *linearization* of $P(\lambda)$ is a *matrix pencil* $K(\lambda) = \lambda X + Y$ which is *equivalent* to $\mathrm{diag}\,(I_{n(k-1)}, P(\lambda))$ [16]. This means that there exist two *unimodular* matrix polynomials (that is, matrix polynomials with nonzero constant determinant) $U(\lambda), V(\lambda)$ such that

$$U(\lambda)K(\lambda)V(\lambda) = \begin{bmatrix} I_{n(k-1)} & 0 \\ 0 & P(\lambda) \end{bmatrix}, \tag{1.2}$$

where here and hereafter I_m denotes the $m \times m$ identity matrix (I_n, where $n \times n$ is the size of $P(\lambda)$, will be denoted simply by I). The linearization $K(\lambda)$ is said to be *strong* if, additionally, $\mathrm{rev}\, K(\lambda)$ is a linearization for $\mathrm{rev}\, P(\lambda)$. This notion was introduced in [15] and named later in [26]. Note that the size of the linearizations in (1.2) is assumed to be exactly $nk \times nk$. Linearizations with smaller sizes have been considered recently in [3], and their minimal possible size has been determined in [7]. Some classes of linearizations, among them the first and second companion forms, are also useful to study singular matrix polynomials, as has been shown in [8, 9, 31].

The use of linearizations in the PEP is justified by the following two facts. First, all linearizations (resp. strong linearizations) of $P(\lambda)$ have the same finite (resp. finite and infinite) elementary divisors as $P(\lambda)$, and so the same eigenvalues [14]. Second, since linearizations transform a PEP into a *generalized Eigenvalue Problem* (GEP), then well established algorithms for the GEP may be used on linearizations both for regular and singular polynomials [5, 6, 11, 17, 30, 31]. However, note that in the case of a regular polynomial $P(\lambda)$, right and left eigenvectors of a linearization $K(\lambda)$ for a certain eigenvalue λ_0 of $P(\lambda)$ have length nk, and they are not eigenvectors of $P(\lambda)$, which have length n. As a consequence, it is needed to know how to recover the eigenvectors of $P(\lambda)$ from those of $K(\lambda)$ to numerically solve the whole PEP through the linearization $K(\lambda)$. Of course, recovery procedures of eigenvectors are well known since many years ago for the first and second Frobenius companion forms [16], which have been the linearizations traditionally used in practice. An analogue discussion to the one for eigenvectors can be made for minimal bases and indices of singular matrix polynomials.

However, the first and second companion forms are not always satisfactory, and, in particular, they usually do not share any algebraic structure that $P(\lambda)$ might have. For example, if $P(\lambda)$ is symmetric, Hermitian, alternating, or palindromic, then the companion forms will not retain any of these structures. Consequently, if the companion forms are used to numerically solve the PEP, then the rounding errors inherent to numerical computations may destroy qualitative aspects of the spectrum of structured matrix polynomials that appear very often in applications. This has motivated a recent intense activity towards the development of new classes of linearizations. Several classes have been introduced for regular matrix polynomials in [2] and [27], generalizing the Frobenius companion forms in a number of different ways. Other classes of linearizations were introduced and studied in [1], motivated by the use of non-monomial bases for the space of polynomials. The extension of all these classes of linearizations to square singular matrix polynomials have been studied in [8, 9]. The numerical properties of the linearizations in [27] have been analyzed in [18, 19, 21], while the exploitation of these linearizations for the preservation of structure in a variety of contexts has been developed in [20, 28]. Unfortunately, none of the structure preserving pencils in [20, 27, 28] is a linearization for singular matrix polynomials [8]. In addition, simple recovery procedures for the eigenvectors, minimal bases and minimal indices of $P(\lambda)$ from the corresponding

magnitudes of the linearization have been developed for all the classes, except one, of the linearizations mentioned above [8, 9, 18, 27]. The exception is the class of what we name *generalized Fiedler linearizations, and the development of very easy recovery procedures for eigenvectors, minimal bases and minimal indices of $P(\lambda)$ from the linearizations in this class is the main contribution in this work*. The interest of generalized Fiedler linearizations is discussed in the next paragraph.

Two classes of linearizations of matrix polynomials were introduced in [2] and both of them are extensions of linearizations previously developed by Fiedler for scalar polynomials in [12]. The first class received the name of *Fiedler linearizations* in [9], and we will refer to the second class as *generalized Fiedler linearizations*. We will describe in detail these two classes in Section 2. The algebraic properties of Fiedler linearizations are well understood [9] and are excellent: (i) they are constructed, as the classical companion forms, simply by placing the coefficients of the matrix polynomial together with $2(k-1)$ identity blocks in certain blocks entries and setting the remaining blocks to zero; (ii) they are strong linearizations for every matrix polynomial, which is in contrast with the pencils introduced in [1, 27] that are not linearizations for certain regular and singular polynomials; and (iii) eigenvectors, minimal bases and minimal indices of the polynomial are easily recovered from those of any Fiedler linearization. However, the class of Fiedler linearizations does not contain pencils that are symmetric or palindromic when $P(\lambda)$ is, respectively, symmetric or palindromic. In contrast, the wider class of generalized Fiedler linerizations does contain linearizations that preserve the symmetric structure [2] and the palindromic structure [10], and, moreover, the linearizations in this class retain the excellent properties of Fiedler linerizations: (i) they are also very easily constructible from the coefficients of the polynomial, again, in most cases, simply by placing the coefficients in certain blocks entries (see examples in [2, 10]); and (ii) most of them are strong linearizations for every matrix polynomial, and for those that do not satisfy this property, the only polynomials for which they are not linearizations are the ones with singular leading and/or zero degree coefficients. It should be remarked that for odd degree matrix polynomials, the structure preserving generalized Fiedler pencils presented in [2, 10] for symmetric and palindromic polynomials are always strong linearizations, which is again in contrast with the structured pencils developed in [20, 27, 28] that are not linearizations for certain regular polynomials and, in fact, are never linearizations for singular polynomials. All these properties make generalized Fiedler pencils particularly relevant and give, in our opinion, a strong motivation for obtaining the results presented in this work.

The paper is organized as follows. In Section 2 we revise the families of Fiedler and generalized Fiedler pencils and establish some of their properties. Section 3 includes the main results of this work, that is, recovery procedures of the eigenvectors of regular matrix polynomials from the eigenvectors of any of its generalized Fiedler linearizations. Except for a few particular pencils, these recovery procedures consist simply in extracting adequate blocks from the eigenvectors of the linearizations and, therefore, do not require any computational effort. We consider the recovery of minimal bases and indices of square singular matrix polynomials in Section 4. Special attention is paid to structure preserving generalized Fiedler linearizations in Sections 3 and 4. Some conclusions are presented in Section 5.

2. Generalized Fiedler pencils. In this section we recall the families of Fiedler and generalized Fiedler (GF) pencils of a given matrix polynomial and some of their properties. These families were introduced in [2] for regular matrix polynomials (without giving them the names of Fiedler and GF pencils). Fiedler pencils (not GF) of singular polynomials were studied in [9] together with recovery procedures of eigenvectors, minimal bases and minimal indices. We also establish in this section the basic Lemma 2.6 that relates both families in a particular way and is the basis of the recovery results proved in Sections 3 and 4. Finally,

we briefly recall two types of structure preserving linearizations related to GF pencils: the symmetric GF pencils introduced in [2] and the palindromic pencils presented in [10].

2.1. The Fiedler pencils. To introduce the Fiedler family of the matrix polynomial $P(\lambda)$ in (1.1), we need the following block-partitioned matrices:

$$M_k := \begin{bmatrix} A_k & \\ & I_{(k-1)n} \end{bmatrix}, \quad M_0 := \begin{bmatrix} I_{(k-1)n} & \\ & -A_0 \end{bmatrix}, \tag{2.1}$$

and

$$M_i := \begin{bmatrix} I_{(k-i-1)n} & & & \\ & -A_i & I & \\ & I & 0 & \\ & & & I_{(i-1)n} \end{bmatrix}, \quad i = 1, \ldots, k-1\,. \tag{2.2}$$

We will often consider these matrices partitioned into $k \times k$ blocks of size $n \times n$, and are the basic factors used to build the Fiedler pencils of $P(\lambda)$. In [2] these pencils are constructed as

$$\lambda M_k - M_{i_0} M_{i_1} \cdots M_{i_{k-1}},$$

where $(i_0, i_1 \ldots, i_{k-1})$ is any possible permutation of the k-tuple $(0, 1, \ldots, k-1)$. In order to better express certain key properties of this permutation, the product of the M_i-factors was indexed in [9] as follows: given any bijection $\sigma : \{0, 1, \ldots, k-1\} \to \{1, \ldots, k\}$, the *Fiedler pencil of* $P(\lambda)$ *associated with* σ is the $nk \times nk$ matrix pencil

$$F_\sigma(\lambda) := \lambda M_k - M_{\sigma^{-1}(1)} \cdots M_{\sigma^{-1}(k)}\,. \tag{2.3}$$

Note that $\sigma(i)$ describes the position of the factor M_i in the product $M_{\sigma^{-1}(1)} \cdots M_{\sigma^{-1}(k)}$, i.e., $\sigma(i) = j$ means that M_i is the jth factor in the product.

Fiedler pencils include the first and second companion forms of $P(\lambda)$, which are, respectively, $\lambda M_k - M_{k-1} M_{k-2} \cdots M_1 M_0$ and $\lambda M_k - M_0 M_1 \cdots M_{k-2} M_{k-1}$ [9, p. 2186].

It is obvious to check the commutativity relations

$$M_i M_j = M_j M_i \quad \text{for} \quad |i - j| \neq 1\,, \tag{2.4}$$

which imply that some Fiedler pencils associated with different bijections σ are equal.

The M_i matrices in (2.2) are always invertible for $i = 1, \ldots, k-1$ and the inverses are

$$M_i^{-1} = \begin{bmatrix} I_{(k-i-1)n} & & & \\ & 0 & I & \\ & I & A_i & \\ & & & I_{(i-1)n} \end{bmatrix}, \tag{2.5}$$

which satisfy commutativity relations analogous to (2.4). The same holds for $M_i^{-1} M_j$.

2.2. The generalized Fiedler pencils. We will use the following notation introduced in [2, p. 82]: if $\mathcal{E} = \{i_1, \ldots, i_p\} \subseteq \{1, \ldots, k-1\}$ is an ordered set, where $i_j \neq i_l$ if $j \neq l$, then we define

$$M_{\mathcal{E}} := M_{i_1} M_{i_2} \cdots M_{i_p} \quad \text{and} \quad M_\emptyset := I_{nk}. \tag{2.6}$$

Given four sets of this type $\mathcal{E}_j$, $j = 1, 2, 3, 4$, such that $\mathcal{E}_i \cap \mathcal{E}_j = \emptyset$ if $i \neq j$, and $\cup_{i=1}^4 \mathcal{E}_i = \{1, \ldots, k-1\}$, the following pencil was introduced in [2, Corollary 2.4]

$$T(\lambda) = \lambda\, M_{\mathcal{E}_1}^{-1} M_k M_{\mathcal{E}_2}^{-1} - M_{\mathcal{E}_3} M_0 M_{\mathcal{E}_4}. \tag{2.7}$$

We refer to pencils of this type as *proper generalized Fiedler pencils* of the matrix polynomial $P(\lambda)$ in (1.1). Note that $T(\lambda)$ is always defined for $P(\lambda)$, even if $P(\lambda)$ is singular. In addition, $T(\lambda)$ is *strictly equivalent* to the Fiedler pencil $\lambda M_k - M_{\mathcal{E}_1} M_{\mathcal{E}_3} M_0 M_{\mathcal{E}_4} M_{\mathcal{E}_2}$ (recall that two pencils $K_1(\lambda)$ and $K_2(\lambda)$ are strictly equivalent if $K_1(\lambda) = EK_2(\lambda)F$, where E and F are nonsingular constant matrices). A particular but interesting family of proper generalized Fiedler pencils has been presented in [10].

The commutativity relations (2.4) allow us to express $T(\lambda)$ in (2.7) in different forms. In particular, we may shift M_k to the first position in $M_{\mathcal{E}_1}^{-1} M_k M_{\mathcal{E}_2}^{-1}$ if $k-1 \notin \mathcal{E}_1$, or to the last one if $k-1 \notin \mathcal{E}_2$. Analogously, we may shift M_0 to the first or last position in $M_{\mathcal{E}_3} M_0 M_{\mathcal{E}_4}$. To be precise let us assume that $T(\lambda) = \lambda M_k M_{\mathcal{E}_1}^{-1} M_{\mathcal{E}_2}^{-1} - M_0 M_{\mathcal{E}_3} M_{\mathcal{E}_4}$, then, if A_k and/or A_0 are nonsingular[1], we can create the pencils $M_k^{-1} T(\lambda) = \lambda M_{\mathcal{E}_1}^{-1} M_{\mathcal{E}_2}^{-1} - M_k^{-1} M_0 M_{\mathcal{E}_3} M_{\mathcal{E}_4}$, $M_0^{-1} T(\lambda) = \lambda M_0^{-1} M_k M_{\mathcal{E}_1}^{-1} M_{\mathcal{E}_2}^{-1} - M_{\mathcal{E}_3} M_{\mathcal{E}_4}$ and $M_0^{-1} M_k^{-1} T(\lambda) = \lambda M_0^{-1} M_{\mathcal{E}_1}^{-1} M_{\mathcal{E}_2}^{-1} - M_k^{-1} M_{\mathcal{E}_3} M_{\mathcal{E}_4}$ (recall that $k \geq 2$ and so M_k and M_0 commute). All these pencils are examples of what we call *generalized Fiedler pencils* of $P(\lambda)$. Observe that they are strictly equivalent to $T(\lambda)$ and to the Fiedler pencil $\lambda M_k - M_{\mathcal{E}_1} M_{\mathcal{E}_3} M_0 M_{\mathcal{E}_4} M_{\mathcal{E}_2}$. Some interesting generalized Fiedler pencils have been presented in [2, Theorem 3.1].

In Definition 2.1, we use bijections (as in the case of Fiedler pencils) to make precise the notion of generalized Fiedler pencils and the subset of proper generalized Fiedler pencils.

DEFINITION 2.1. *Let $P(\lambda)$ be the matrix polynomial in* (1.1) *and let M_i for $i = 0, 1, \ldots, k$ be the matrices defined in* (2.1)-(2.2)*. Let $\{C_0, C_1\}$ be a partition*[2] *of $\{0, 1, ..., k\}$ with $m_i = |C_i|$ for $i = 0, 1$. Given any pair of bijections $\mu_i : C_i \to \{1, 2, ..., m_i\}$, $i = 0, 1$, we denote $\mu = (\mu_0, \mu_1)$. Then the* generalized Fiedler (GF) *pencil of $P(\lambda)$ associated with μ is the $nk \times nk$ pencil $T_\mu(\lambda) := \lambda T_{\mu_1} - T_{\mu_0}$ with*

$$T_{\mu_i} := \widetilde{M}_{\mu_i^{-1}(1)} \widetilde{M}_{\mu_i^{-1}(2)} \cdots \widetilde{M}_{\mu_i^{-1}(m_i)}, \qquad i = 0, 1,$$

where the factors $\widetilde{M}_j$ are defined, in a different way for $i = 0$ than for $i = 1$, as follows

(a) if $i = 0$ and $j \in C_0$, then $\widetilde{M}_j = M_k^{-1}$ for $j = k$, and $\widetilde{M}_j = M_j$ for $j \neq k$;

(b) if $i = 1$ and $j \in C_1$, then $\widetilde{M}_j = M_k$ for $j = k$, and $\widetilde{M}_j = M_j^{-1}$ for $j \neq k$.

Note that $\mu_i(j)$ describes the position of $\widetilde{M}_j$ in the product $\widetilde{M}_{\mu_i^{-1}(1)} \widetilde{M}_{\mu_i^{-1}(2)} \cdots \widetilde{M}_{\mu_i^{-1}(m_i)}$.

If $0 \in C_0$ and $k \in C_1$, then the pencil $T_\mu(\lambda)$ is said to be a proper generalized Fiedler (PGF) *pencil of $P(\lambda)$.*

It is obvious that any Fiedler pencil $F_\sigma(\lambda)$ of $P(\lambda)$ is a particular case of GF pencil with $C_0 = \{0, 1, \ldots, k-1\}$, $C_1 = \{k\}$, $\mu_0 = \sigma$ and $\mu_1(k) = 1$. We stress the fact that GF pencils that are not proper are defined only if A_k and/or A_0 are not singular.

It is straightforward to prove that any GF pencil of $P(\lambda)$ is strictly equivalent to a Fiedler pencil of $P(\lambda)$ by using the commutativity relations (2.4). This fact and Theorem 4.6 in [9] imply directly the following result.

THEOREM 2.2. *Let $P(\lambda)$ be an $n \times n$ matrix polynomial. Then any generalized Fiedler pencil of $P(\lambda)$ is a strong linearization for $P(\lambda)$.*

Theorem 2.2 holds for singular polynomials $P(\lambda)$, but in this case recall that the only GF pencils that are defined are the PGF pencils. The fact that they are strong linearizations for any square matrix polynomial makes PGF pencils the most interesting class of GF pencils.

[1] Note that A_k and A_0 are necessarily singular if $P(\lambda)$ is singular, and that they can be singular even if $P(\lambda)$ is regular.

[2] We admit $C_i = \emptyset$, with $T_{\mu_i} = I_{nk}$ in this case.

2.3. Consecutions and inversions. The commutativity relations (2.4) and the results in [9] suggest that the relative positions of factors $\widetilde{M}_j$ and $\widetilde{M}_{j+1}$ in the products T_{μ_1} and T_{μ_0} defining a GF pencil are of fundamental interest in studying GF pencils. This motivates Definition 2.3 that is related to Definition 3.3 in [9].

DEFINITION 2.3. *Let $\{C_0, C_1\}$ be a partition of $\{0, 1, ..., k\}$ with $m_i = |C_i|$ for $i = 0, 1$, and let $\mu_i : C_i \to \{1, 2, ..., m_i\}$, $i = 0, 1$, be a pair of bijections.*

(a) *We say that μ_i has a* consecution *at j if $\{j, j+1\} \subset C_i$ and $\mu_i(j) < \mu_i(j+1)$. We say that μ_i has an* inversion *at j if $\{j, j+1\} \subset C_i$ and $\mu_i(j) > \mu_i(j+1)$.*

(b) *We say that μ_i has $\mathfrak{c}_j$ (resp. $\mathfrak{i}_j$) consecutions (resp. inversions) at j if μ_i has consecutions (resp. inversions) at $j, j+1, \ldots, j+\mathfrak{c}_j-1$ (resp. at $j, j+1, \ldots, j+\mathfrak{i}_j-1$) and it has not a consecution (resp. inversion) at $j+\mathfrak{c}_j$ (resp. $j+\mathfrak{i}_j$).*

(c) *We say that μ_i has $\mathfrak{c}_f$ (resp. $\mathfrak{i}_f$) final consecutions (resp. inversions) if μ_i has consecutions (resp. inversions) at $k-\mathfrak{c}_f, k-\mathfrak{c}_f+1, \ldots, k-2, k-1$ (resp. at $k-\mathfrak{i}_f, k-\mathfrak{i}_f+1, \ldots, k-2, k-1$) and it has not a consecution (resp. inversion) at $k-\mathfrak{c}_f-1$ (resp. $k-\mathfrak{i}_f-1$).*

REMARK 2.4. *The following remarks will be often used in the rest of the paper.*

(a) *For $i = 0, 1$, the bijection μ_i in Definition 2.1 has a consecution (resp. inversion) at j if and only if $\widetilde{M}_j$ and $\widetilde{M}_{j+1}$ are both factors of the product defining T_{μ_i} and $\widetilde{M}_j$ is to the left (resp. right) of $\widetilde{M}_{j+1}$ in T_{μ_i}.*

(b) *Given a GF pencil $T_\mu(\lambda) = \lambda T_{\mu_1} - T_{\mu_0}$, the commutativity relations* (2.4) *may allow us to change the order of the factors $\widetilde{M}_j$ in T_{μ_1} and T_{μ_0}. The new order will be related to a pair of bijections $\mu' = (\mu'_0, \mu'_1)$ such that $\mu' \neq \mu$ and $T_{\mu'}(\lambda) = T_\mu(\lambda)$. However, the use of the commutativity relations* (2.4) *cannot change the relative positions of $\widetilde{M}_j$ and $\widetilde{M}_{j+1}$, and so, for $i = 0, 1$, μ'_i has a consecution (resp. inversion) at j if and only if μ_i has a consecution (resp. inversion) at j.*

(c) *In [9, Definition 3.3] the quantities c_1 and i_1 are defined for the bijection σ defining the Fiedler pencil $F_\sigma(\lambda)$. In the case of Fiedler pencils, we know that $\sigma = \mu_0$, and it is obvious to see that if σ has $\mathfrak{c}_0$ consecutions at 0, then $\mathfrak{c}_0 = c_1$. However, we stress that if σ has $\mathfrak{i}_0$ inversions at 0, then $\mathfrak{i}_0 \neq i_1$ in general, but that $\mathfrak{i}_0 = i_1$ if $\mathfrak{c}_0 = c_1 = 0$.*

2.4. Fundamental lemma. There may be more than one Fiedler pencil strictly equivalent to a given GF pencil. Moreover, the Fiedler pencils strictly equivalent to a GF pencil may have quite different structures. Let us illustrate this fact with an example.

EXAMPLE 2.5. *Let the degree of the polynomial $P(\lambda)$ in* (1.1) *be $k = 5$. Consider the PGF pencil of $P(\lambda)$*

$$T_\mu(\lambda) = \lambda M_5 M_3^{-1} M_1^{-1} - M_0 M_2 M_4 = \lambda M_1^{-1} M_5 M_3^{-1} - M_0 M_2 M_4,$$

where the last equality follows from (2.4)*. Therefore $T_\mu(\lambda)$ is strictly equivalent to the Fiedler pencil*

$$F_\sigma(\lambda) = T_\mu(\lambda) M_1 M_3 = \lambda M_5 - M_0 M_2 M_4 M_1 M_3,$$

and also strictly equivalent to the Fiedler pencil

$$F_{\sigma'}(\lambda) = M_1 T_\mu(\lambda) M_3 = \lambda M_5 - M_1 M_0 M_2 M_4 M_3.$$

The reader is invited to check by direct multiplication that $F_\sigma(\lambda)$ is quite different than $F_{\sigma'}(\lambda)$. Observe in addition that, with the notation of Definition 2.1, μ_0 has 0 consecutions at 0, σ has 1 consecution at 0, and σ' has 0 consecutions at 0. In plain words, $F_{\sigma'}(\lambda)$ preserves the consecutions at 0 of T_{μ_0}, but $F_\sigma(\lambda)$ does not.

Our goal in this section is to prove in Lemma 2.6 below that any PGF pencil $T_\mu(\lambda)$ of $P(\lambda)$ in (1.1) is strictly equivalent to a Fiedler pencil that preserves the consecutions at 0 of μ_0, except in the following very particular cases

$$T_\tau(\lambda) = \lambda M_k M_{k-1}^{-1} \cdots M_{\mathfrak{c}_0+1}^{-1} - M_0 M_1 \cdots M_{\mathfrak{c}_0}, \tag{2.8}$$

where $\mathfrak{c}_0 \in \{0, 1, ..., k-2\}$. Note that in (2.8) the pair of bijections $\tau = (\tau_0, \tau_1)$ is defined by

$$(\,\tau_0^{-1}(1), \tau_0^{-1}(2), \ldots, \tau_0^{-1}(\mathfrak{c}_0+1)\,) = (0, 1, \ldots, \mathfrak{c}_0), \tag{2.9}$$

$$(\,\tau_1^{-1}(1), \tau_1^{-1}(2), \ldots, \tau_1^{-1}(k-\mathfrak{c}_0)\,) = (k, k-1, \ldots, \mathfrak{c}_0+1). \tag{2.10}$$

LEMMA 2.6. *Let $P(\lambda)$ be the matrix polynomial in* (1.1) *and let $T_\mu(\lambda)$ be the PGF pencil of $P(\lambda)$ associated with the pair of bijections $\mu = (\mu_0, \mu_1)$. If μ_0 has $\mathfrak{c}_0$ consecutions at 0 and $(\mu_0, \mu_1) \neq (\tau_0, \tau_1)$, where τ_0 and τ_1 are defined in* (2.9)-(2.10)*, then there exist two ordered subsets $\mathcal{E}_1$ and $\mathcal{E}_2$ of $\{1, \ldots, k-1\}$ such that*

(a) *$\mathfrak{c}_0 \notin \mathcal{E}_2$ and $\mathfrak{c}_0 + 1 \notin \mathcal{E}_2$; and,*
(b) *$F_\sigma(\lambda) = M_{\mathcal{E}_1}\, T_\mu(\lambda)\, M_{\mathcal{E}_2}$ is a Fiedler pencil of $P(\lambda)$ associated with a bijection σ that has $\mathfrak{c}_0$ consecutions at 0.*

Proof. If $T_\mu(\lambda)$ is a Fiedler pencil of $P(\lambda)$, then the result follows trivially by taking $\mathcal{E}_1 = \mathcal{E}_2 = \emptyset$, because $M_{\mathcal{E}_1} = M_{\mathcal{E}_2} = I_{nk}$ by (2.6). Therefore, we will assume in the rest of the proof that $T_\mu(\lambda) = \lambda T_{\mu_1} - T_{\mu_0}$ is not a Fiedler pencil. Recall that the fact that $\mu_0 : C_0 \to \{1, 2, \ldots, m_0\}$ has $\mathfrak{c}_0$ consecutions at 0 implies that $\{0, 1, \ldots, \mathfrak{c}_0\} \subseteq C_0$, or, equivalently, that $M_0, M_1, \ldots, M_{\mathfrak{c}_0}$ are among the factors of the product defining T_{μ_0} (moreover in this precise relative order). As a consequence, $\mathfrak{c}_0 \leq k-2$, because otherwise the pencil would be a Fiedler pencil. We will separate the proof in three cases.

Case 1: $\mathfrak{c}_0 + 1 \in C_0$. Then $M_{\mathfrak{c}_0+1}$ is to the left of $M_{\mathfrak{c}_0}$ in the product defining T_{μ_0}, because otherwise μ_0 would have more than $\mathfrak{c}_0$ consecutions at 0. Therefore,

$$T_\mu(\lambda) = \lambda\, M_{\mathcal{E}_1}^{-1} M_k M_{\mathcal{E}_2}^{-1} - T_{\mu_0},$$

where $\mathfrak{c}_0 \notin \mathcal{E}_2$ and $\mathfrak{c}_0 + 1 \notin \mathcal{E}_2$, and $\{\mathcal{E}_1, \mathcal{E}_2, C_0\}$ is a partition of $\{0, 1, \ldots, k-1\}$. Then $F_\sigma(\lambda) = M_{\mathcal{E}_1}\, T_\mu(\lambda)\, M_{\mathcal{E}_2} = \lambda M_k - M_{\mathcal{E}_1}\, T_{\mu_0}\, M_{\mathcal{E}_2}$ is a Fiedler pencil of $P(\lambda)$ and σ has $\mathfrak{c}_0$ consecutions at 0, because these consecutions are determined only by the factors in T_{μ_0}.

Case 2: $\mathfrak{c}_0 + 1 \in C_1$ and $\mu_1(\mathfrak{c}_0 + 1) < \mu_1(k)$. This last condition is equivalent to say that $M_{\mathfrak{c}_0+1}^{-1}$ is to the left of M_k in the product defining T_{μ_1}. Therefore,

$$T_\mu(\lambda) = \lambda\, M_{\mathcal{E}_1}^{-1} M_k M_{\mathcal{E}_2}^{-1} - T_{\mu_0}, \tag{2.11}$$

where $\mathfrak{c}_0+1 \in \mathcal{E}_1$, $\mathfrak{c}_0 \notin \mathcal{E}_2$ and $\mathfrak{c}_0+1 \notin \mathcal{E}_2$, and $\{\mathcal{E}_1, \mathcal{E}_2, C_0\}$ is a partition of $\{0, 1, \ldots, k-1\}$. Then $F_\sigma(\lambda) = M_{\mathcal{E}_1}\, T_\mu(\lambda)\, M_{\mathcal{E}_2} = \lambda M_k - M_{\mathcal{E}_1}\, T_{\mu_0}\, M_{\mathcal{E}_2}$ is a Fiedler pencil of $P(\lambda)$ and σ has $\mathfrak{c}_0$ consecutions at 0, because these consecutions are determined only by the factors in T_{μ_0} and the fact that $M_{\mathfrak{c}_0+1}$ is a factor of $M_{\mathcal{E}_1}$, so σ has an inversion at $\mathfrak{c}_0$.

Case 3: $\mathfrak{c}_0 + 1 \in C_1$ and $\mu_1(k) < \mu_1(\mathfrak{c}_0 + 1)$. This last condition is equivalent to say that $M_{\mathfrak{c}_0+1}^{-1}$ is to the right of M_k in the product defining T_{μ_1}. Therefore,

$$T_\mu(\lambda) = \lambda\, M_{\mathcal{E}_1'}^{-1} M_k M_{\mathcal{E}_2'}^{-1} - T_{\mu_0},$$

where $\mathfrak{c}_0 \notin \mathcal{E}_2'$ but $\mathfrak{c}_0 + 1 \in \mathcal{E}_2'$, and $\{\mathcal{E}_1', \mathcal{E}_2', C_0\}$ is a partition of $\{0, 1, \ldots, k-1\}$. Our strategy consists in using the commutativity relations (2.4) to rearrange the order of the factors defining T_{μ_1} and shift $M_{\mathfrak{c}_0+1}^{-1}$ to the left of M_k. To this purpose let $s \geq 0$ be the integer such

that μ_1 has inversions at $\mathfrak{c}_0+1, \mathfrak{c}_0+2, \ldots, \mathfrak{c}_0+s$, but not at $\mathfrak{c}_0+1+s$. Note that $\mathfrak{c}_0+1+s < k$, because otherwise the pencil $T_\mu(\lambda)$ must be

$$\lambda M_k M_{k-1}^{-1} \cdots M_{\mathfrak{c}_0+1}^{-1} - M_0 M_1 \cdots M_{\mathfrak{c}_0}, \tag{2.12}$$

which is not possible because $(\mu_0, \mu_1) \neq (\tau_0, \tau_1)$. The commutativity relations (2.4) allow us to shift in T_{μ_1} the factors $M_{\mathfrak{c}_0+1}^{-1}, M_{\mathfrak{c}_0+2}^{-1}, \ldots, M_{\mathfrak{c}_0+s}^{-1}$ to the left and group together

$$M_{\mathcal{B}}^{-1} := M_{\mathfrak{c}_0+1+s}^{-1} M_{\mathfrak{c}_0+s}^{-1} \cdots M_{\mathfrak{c}_0+2}^{-1} M_{\mathfrak{c}_0+1}^{-1}.$$

It may happen that in this shifting process $M_{\mathfrak{c}_0+1}^{-1}$ moves to the left of M_k, then

$$T_{\mu_1} = \cdots M_{\mathcal{B}}^{-1} \cdots M_k \cdots, \tag{2.13}$$

or that $M_{\mathfrak{c}_0+1}^{-1}$ stays to the right of M_k, then

$$T_{\mu_1} = \cdots M_k \cdots M_{\mathcal{B}}^{-1} \cdots. \tag{2.14}$$

In the scenario (2.14), $\mathfrak{c}_0 + 1 + s < k - 1$ (otherwise the pencil must be (2.12), which is not possible) and $M_{\mathfrak{c}_0+2+s}^{-1}$ is not between M_k and $M_{\mathcal{B}}^{-1}$ (since μ_1 has not an inversion at $\mathfrak{c}_0 + 1 + s$). Therefore, the relations (2.4) allow us to write T_{μ_1} in (2.14) as in (2.13). Thus, in any situation, $T_\mu(\lambda)$ can be written in turn as in (2.11) and we obtain the result proceeding as in *Case 2*. □

2.5. Symmetric GF pencils. In [2, Theorem 3.1] two particular GF pencils of $P(\lambda)$ in (1.1) were considered. These GF pencils have the key property of being symmetric if $P(\lambda)$ is symmetric, that is, if $A_i^T = A_i$ for $i = \{0, 1, ..., k\}$. These GF pencils are defined as follows

$$S(\lambda) = \begin{cases} \lambda M_k M_{k-2}^{-1} \cdots M_3^{-1} M_1^{-1} - M_0 M_2 \cdots M_{k-3} M_{k-1}, & \text{if k is odd} \\ \lambda M_{k-1}^{-1} M_{k-3}^{-1} \cdots M_3^{-1} M_1^{-1} - M_0 M_2 \cdots M_{k-2} M_k^{-1}, & \text{if k is even.} \end{cases} \tag{2.15}$$

Note that $S(\lambda)$ is a PGF pencil if the degree k is odd. In addition, the pencils in (2.15) have a simple block-tridiagonal structure.

2.6. Palindromic linearizations based on PGF pencils. The matrix polynomial $P(\lambda)$ in (1.1) is said to be *palindromic* [28] if $A_i^T = A_{k-i}$ for $i = 0, 1, \ldots, k$, or in other words if $\operatorname{rev} P(\lambda) = P(\lambda)^T$. Palindromic polynomials arise in a number of application areas and are receiving considerable attention in the last years [4, 22, 23, 24, 28, 32]. As far as we know, GF pencils of $P(\lambda)$ that are palindromic when $P(\lambda)$ is palindromic have not been found. However, a family of linearizations with this property has been introduced in [10] for odd degree matrix polynomials. The pencils in this family are obtained by multiplying by two constant simple matrices the following PGF pencils.

DEFINITION 2.7. [10] (Admissible index set and associated pencils) *Let $P(\lambda)$ be the matrix polynomial in* (1.1) *with odd degree k and let $h := (k+1)/2$. A subset with h elements $C_0 = \{j_1, \ldots, j_h\} \subset \{0, 1, \ldots, k-1\}$ is said to be an* admissible index set *if $0 \in C_0$ and $C_0 \cap \{k - j_1, \ldots, k - j_h\} = \emptyset$. Given any bijection $\tau_0 : C_0 \to \{1, 2, \ldots, h\}$, the pencil of $P(\lambda)$ associated with C_0 and τ_0 is the $nk \times nk$ pencil*

$$L_{\tau_0}(\lambda) := \lambda \widetilde{M}_{k-\tau_0^{-1}(h)} \cdots \widetilde{M}_{k-\tau_0^{-1}(2)} \widetilde{M}_{k-\tau_0^{-1}(1)} - M_{\tau_0^{-1}(1)} M_{\tau_0^{-1}(2)} \cdots M_{\tau_0^{-1}(h)}, \tag{2.16}$$

where $\widetilde{M}_j$ are the matrices defined in part (b) of Definition 2.1.

If $C_1 := \{k - j_1, \ldots, k - j_h\}$ and we define the bijection $\tau_1 : C_1 \to \{1, 2, \ldots, h\}$ as $(\tau_1^{-1}(1), \tau_1^{-1}(2), \ldots, \tau_1^{-1}(h)) := (k - \tau_0^{-1}(h), \ldots, k - \tau_0^{-1}(2), k - \tau_0^{-1}(1))$, then observe

that (2.16) is the PGF pencil associated with the pair of bijections (τ_0, τ_1). As explained in [10], admissible index sets are easy to construct: simply take exactly one element from $\{j, k-j\}$ for each $j = 1, 2, \ldots, (k-1)/2$, and then add 0.

We still need another two matrices to construct the linearizations presented in [10]. One is the $k \times k$ block reverse identity matrix $R \in \mathbb{F}^{nk \times nk}$ defined as

$$R := \begin{bmatrix} & & I \\ & \iddots & \\ I & & \end{bmatrix}. \tag{2.17}$$

The other one is a $k \times k$ block-diagonal matrix $S \in \mathbb{F}^{nk \times nk}$ whose $n \times n$ diagonal blocks are

$$S(i,i) := \begin{cases} -I & \text{if} \quad \begin{cases} \tau_0 \text{ has an inversion at } i-1, \text{ or} \\ \tau_0 \text{ has a consecution at } k-i, \text{ or} \\ i \in C_0 \text{ and } i-1 \notin C_0 \end{cases} \\ I & \text{otherwise} \end{cases} . \tag{2.18}$$

It was proved in [10, Theorem 4.7] that the pencil $S \cdot R \cdot L_{\tau_0}(\lambda)$ is a strong linearization of $P(\lambda)$ which is palindromic whenever $P(\lambda)$ is palindromic. Although the definition of these pencils seems complicated, they are very easy to construct and some of them have a simple block anti-tridiagonal structure. See examples in [10].

3. Recovery of eigenvectors of regular matrix polynomials from GF pencils. This section includes the two main results in this paper: the recovery of eigenvectors corresponding to a finite eigenvalue of a regular matrix polynomial from the eigenvectors of any of its GF pencils (Theorem 3.2), and the recovery of eigenvectors corresponding to the infinite eigenvalue (Theorem 3.4). As corollaries of these results we present eigenvector recovery procedures from the structure preserving linearizations discussed in Sections 2.5 and 2.6.

Instead of dealing with individual eigenvectors, we will consider the general problem of the recovery of bases of the right and left eigenspaces for an eigenvalue λ_0 of a regular matrix polynomial $P(\lambda)$. These eigenspaces are the right and left null spaces of $P(\lambda_0)$, i.e.,

$$\begin{aligned} \mathcal{N}_r(P(\lambda_0)) &:= \left\{ x \in \mathbb{F}^{n \times 1} \, : \, P(\lambda_0)x = 0 \right\}, \\ \mathcal{N}_\ell(P(\lambda_0)) &:= \left\{ y^T \in \mathbb{F}^{1 \times n} \, : \, y^T P(\lambda_0) = 0^T \right\}. \end{aligned}$$

The key ideas behind the recovery results we present are very simple. First, consider two $nk \times nk$ linearizations $L(\lambda)$ and $K(\lambda)$ of a regular matrix polynomial $P(\lambda)$ that are *strictly equivalent*, i.e, $L(\lambda) = UK(\lambda)V$ where $U, V \in \mathbb{F}^{nk \times nk}$ are nonsingular constant matrices. Assume that $\lambda_0 \in \mathbb{F}$ is a finite eigenvalue of $P(\lambda)$ (and so also of $L(\lambda)$ and $K(\lambda)$). It is straightforward to prove that the right eigenspaces of $L(\lambda)$ and $K(\lambda)$ for λ_0 are related by the following isomorphism

$$\begin{array}{ccc} \mathcal{N}_r(L(\lambda_0)) & \longrightarrow & \mathcal{N}_r(K(\lambda_0)) \\ x & \longmapsto & V \cdot x \end{array} \tag{3.1}$$

Second, recall that almost any PGF pencil $T_\mu(\lambda)$ of $P(\lambda)$ is strictly equivalent to a particular Fiedler pencil $F_\sigma(\lambda)$ of $P(\lambda)$ (see Lemma 2.6) and that it is known how to recover very easily the eigenvectors of $P(\lambda)$ with eigenvalue λ_0 from the eigenvectors of $F_\sigma(\lambda)$ with eigenvalue λ_0 (see [9, Corollary 7.1]). Then, the specific isomorphism between $\mathcal{N}_r(T_\mu(\lambda_0))$ and $\mathcal{N}_r(F_\sigma(\lambda_0))$ will allow us to recover also very easily the eigenvectors of $P(\lambda)$ from the eigenvectors of $T_\mu(\lambda)$. The recovery results for GF pencils that are not proper will require a little bit of extra work, and we need to distinguish in Theorem 3.2 if the matrix $\widetilde{M}_0$ is a factor of the one or the zero degree coefficient of the GF pencil.

REMARK 3.1. *We warn the reader that the situations covered in parts* (a2), (b2), (c2), (d2), (e2) *and* (f2) *of Theorem 3.2 correspond to very particular pencils that, as far as we know, are not used in any application. For instance:*

(a2) *corresponds only to the pencil* $\lambda I_{nk} - M_0 M_1 \cdots M_{k-1} M_k^{-1}$*;*

(b2) *corresponds to the pencils* $\lambda M_{\mathfrak{i}_0'}^{-1} \cdots M_1^{-1} M_0^{-1} - M_{\mathfrak{i}_0'+1} \cdots M_{k-1} M_k^{-1}$ *for* $\mathfrak{i}_0' = 0, 1, \ldots, k-1$*; and*

(c2) *corresponds only to the pencil* $\lambda M_k M_{k-1}^{-1} \cdots M_1^{-1} M_0^{-1} - I_{nk}$.

These cases are included in Theorem 3.2 for completeness, and note that for them the recovery of eigenvectors is more complicated than for the rest of GF pencils.

THEOREM 3.2 (**eigenvector recovery from generalized Fiedler pencils**). *Let* $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ *be an* $n \times n$ *regular matrix polynomial with degree* $k \geq 2$*, and let* $T_\mu(\lambda) = \lambda T_{\mu_1} - T_{\mu_0}$ *be the GF pencil of* $P(\lambda)$ *associated with the pair of bijections* $\mu = (\mu_0, \mu_1)$*, where* $\mu_i : C_i \to \{1, \ldots, m_i\}$*,* $i = 0, 1$*. Let* λ_0 *be a finite eigenvalue of* $P(\lambda)$.

Right eigenvectors: *Let* $\{z_1, \ldots, z_p\} \subset \mathbb{F}^{nk \times 1}$ *be a basis of* $\mathcal{N}_r(T_\mu(\lambda_0))$*, partition the vectors* $z_j \in \mathbb{F}^{nk \times 1}$ *as* $k \times 1$ *block vectors with* $n \times 1$ *blocks, denote by* $z_j^{(q)} \in \mathbb{F}^{n \times 1}$ *the qth block of* z_j*, and assume that* μ_0 *has* $\mathfrak{c}_j$ *consecutions at* j *and* μ_1 *has* $\mathfrak{i}_j'$ *inversions at* j.

(a) *Suppose* $0 \in C_0$:

(a1) *If* $\mathfrak{c}_0 < k$*, then* $\{z_1^{(k-\mathfrak{c}_0)}, \ldots, z_p^{(k-\mathfrak{c}_0)}\}$ *is a basis of* $\mathcal{N}_r(P(\lambda_0))$.
Note that all PGF pencils are included in part (a1).

(a2) *If* $\mathfrak{c}_0 = k$*, then* $\{A_k^{-1} z_1^{(1)}, \ldots, A_k^{-1} z_p^{(1)}\}$ *is a basis of* $\mathcal{N}_r(P(\lambda_0))$.

Part (a) includes all GF pencils that have been used so far in applications.

(b) *Suppose* $0 \in C_1$ *and* $\mathfrak{i}_0' + 1 \in C_0$:

(b1) *If* $s = \mathfrak{i}_0' + \mathfrak{c}_{\mathfrak{i}_0'+1} + 1 < k$*, then* $\{z_1^{(k-s)}, \ldots, z_p^{(k-s)}\}$ *is a basis of* $\mathcal{N}_r(P(\lambda_0))$.

(b2) *If* $\mathfrak{i}_0' + \mathfrak{c}_{\mathfrak{i}_0'+1} + 1 = k$*, then* $\{A_k^{-1} z_1^{(1)}, \ldots, A_k^{-1} z_p^{(1)}\}$ *is a basis of* $\mathcal{N}_r(P(\lambda_0))$.

(c) *Suppose* $0 \in C_1$ *and* $\mathfrak{i}_0' + 1 \notin C_0$:

(c1) *If* $\mathfrak{i}_0' < k$*, then* $\{z_1^{(k-\mathfrak{i}_0')}, \ldots, z_p^{(k-\mathfrak{i}_0')}\}$ *is a basis of* $\mathcal{N}_r(P(\lambda_0))$.

(c2) *If* $\mathfrak{i}_0' = k$*, then* $\{A_k^{-1} z_1^{(1)}, \ldots, A_k^{-1} z_p^{(1)}\}$ *is a basis of* $\mathcal{N}_r(P(\lambda_0))$.

Left eigenvectors: *Let* $\{w_1^T, \ldots, w_p^T\} \subset \mathbb{F}^{1 \times nk}$ *be a basis of* $\mathcal{N}_\ell(T_\mu(\lambda_0))$*, partition the vectors* $w_j^T \in \mathbb{F}^{1 \times nk}$ *as* $1 \times k$ *block vectors with* $1 \times n$ *blocks, denote by* $(w_j^T)^{(q)} \in \mathbb{F}^{1 \times n}$ *the qth block of* w_j^T*, and assume that* μ_0 *has* $\mathfrak{i}_j$ *inversions at* j *and* μ_1 *has* $\mathfrak{c}_j'$ *consecutions at* j.

(d) *Suppose* $0 \in C_0$:

(d1) *If* $\mathfrak{i}_0 < k$*, then* $\{(w_1^T)^{(k-\mathfrak{i}_0)}, \ldots, (w_p^T)^{(k-\mathfrak{i}_0)}\}$ *is a basis of* $\mathcal{N}_\ell(P(\lambda_0))$.
Note that all PGF pencils are included in part (d1).

(d2) *If* $\mathfrak{i}_0 = k$*, then* $\{(w_1^T)^{(1)} A_k^{-1}, \ldots, (w_p^T)^{(1)} A_k^{-1}\}$ *is a basis of* $\mathcal{N}_\ell(P(\lambda_0))$.

Part (d) includes all GF pencils that have been used so far in applications.

(e) *Suppose* $0 \in C_1$ *and* $\mathfrak{c}_0' + 1 \in C_0$:

(e1) *If* $s = \mathfrak{c}_0' + \mathfrak{i}_{\mathfrak{c}_0'+1} + 1 < k$*, then* $\{(w_1^T)^{(k-s)}, \ldots, (w_p^T)^{(k-s)}\}$ *is a basis of* $\mathcal{N}_\ell(P(\lambda_0))$.

(e2) *If* $\mathfrak{c}_0' + \mathfrak{i}_{\mathfrak{c}_0'+1} + 1 = k$*, then* $\{(w_1^T)^{(1)} A_k^{-1}, \ldots, (w_p^T)^{(1)} A_k^{-1}\}$ *is a basis of* $\mathcal{N}_\ell(P(\lambda_0))$.

(f) *Suppose* $0 \in C_1$ *and* $\mathfrak{c}_0' + 1 \notin C_0$:

(f1) *If* $\mathfrak{c}_0' < k$*, then* $\{(w_1^T)^{(k-\mathfrak{c}_0')}, \ldots, (w_p^T)^{(k-\mathfrak{c}_0')}\}$ *is a basis of* $\mathcal{N}_\ell(P(\lambda_0))$.

(f2) *If* $\mathfrak{c}_0' = k$*, then* $\{(w_1^T)^{(1)} A_k^{-1}, \ldots, (w_p^T)^{(1)} A_k^{-1}\}$ *is a basis of* $\mathcal{N}_\ell(P(\lambda_0))$.

Proof. Part (a1) *for PGF pencils.* This is the key result. The rest of the theorem follows easily from it. Note that PGF pencils have less than $k+1$ factors M_j in the product defining T_{μ_0}, because M_k is a factor of T_{μ_1}, so necessarily $\mathfrak{c}_0 < k$. Assume first that $\mu \neq \tau$, where

the pair of bijections τ was defined in (2.9)-(2.10), and let $F_\sigma(\lambda) = M_{\mathcal{E}_1}\, T_\mu(\lambda)\, M_{\mathcal{E}_2}$ be the Fiedler pencil in part (b) of Lemma 2.6. This means that σ has $\mathfrak{c}_0$ consecutions at 0. Then $x \mapsto M_{\mathcal{E}_2} x$ is an isomorphism from $\mathcal{N}_r(F_\sigma(\lambda_0))$ to $\mathcal{N}_r(T_\mu(\lambda_0))$, which means that

$$\{z_1, \ldots, z_p\} = \{M_{\mathcal{E}_2} v_1, \ldots, M_{\mathcal{E}_2} v_p\}, \tag{3.2}$$

for some basis $\{v_1, \ldots, v_p\} \subset \mathbb{F}^{nk\times 1}$ of $\mathcal{N}_r(F_\sigma(\lambda_0))$. From [9, Corollary 7.1], we know that $\{v_1^{(k-\mathfrak{c}_0)}, \ldots, v_p^{(k-\mathfrak{c}_0)}\} \subset \mathbb{F}^{n\times 1}$ is a basis of $\mathcal{N}_r(P(\lambda_0))$. The result follows by noting that $\{z_1^{(k-\mathfrak{c}_0)}, \ldots, z_p^{(k-\mathfrak{c}_0)}\} = \{v_1^{(k-\mathfrak{c}_0)}, \ldots, v_p^{(k-\mathfrak{c}_0)}\}$, because $M_{\mathcal{E}_2} = M_{i_1} M_{i_2} \ldots M_{i_c}$ with $i_j \neq \mathfrak{c}_0$, $i_j \neq \mathfrak{c}_0 + 1$, which implies that all these factors M_{i_j} have I at block entry $(k - \mathfrak{c}_0, k - \mathfrak{c}_0)$ when they are partitioned into $k \times k$ blocks of size $n \times n$.

Assume next that $\mu = \tau$ and define in this situation $M_{\mathcal{E}_2} := M_{\mathfrak{c}_0+1} M_{\mathfrak{c}_0+2} \ldots M_{k-1}$. Then, according to (2.8),

$$T_\tau(\lambda) M_{\mathcal{E}_2} = \lambda M_k - M_0\, M_1 \cdots M_{k-1} =: F_{\sigma'}(\lambda), \tag{3.3}$$

where the Fiedler pencil $F_{\sigma'}(\lambda)$ is such that σ' has $k-1$ consecutions at 0 (in fact, $F_{\sigma'}(\lambda)$ is the second companion form of $P(\lambda)$). The same argument as above shows that the relation (3.2) holds for some basis $\{v_1, \ldots, v_p\} \subset \mathbb{F}^{nk\times 1}$ of $\mathcal{N}_r(F_{\sigma'}(\lambda_0))$. Use again [9, Corollary 7.1], to prove that $\{v_1^{(1)}, \ldots, v_p^{(1)}\} \subset \mathbb{F}^{n\times 1}$ is a basis of $\mathcal{N}_r(P(\lambda_0))$. The result follows from noting that

$$M_{\mathcal{E}_2} = \left[\begin{array}{ccccc|c} -A_{k-1} & I & \ldots & 0 & 0 & \\ -A_{k-2} & 0 & \ddots & 0 & 0 & \\ \vdots & \vdots & & \ddots & 0 & \\ -A_{\mathfrak{c}_0+1} & 0 & & & I & \\ I & 0 & \ldots & \ldots & 0 & \\ \hline & & & & & I_{\mathfrak{c}_0 n} \end{array}\right], \tag{3.4}$$

which implies that $\{z_1^{(k-\mathfrak{c}_0)}, \ldots, z_p^{(k-\mathfrak{c}_0)}\} = \{v_1^{(1)}, \ldots, v_p^{(1)}\}$. This completes the proof for PGF pencils.

Part (a1) *for GF pencils.* It only remains to prove the result if $T_\mu(\lambda)$ is not a PGF pencil. This happens when $k \in C_0$. The commutativity relations (2.4) and the fact that μ_0 has $\mathfrak{c}_0$ consecutions at 0 allow us to shift $M_0, M_1, \ldots M_{\mathfrak{c}_0}$ to the right and write

$$T_{\mu_0} = H_\ell (M_0 M_1 \cdots M_{\mathfrak{c}_0}),$$

where H_ℓ is a product of a certain set of M_j factors ($j \neq 0, j \neq k$) and M_k^{-1}. Therefore H_ℓ is a nonsingular matrix. Observe that $T_{\mu'}(\lambda) := H_\ell^{-1} T_\mu(\lambda)$ is a PGF pencil such that μ_0' has $\mathfrak{c}_0$ consecutions at zero and that, according to (3.1), $\mathcal{N}_r(T_{\mu'}(\lambda_0)) = \mathcal{N}_r(T_\mu(\lambda_0))$. The result follows from applying (a1) to the PGF pencil $T_{\mu'}(\lambda)$.

Part (a2) . The condition $\mathfrak{c}_0 = k$ determines that

$$T_\mu(\lambda) = \lambda I_{nk} - M_0 M_1 M_2 \cdots M_{k-1} M_k^{-1}.$$

Observe that $T_{\mu'}(\lambda) = \lambda T_{\mu_1'} - T_{\mu_0'} := T_\mu(\lambda) M_k$ is a Fiedler pencil (so PGF) such that μ_0' has $k-1$ consecutions at 0 and that, according to (3.1), $\{z_1', \ldots, z_p'\} := \{M_k^{-1} z_1, \ldots, M_k^{-1} z_p\}$ is a basis of $\mathcal{N}_r(T_{\mu'}(\lambda_0))$. Part (a1) applied on $T_{\mu'}(\lambda)$ implies that $\{(z_1')^{(1)}, \ldots, (z_p')^{(1)}\} = \{A_k^{-1}\, z_1^{(1)}, \ldots, A_k^{-1}\, z_p^{(1)}\}$ is a basis of $\mathcal{N}_r(P(\lambda_0))$ (recall (2.1)).

Part (b). In this situation M_0^{-1} is a factor in the product defining T_{μ_1}. The commutativity relations (2.4) and the fact that μ_1 has $\mathfrak{i}_0'$ inversions at 0 allow us to shift $M_0^{-1}, M_1^{-1}, \dots M_{\mathfrak{i}_0'}^{-1}$ to the left and write

$$T_{\mu_1} = (M_{\mathfrak{i}_0'}^{-1} \cdots M_1^{-1} M_0^{-1}) \cdots .$$

Then the pencil $T_{\mu'}(\lambda) = \lambda T_{\mu_1'} - T_{\mu_0'} := (M_0 M_1 \cdots M_{\mathfrak{i}_0'}) T_\mu(\lambda)$ is a GF pencil that satisfies: (i) M_0 and $\widetilde{M}_{\mathfrak{i}_0'+1}$ are factors of $T_{\mu_0'}$; (ii) the bijection μ_0' has $s = \mathfrak{i}_0' + \mathfrak{c}_{\mathfrak{i}_0'+1} + 1$ consecutions at 0, because $M_0 M_1 \cdots M_{\mathfrak{i}_0'}$ are the first factors of $T_{\mu_0'}$; and (iii) $\mathcal{N}_r(T_{\mu'}(\lambda_0)) = \mathcal{N}_r(T_\mu(\lambda_0))$ by (3.1), since $(M_0 M_1 \cdots M_{\mathfrak{i}_0'})$ is invertible. The result follows from applying part (a) to $T_{\mu'}(\lambda)$.

Part (c). We follow the proof of part (b) to construct the GF pencil $T_{\mu'}(\lambda) = \lambda T_{\mu_1'} - T_{\mu_0'} := (M_0 M_1 \cdots M_{\mathfrak{i}_0'}) T_\mu(\lambda)$, but now $T_{\mu'}(\lambda)$ satisfies: (i) M_0 is a factor of $T_{\mu_0'}$ but $\widetilde{M}_{\mathfrak{i}_0'+1}$ is not; (ii) the bijection μ_0' has $\mathfrak{i}_0'$ consecutions at 0; and (iii) $\mathcal{N}_r(T_{\mu'}(\lambda_0)) = \mathcal{N}_r(T_\mu(\lambda_0))$ by (3.1). The result follows again from applying part (a) to $T_{\mu'}(\lambda)$.

Proof of the recovery of left eigenvectors. Note that for any matrix polynomial $Q(\lambda)$ (of any size) and for any λ_0, the mapping $x^T \mapsto x$ establishes an isomorphism from $\mathcal{N}_\ell(Q(\lambda_0))$ to $\mathcal{N}_r(Q(\lambda_0)^T)$. Then, $\{w_1^T, \dots, w_p^T\} \subset \mathbb{F}^{1\times nk}$ is a basis of $\mathcal{N}_\ell(T_\mu(\lambda_0))$ if and only if $\{w_1, \dots, w_p\} \subset \mathbb{F}^{nk\times 1}$ is a basis of $\mathcal{N}_r(T_\mu(\lambda_0)^T)$. In addition, $T_\mu(\lambda)^T$ is a GF pencil of $P(\lambda)^T$, as a consequence of the structure of the M_j and M_j^{-1} matrices defined in (2.1), (2.2) and (2.5). Note also that the action of taking transposes reverses the order of the factors of a product, so if μ' is the pair of bijections corresponding to $T_\mu(\lambda)^T$ viewed as a GF pencil for $P(\lambda)^T$ (symbolically, $T_{\mu'}(P^T)(\lambda) := T_\mu(\lambda)^T$), then, for $i = 0, 1$, μ_i' has a consecution (resp. inversion) at j if and only if μ_i has an inversion (resp. consecution) at j. Finally, the result follows from applying the recovery of bases of right eigenspaces to get a basis of $\mathcal{N}_r(P(\lambda_0)^T)$ from the basis $\{w_1, \dots, w_p\} \subset \mathbb{F}^{nk\times 1}$ of $\mathcal{N}_r(T_\mu(\lambda_0)^T)$. □

We illustrate with a couple of examples the utter simplicity of the "recipes" for recovering eigenvectors given in Theorem 3.2. For brevity, we focus on right eigenvectors.

EXAMPLE 3.3. *Consider the following GF pencils of* $P(\lambda)$ *in* (1.1) *with degree* $k = 6$*:*

$$T_\mu(\lambda) = \lambda M_3^{-1} M_6 M_5^{-1} - M_4 M_0 M_2 M_1, \tag{3.5}$$

$$T_{\mu'}(\lambda) = \lambda M_1^{-1} M_6 M_0^{-1} - M_4 M_5 M_2 M_3, \tag{3.6}$$

and let λ_0 *be a finite eigenvalue of* $P(\lambda)$*. For dealing with the first pencil* $T_\mu(\lambda)$*, we use part* (a) *of Theorem 3.2 and observe that* $\mathfrak{c}_0 = 1$ *in this case. So, if* $\{z_1, \dots, z_p\} \subset \mathbb{F}^{6n\times 1}$ *is a basis of* $\mathcal{N}_r(T_\mu(\lambda_0))$*, then* $\{z_1^{(5)}, \dots, z_p^{(5)}\} \subset \mathbb{F}^{n\times 1}$ *is a basis of* $\mathcal{N}_r(P(\lambda_0))$*.*

For dealing with the second pencil $T_{\mu'}(\lambda)$*, we use part* (b) *of Theorem 3.2, since* $\mathfrak{i}_0' = 1$ *and* M_2 *is a factor of* $T_{\mu_0'}$*. Observe that* $\mathfrak{c}_2 = 1$*, so* $\mathfrak{i}_0' + \mathfrak{c}_{\mathfrak{i}_0'+1} + 1 = 3$*. Therefore, if* $\{v_1, \dots, v_p\} \subset \mathbb{F}^{6n\times 1}$ *is a basis of* $\mathcal{N}_r(T_{\mu'}(\lambda_0))$*, then* $\{v_1^{(3)}, \dots, v_p^{(3)}\} \subset \mathbb{F}^{n\times 1}$ *is a basis of* $\mathcal{N}_r(P(\lambda_0))$*.*

Next, we consider the recovery of left and right eigenvectors corresponding to the eigenvalue ∞ from GF pencils. This is simpler than the recovery for finite eigenvalues, because if $P(\lambda)$ in (1.1) has the eigenvalue ∞, then A_k is singular. This limits the set of GF pencils for $P(\lambda)$, since M_k has to be necessarily a factor of the first degree term of the GF pencil. To understand why A_k is singular, recall that $P(\lambda)$ has an infinite eigenvalue if and only if $\operatorname{rev} P(\lambda)$ has the eigenvalue 0, and that the right and left eigenspaces at ∞ of $P(\lambda)$ are the right and left eigenspaces of $\operatorname{rev} P(\lambda)$ for the eigenvalue 0, that is, the right and left null spaces of the matrix $\operatorname{rev} P(0) = A_k$.

THEOREM 3.4 (**eigenvector recovery at ∞ from generalized Fiedler pencils**). *Let $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ be an $n \times n$ regular matrix polynomial with degree $k \geq 2$, and let $T_\mu(\lambda) = \lambda T_{\mu_1} - T_{\mu_0}$ be the GF pencil of $P(\lambda)$ associated with the pair of bijections $\mu = (\mu_0, \mu_1)$. Suppose ∞ is an eigenvalue of $P(\lambda)$, and assume that μ_1 has $\mathfrak{c}_f$ and $\mathfrak{i}_f$ final consecutions and inversions, respectively.*

Right eigenvectors at ∞: *Let $\{z_1, \ldots, z_p\} \subset \mathbb{F}^{nk\times 1}$ be a basis of the right eigenspace of $T_\mu(\lambda)$ at ∞, partition the vectors $z_j \in \mathbb{F}^{nk\times 1}$ as $k \times 1$ block vectors with $n \times 1$ blocks, and denote by $z_j^{(q)} \in \mathbb{F}^{n\times 1}$ the qth block of z_j.*

(a) *If $\mathfrak{i}_f < k$, then $\{z_1^{(\mathfrak{i}_f+1)}, \ldots, z_p^{(\mathfrak{i}_f+1)}\}$ is a basis of the right eigenspace of $P(\lambda)$ at ∞.* Note that all PGF pencils are included in part (a).

(b) *If $\mathfrak{i}_f = k$, then $\{A_0^{-1} z_1^{(k)}, \ldots, A_0^{-1} z_p^{(k)}\}$ is a basis of the right eigenspace of $P(\lambda)$ at ∞.*

Left eigenvectors at ∞: *Let $\{w_1^T, \ldots, w_p^T\} \subset \mathbb{F}^{1\times nk}$ be a basis of the left eigenspace of $T_\mu(\lambda)$ at ∞, partition the vectors $w_j^T \in \mathbb{F}^{1\times nk}$ as $1 \times k$ block vectors with $1 \times n$ blocks, and denote by $(w_j^T)^{(q)} \in \mathbb{F}^{1\times n}$ the qth block of w_j^T.*

(c) *If $\mathfrak{c}_f < k$, then $\{(w_1^T)^{(\mathfrak{c}_f+1)}, \ldots, (w_p^T)^{(\mathfrak{c}_f+1)}\}$ is a basis of the left eigenspace of $P(\lambda)$ at ∞.* Note that all PGF pencils are included in part (c).

(d) *If $\mathfrak{c}_f = k$, then $\{(w_1^T)^{(k)} A_0^{-1}, \ldots, (w_p^T)^{(k)} A_0^{-1}\}$ is a basis of the left eigenspace of $P(\lambda)$ at ∞.*

Proof. Part (a). Note that $\{z_1, \ldots, z_p\} \subset \mathbb{F}^{nk\times 1}$ is a basis of $\mathcal{N}_r(\operatorname{rev} T_\mu(0)) = \mathcal{N}_r(T_{\mu_1})$. The relations (2.4) and the fact that μ_1 has $\mathfrak{i}_f$ final inversions allow us to write

$$T_{\mu_1} = H_\ell M_k M_{k-1}^{-1} \cdots M_{k-\mathfrak{i}_f}^{-1},$$

where H_ℓ is a product of a certain set of M_j^{-1} factors. Therefore H_ℓ is a nonsingular matrix and $\mathcal{N}_r(\operatorname{rev} T_\mu(0)) = \mathcal{N}_r(T_{\mu_1}) = \mathcal{N}_r(M_k M_{k-1}^{-1} \cdots M_{k-\mathfrak{i}_f}^{-1})$. Note that if $\mathfrak{i}_f < k$, then

$$M_k M_{k-1}^{-1} \cdots M_{k-\mathfrak{i}_f}^{-1} = \left[\begin{array}{ccccc|ccc} 0 & \cdots & \cdots & 0 & A_k & & & \\ I & \cdots & \cdots & 0 & A_{k-1} & & & \\ & \ddots & & \vdots & \vdots & & & \\ & & \ddots & 0 & A_{k-\mathfrak{i}_f+1} & & & \\ & & & I & A_{k-\mathfrak{i}_f} & & & \\ \hline & & & & & I & \cdots & 0 \\ & & & & & \vdots & \ddots & \vdots \\ & & & & & 0 & \cdots & I \end{array}\right]. \tag{3.7}$$

Observe that the first block of $(M_k M_{k-1}^{-1} \cdots M_{k-\mathfrak{i}_f}^{-1})\, z_j = 0$ is $A_k z_j^{(\mathfrak{i}_f+1)} = 0$, for $j = 1, \ldots, p$. This means that $\{z_1^{(\mathfrak{i}_f+1)}, \ldots, z_p^{(\mathfrak{i}_f+1)}\}$ is contained in the right eigenspace of $P(\lambda)$ at ∞, i.e., $\mathcal{N}_r(A_k)$, but not yet that it is basis. To prove that it is a basis, note first that $\dim \mathcal{N}_r(\operatorname{rev} P(0)) = \dim \mathcal{N}_r(\operatorname{rev} T_\mu(0))$, since $T_\mu(\lambda)$ is a strong linearization for $P(\lambda)$, so we only have to prove that $\{z_1^{(\mathfrak{i}_f+1)}, \ldots, z_p^{(\mathfrak{i}_f+1)}\}$ is a linearly independent set. We proceed by contradiction: assume that it is linearly dependent, then there exists a nonzero vector $x \in \mathbb{F}^{p\times 1}$ such that $[\, z_1^{(\mathfrak{i}_f+1)} | \cdots | z_p^{(\mathfrak{i}_f+1)} \,]\, x = 0$. Then the vector $v = [\, z_1 | \cdots | z_p \,] x \in \mathbb{F}^{nk\times 1}$ satisfies: (i) $v \neq 0$, since $\{z_1, \ldots, z_p\}$ are linearly independent; (ii) $v^{(\mathfrak{i}_f+1)} = 0$; and (iii) $(M_k M_{k-1}^{-1} \cdots M_{k-\mathfrak{i}_f}^{-1})\, v = 0$. But (ii), (iii) and (3.7) imply $v = 0$, which is in contradiction with (i). Therefore $\{z_1^{(\mathfrak{i}_f+1)}, \ldots, z_p^{(\mathfrak{i}_f+1)}\}$ is a linearly independent set.

Part (b). It is similar to the proof of part (a). Simply note that if $\mathfrak{i}_f = k$, then $T_{\mu_1} = M_k M_{k-1}^{-1} \cdots M_1^{-1} M_0^{-1}$ and

$$M_k M_{k-1}^{-1} ... M_1^{-1} M_0^{-1} = \begin{bmatrix} 0 & \cdots & \cdots & -A_k A_0^{-1} \\ I & \cdots & \cdots & -A_{k-1} A_0^{-1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I & -A_1 A_0^{-1} \end{bmatrix}.$$

Thus, the first block of $(M_k M_{k-1}^{-1} \cdots M_1^{-1} M_0^{-1})\, z_j = 0$ is $A_k (A_0^{-1})\, z_j^{(k)} = 0$, for $j = 1, \ldots, p$. This means that $\{A_0^{-1} z_1^{(k)}, \ldots, A_0^{-1} z_p^{(k)}\}$ is contained in the right eigenspace of $P(\lambda)$ at ∞, i.e., $\mathcal{N}_r(A_k)$. The proof that it is a basis is essentially the same as the one in part (a) and is omitted.

Parts (c) *and* (d). As in the proof of Theorem 3.2, the recovery of left eigenvectors at ∞ follows from the recovery of right eigenvectors at ∞ from the GF pencil $T_\mu(\lambda)^T$. The idea is the same as in Theorem 3.2 and we invite the reader to complete the details. □

We illustrate the recovery of right eigenvectors at ∞ with a couple of examples.

EXAMPLE 3.5. *Let $T_\mu(\lambda)$ be the GF pencil in* (3.5)*, then $\mathfrak{i}_f = 1$. So, if $\{z_1, \ldots, z_p\} \subset \mathbb{F}^{nk \times 1}$ is a basis of the right eigenspace of $T_\mu(\lambda)$ at ∞, then $\{z_1^{(2)}, \ldots, z_p^{(2)}\} \subset \mathbb{F}^{n \times 1}$ is a basis of the right eigenspace of $P(\lambda)$ at ∞.*

Let $T_{\mu'}(\lambda)$ be the GF pencil in (3.6)*, then $\mathfrak{i}_f = 0$. So, if $\{z_1, \ldots, z_p\} \subset \mathbb{F}^{nk \times 1}$ is a basis of the right eigenspace of $T_{\mu'}(\lambda)$ at ∞, then $\{z_1^{(1)}, \ldots, z_p^{(1)}\} \subset \mathbb{F}^{n \times 1}$ is a basis of the right eigenspace of $P(\lambda)$ at ∞.*

3.1. Recovery of eigenvectors from structure preserving linearizations. Theorems 3.2 and 3.4 lead to very simple eigenvector recovery "recipes" if they are applied to the linearizations presented in Subsection 2.5.

COROLLARY 3.6 (**eigenvector recovery from symmetric GF linearizations**). *Let $P(\lambda)$ be an $n \times n$ regular matrix polynomial with degree $k \geq 2$ and let $S(\lambda)$ be the GF linearization of $P(\lambda)$ defined in (2.15). Let λ_0 be an eigenvalue of $P(\lambda)$ that may be finite or infinite. Observe that if $\lambda_0 = \infty$ and k is even, then $S(\lambda)$ is not defined.*

Right eigenvectors: *Let $\{z_1, \ldots, z_p\} \subset \mathbb{F}^{nk \times 1}$ be a basis of the right eigenspace of $S(\lambda)$ for the eigenvalue λ_0, partition the vectors $z_j \in \mathbb{F}^{nk \times 1}$ as $k \times 1$ block vectors with $n \times 1$ blocks, and denote by $z_j^{(q)} \in \mathbb{F}^{n \times 1}$ the qth block of z_j.*

(a) *If λ_0 is finite, then $\{z_1^{(k)}, \ldots, z_p^{(k)}\}$ is a basis of the right eigenspace of $P(\lambda)$ for λ_0.*
(b) *If $\lambda_0 = \infty$, then $\{z_1^{(1)}, \ldots, z_p^{(1)}\}$ is a basis of the right eigenspace of $P(\lambda)$ at ∞.*

Left eigenvectors: *Let $\{w_1^T, \ldots, w_p^T\} \subset \mathbb{F}^{1 \times nk}$ be a basis of the left eigenspace of $S(\lambda)$ for the eigenvalue λ_0, partition the vectors $w_j^T \in \mathbb{F}^{1 \times nk}$ as $1 \times k$ block vectors with $1 \times n$ blocks, and denote by $(w_j^T)^{(q)} \in \mathbb{F}^{1 \times n}$ the qth block of w_j^T.*

(c) *If λ_0 is finite, then $\{(w_1^T)^{(k)}, \ldots, (w_p^T)^{(k)}\}$ is a basis of the left eigenspace of $P(\lambda)$ for λ_0.*
(d) *If $\lambda_0 = \infty$, then $\{(w_1^T)^{(1)}, \ldots, (w_p^T)^{(1)}\}$ is a basis of the left eigenspace of $P(\lambda)$ at ∞.*

Proof. The matrix M_0 is a factor of the zero degree term of $S(\lambda)$. So the recovery of eigenvectors corresponding to finite eigenvalues is given by parts (a) and (d) of Theorem 3.2. In addition, the magnitudes $\mathfrak{c}_0$ and $\mathfrak{i}_0$ in Theorem 3.2 are $\mathfrak{c}_0 = \mathfrak{i}_0 = 0$ for $S(\lambda)$, which implies parts (a) and (c). If $\lambda_0 = \infty$, then the magnitudes $\mathfrak{c}_f$ and $\mathfrak{i}_f$ in Theorem 3.4 are $\mathfrak{c}_f = \mathfrak{i}_f = 0$, which implies parts (b) and (d). □

REMARK 3.7. *We know that the linearization $S(\lambda)$ in* (2.15) *is symmetric if $P(\lambda)$ is symmetric. In addition, left eigenvectors of any symmetric polynomial are the transpose of the corresponding right eigenvectors. Therefore, for symmetric matrix polynomials, it is only needed to recover right eigenvectors. Recall, however, that $S(\lambda)$ is a linearization for arbitrary matrix polynomials $P(\lambda)$. For this reason, we present in Corollary 3.6 recovery procedures both for left and right eigenvectors.*

Theorems 3.2 and 3.4 also lead to simple eigenvector recovery procedures from the linearizations presented in Subsection 2.6.

COROLLARY 3.8 (**eigenvector recovery from palindromic linearizations based on PGF pencils**). *Let $P(\lambda)$ be an $n \times n$ regular matrix polynomial with odd degree $k \geq 3$, let S and R be the matrices defined in* (2.18) *and* (2.17)*, and let $L_{\tau_0}(\lambda)$ be the PGF pencil of $P(\lambda)$ defined in* (2.16)*. Suppose that the bijection τ_0 has $\mathfrak{c}_0$ and $\mathfrak{i}_0$ consecutions and inversions at 0, respectively. Let λ_0 be an eigenvalue of $P(\lambda)$ that may be finite or infinite.*

Right eigenvectors: *Let $\{z_1, \ldots, z_p\} \subset \mathbb{F}^{nk\times 1}$ be a basis of the right eigenspace of $SRL_{\tau_0}(\lambda)$ for the eigenvalue λ_0, partition the vectors $z_j \in \mathbb{F}^{nk\times 1}$ as $k \times 1$ block vectors with $n \times 1$ blocks, and denote by $z_j^{(q)} \in \mathbb{F}^{n\times 1}$ the qth block of z_j.*

(a) *If λ_0 is finite, then $\{z_1^{(k-\mathfrak{c}_0)}, \ldots, z_p^{(k-\mathfrak{c}_0)}\}$ is a basis of the right eigenspace of $P(\lambda)$ for λ_0.*

(b) *If $\lambda_0 = \infty$, then $\{z_1^{(\mathfrak{i}_0+1)}, \ldots, z_p^{(\mathfrak{i}_0+1)}\}$ is a basis of the right eigenspace of $P(\lambda)$ at ∞.*

Left eigenvectors: *Let $\{w_1^T, \ldots, w_p^T\} \subset \mathbb{F}^{1\times nk}$ be a basis of the left eigenspace of $SRL_{\tau_0}(\lambda)$ for the eigenvalue λ_0, partition the vectors $w_j^T \in \mathbb{F}^{1\times nk}$ as $1 \times k$ block vectors with $1 \times n$ blocks, and denote by $(w_j^T)^{(q)} \in \mathbb{F}^{1\times n}$ the qth block of w_j^T.*

(c) *If λ_0 is finite, then $\{(w_1^T)^{(\mathfrak{i}_0+1)}, \ldots, (w_p^T)^{(\mathfrak{i}_0+1)}\}$ is a basis of the left eigenspace of $P(\lambda)$ for λ_0.*

(d) *If $\lambda_0 = \infty$, then $\{(w_1^T)^{(k-\mathfrak{c}_0)}, \ldots, (w_p^T)^{(k-\mathfrak{c}_0)}\}$ is a basis of the left eigenspace of $P(\lambda)$ at ∞.*

Proof. The right eigenspace of $SRL_{\tau_0}(\lambda)$ corresponding to λ_0 is equal to the right eigenspace of the PGF pencil $L_{\tau_0}(\lambda)$ corresponding to λ_0. So parts (a) and (b) follow from applying Theorem 3.1-(a1) and Theorem 3.4-(a) to the PGF pencil $L_{\tau_0}(\lambda)$. The application of Theorem 3.4-(a) is particularly simple in this case, because, as we explained in Subsection 2.6, $L_{\tau_0}(\lambda)$ is the PGF pencil associated with the pair of bijections (τ_0, τ_1), where $(\tau_1^{-1}(1), \tau_1^{-1}(2), \ldots, \tau_1^{-1}(h)) := (k - \tau_0^{-1}(h), \ldots, k - \tau_0^{-1}(2), k - \tau_0^{-1}(1))$. Therefore, τ_1 has $\mathfrak{c}_f$ (resp. $\mathfrak{i}_f$) final consecutions (resp. inversions) if and only if τ_0 has $\mathfrak{c}_0$ (resp. $\mathfrak{i}_0$) consecutions (resp. inversions) at 0. Before proving this property, we invite the reader to consider, for $k = 5$, the example $L_{\tau_0}(\lambda) = \lambda M_3^{-1} M_4^{-1} M_5 - M_0 M_1 M_2$.

For the left eigenspaces, note that $y^T \mapsto y^T SR$ establishes an isomorphism from the left eigenspace of $SRL_{\tau_0}(\lambda)$ corresponding to λ_0 to the left eigenspace of the PGF pencil $L_{\tau_0}(\lambda)$ corresponding to λ_0. Therefore, $\{w_1^T S\, R, \ldots, w_p^T S\, R\} \subset \mathbb{F}^{1\times nk}$ is a basis of the left eigenspace of $L_{\tau_0}(\lambda)$ for the eigenvalue λ_0. Equations (2.17) and (2.18) give the following relationships between blocks: $(w_j^T S\, R)^{(q)} = \pm (w_j^T)^{(k-q+1)}$, for $j = 1, \ldots, p$ and $q = 1, \ldots, k$. Use these relationships, apply Theorem 3.1-(d1) and Theorem 3.4-(c) to $L_{\tau_0}(\lambda)$, and get parts (c) and (d) of Corollary 3.8. □

REMARK 3.9. *We know that the linearization $SRL_{\tau_0}(\lambda)$ is palindromic if $P(\lambda)$ is palindromic. In addition, left eigenvectors of an eigenvalue λ_0 of any palindromic polynomial are the transpose of the right eigenvectors for the eigenvalue $1/\lambda_0$. Therefore, for palindromic matrix polynomials, it is only needed to recover right eigenvectors. Recall, however, that $SRL_{\tau_0}(\lambda)$ is a linearization for arbitrary matrix polynomials $P(\lambda)$. For this reason, we*

present in Corollary 3.8 recovery procedures both for left and right eigenvectors.

4. Recovery of minimal bases and minimal indices of singular matrix polynomials from PGF pencils. In this section we consider square *singular* matrix polynomials, that is, matrix polynomials $P(\lambda)$ such that all the coefficients of $\det P(\lambda)$ as a scalar polynomial in λ are zero. We focus on the recovery of the minimal bases and indices of $P(\lambda)$ from those of its PGF pencils. We explained in Section 2 that PGF are the only GF pencils defined for singular polynomials, and this fact makes very simple the recovery of minimal bases. Minimal bases and indices are magnitudes of fundamental importance in different control problems [13]. Next, we briefly recall their definitions. The reader may found a more complete summary in [8].

Let $\mathbb{F}(\lambda)$ denote the field of rational functions with coefficients in $\mathbb{F}$. An $n \times n$ singular matrix polynomial $P(\lambda)$ has *left* and *right nullspaces* that are $\mathbb{F}(\lambda)$-vector spaces. These are, respectively,

$$\mathcal{N}_\ell(P) := \left\{ y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times n} \, : \, y(\lambda)^T P(\lambda) \equiv 0^T \right\} ,$$
$$\mathcal{N}_r(P) := \left\{ x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} \, : \, P(\lambda)x(\lambda) \equiv 0 \right\} .$$

A *left* (resp. *right*) minimal basis of $P(\lambda)$ is a basis of $\mathcal{N}_\ell(P)$ (resp. $\mathcal{N}_r(P)$) consisting of vectors polynomials, i.e., vectors with polynomial entries, and such that the sum of the degrees of the vectors in this basis is minimal among all polynomial bases of $\mathcal{N}_\ell(P)$ (resp. $\mathcal{N}_r(P)$) [13]. It can be shown [13] that the ordered list of degrees of the vectors in any left (resp. right) minimal basis of $P(\lambda)$ is always the same. These degrees are then called the *left (resp. right) minimal indices* of $P(\lambda)$.

Minimal bases of $P(\lambda)$ can be recovered from minimal bases of any of its PGF pencils as explained in Theorem 4.1.

THEOREM 4.1 (**recovery of minimal bases from PGF pencils**). *Let $P(\lambda)$ be an $n \times n$ singular matrix polynomial with degree $k \geq 2$ and let $T_\mu(\lambda) = \lambda T_{\mu_1} - T_{\mu_0}$ be the PGF pencil of $P(\lambda)$ associated with the pair of bijections $\mu = (\mu_0, \mu_1)$. Assume that μ_0 has $\mathfrak{c}_0$ and $\mathfrak{i}_0$ consecutions and inversions at 0, respectively.*

Right minimal bases: *Let $\{z_1(\lambda), \ldots, z_p(\lambda)\} \subset \mathbb{F}(\lambda)^{nk\times 1}$ be a right minimal basis of $T_\mu(\lambda)$, partition the vectors $z_j(\lambda) \in \mathbb{F}(\lambda)^{nk\times 1}$ as $k \times 1$ block vectors with $n \times 1$ blocks, and denote by $x_j(\lambda) \in \mathbb{F}^{n\times 1}$ the $(k - \mathfrak{c}_0)$th block of $z_j(\lambda)$. Then $\{x_1(\lambda), \ldots, x_p(\lambda)\}$ is a right minimal basis of $P(\lambda)$.*

Left minimal bases: *Let $\{w_1(\lambda)^T, \ldots, w_p(\lambda)^T\} \subset \mathbb{F}(\lambda)^{1\times nk}$ be a left minimal basis of $T_\mu(\lambda)$, partition the vectors $w_j(\lambda)^T \in \mathbb{F}(\lambda)^{1\times nk}$ as $1 \times k$ block vectors with $1 \times n$ blocks, denote by $y_j(\lambda)^T \in \mathbb{F}^{1\times n}$ the $(k - \mathfrak{i}_0)$th block of $w_j(\lambda)^T$, then $\{y_1(\lambda)^T, \ldots, y_p(\lambda)^T\}$ is a left minimal basis of $P(\lambda)$.*

Proof. The proof for right minimal bases is similar to the proof of Theorem 3.2-(a1). We only sketch the main ideas. Assume first that $\mu \neq \tau$, where the pair of bijections τ was defined in (2.9)-(2.10), and let $F_\sigma(\lambda) = M_{\mathcal{E}_1}\, T_\mu(\lambda)\, M_{\mathcal{E}_2}$ be the Fiedler pencil in part (b) of Lemma 2.6. This means that σ has $\mathfrak{c}_0$ consecutions at 0. Then $v(\lambda) \mapsto M_{\mathcal{E}_2} v(\lambda)$ is an isomorphism from $\mathcal{N}_r(F_\sigma)$ to $\mathcal{N}_r(T_\mu)$, that are $\mathbb{F}(\lambda)$-vector spaces. In addition, this isomorphism induces a *degree-preserving bijection* between the subsets of vector polynomials in $\mathcal{N}_r(F_\sigma)$ and $\mathcal{N}_r(T_\mu)$, because $M_{\mathcal{E}_2}$ is a constant nonsingular matrix. This means that

$$\{z_1(\lambda), \ldots, z_p(\lambda)\} = \{M_{\mathcal{E}_2} v_1(\lambda), \ldots, M_{\mathcal{E}_2} v_p(\lambda)\}, \tag{4.1}$$

for some right minimal basis $\{v_1(\lambda), \ldots, v_p(\lambda)\}$ of $F_\sigma(\lambda)$. From [9, Corollary 5.8], we know that the $(k - \mathfrak{c}_0)$th blocks of $\{v_1(\lambda), \ldots, v_p(\lambda)\}$ form a right minimal basis of $P(\lambda)$.

These $(k-\mathfrak{c}_0)$th blocks are precisely $\{x_1(\lambda),\dots,x_p(\lambda)\}$, because $M_{\mathcal{E}_2}=M_{i_1}M_{i_2}\dots M_{i_c}$ with $i_j\neq\mathfrak{c}_0$, $i_j\neq\mathfrak{c}_0+1$, which implies that all these factors M_{i_j} have I at block entry $(k-\mathfrak{c}_0,k-\mathfrak{c}_0)$. If $\mu=\tau$, then use (3.3) and (3.4) and follow the same argument.

As in the proof of Theorem 3.2, the recovery of left minimal bases follows from the recovery of right minimal bases for $T_\mu(\lambda)^T$, which is a PGF pencil for $P(\lambda)^T$. We only remark that for any singular matrix polynomial $Q(\lambda)$, the mapping $y(\lambda)^T\mapsto y(\lambda)$ transforms left minimal bases of $Q(\lambda)$ into right minimal bases of $Q(\lambda)^T$, and viceversa. □

The recovery of minimal indices from PGF pencils is also very simple. We use for this purpose the notation introduced in (2.6) and (2.7).

THEOREM 4.2 (**recovery of minimal indices from PGF pencils**). *Let $P(\lambda)$ be an $n\times n$ singular matrix polynomial with degree $k\geq 2$ and consider four ordered sets $\mathcal{E}_j$, $j=1,2,3,4$, such that $\mathcal{E}_i\cap\mathcal{E}_j=\emptyset$ if $i\neq j$, and $\cup_{i=1}^4\mathcal{E}_i=\{1,\dots,k-1\}$. Let*

$$T_\mu(\lambda)=\lambda\, M_{\mathcal{E}_1}^{-1}M_kM_{\mathcal{E}_2}^{-1}-M_{\mathcal{E}_3}M_0M_{\mathcal{E}_4}$$

be a PGF pencil of $P(\lambda)$ and consider the related Fiedler pencil

$$F_\sigma(\lambda)=\lambda\, M_k-M_{\mathcal{E}_1}M_{\mathcal{E}_3}M_0M_{\mathcal{E}_4}M_{\mathcal{E}_2}.$$

Let $0\leq\eta_1\leq\eta_2\leq\cdots\leq\eta_p$ and $0\leq\varepsilon_1\leq\varepsilon_2\leq\cdots\leq\varepsilon_p$ be, respectively, the left and right minimal indices of $P(\lambda)$ and $\mathfrak{c}(\sigma)$ and $\mathfrak{i}(\sigma)$ be, respectively, the total number of consecutions and inversions of σ. Then, the left and right minimal indices of $T_\mu(\lambda)$ are, respectively,

$$\eta_1+\mathfrak{c}(\sigma)\leq\eta_2+\mathfrak{c}(\sigma)\leq\cdots\leq\eta_p+\mathfrak{c}(\sigma)\quad\text{ and }\quad\varepsilon_1+\mathfrak{i}(\sigma)\leq\varepsilon_2+\mathfrak{i}(\sigma)\leq\cdots\leq\varepsilon_p+\mathfrak{i}(\sigma).$$

Proof. It is immediate to see that $T_\mu(\lambda)$ and $F_\sigma(\lambda)$ have equal minimal indices because they are strictly equivalent. The result follows from applying Corollaries 5.8 and 5.11 in [9] to $F_\sigma(\lambda)$. □

We illustrated in Example 2.5 that there may be more than one Fiedler pencil strictly equivalent to a given PGF pencil, and that these Fiedler pencils may have quite different structures. However, Theorem 4.2 implies that all these Fiedler pencils have the same total number of consecutions and the same total number of inversions. We illustrate this fact with an example

EXAMPLE 4.3. *Let $F_\sigma(\lambda)$ and $F_{\sigma'}(\lambda)$ be the Fiedler pencils defined in Example 2.5. Then, $\mathfrak{c}(\sigma)=\mathfrak{c}(\sigma')=2$ and $\mathfrak{i}(\sigma)=\mathfrak{i}(\sigma')=2$. However, σ and σ' do not have neither all the consecutions nor all the inversions at the same indices.*

Theorems 4.1 and 4.2 can be directly applied to the PGF pencil $S(\lambda)$ defined in (2.15) for odd degree, because for $S(\lambda)$ the magnitudes in these theorems are $\mathfrak{i}_0=\mathfrak{c}_0=0$ and $\mathfrak{i}(\sigma)=\mathfrak{c}(\sigma)=(k-1)/2$. So we can state the following corollary.

COROLLARY 4.4 (**recovery of minimal bases and indices from symmetric PGF linerizations**). *Let $P(\lambda)$ be an $n\times n$ singular matrix polynomial with odd degree $k\geq 3$ and let $S(\lambda)$ be the PGF linearization of $P(\lambda)$ defined in (2.15). Then the minimal bases and indices of $P(\lambda)$ can be recovered from the minimal bases and indices of $S(\lambda)$ by setting $\mathfrak{i}_0=\mathfrak{c}_0=0$ and $\mathfrak{i}(\sigma)=\mathfrak{c}(\sigma)=(k-1)/2$ in Theorems 4.1 and 4.2.*

The last result in this paper is Corollary 4.5, which establishes the recovery of minimal indices and bases from the linearizations presented in Subsection 2.6. As in the proof of Corollary 3.8 for eigenvectors, the main idea is to deal first with the PGF pencil $L_{\tau_0}(\lambda)$ and then with the strictly equivalent pencil $SRL_{\tau_0}(\lambda)$. For brevity we omit the straightforward proof. We only remark that the recovery of minimal indices was already presented in [10, Theorem 6.1], and that the argument presented in [10] is, in our opinion, simpler than a direct application of Theorem 4.2.

COROLLARY 4.5 (**recovery of minimal indices and bases from palindromic linearizations based on PGF pencils**). *Let $P(\lambda)$ be an $n \times n$ singular matrix polynomial with odd degree $k \geq 3$, let S and R be the matrices defined in* (2.18) *and* (2.17)*, and let $L_{\tau_0}(\lambda)$ be the PGF pencil of $P(\lambda)$ defined in* (2.16)*. Suppose that the bijection τ_0 has $\mathfrak{c}_0$ and $\mathfrak{i}_0$ consecutions and inversions at* 0*, respectively.*

Right minimal bases: *Let $\{z_1(\lambda), \ldots, z_p(\lambda)\} \subset \mathbb{F}(\lambda)^{nk\times 1}$ be a right minimal basis of $SRL_{\tau_0}(\lambda)$, partition the vectors $z_j(\lambda) \in \mathbb{F}(\lambda)^{nk\times 1}$ as $k \times 1$ block vectors with $n \times 1$ blocks, and denote by $x_j(\lambda) \in \mathbb{F}^{n\times 1}$ the $(k - \mathfrak{c}_0)$th block of $z_j(\lambda)$. Then $\{x_1(\lambda), \ldots, x_p(\lambda)\}$ is a right minimal basis of $P(\lambda)$.*

Left minimal bases: *Let $\{w_1(\lambda)^T, \ldots, w_p(\lambda)^T\} \subset \mathbb{F}(\lambda)^{1\times nk}$ be a left minimal basis of $SRL_{\tau_0}(\lambda)$, partition the vectors $w_j(\lambda)^T \in \mathbb{F}(\lambda)^{1\times nk}$ as $1\times k$ block vectors with $1\times n$ blocks, and denote by $y_j(\lambda)^T \in \mathbb{F}^{1\times n}$ the $(\mathfrak{i}_0 + 1)$th block of $w_j(\lambda)^T$. Then $\{y_1(\lambda)^T, \ldots, y_p(\lambda)^T\}$ is a left minimal basis of $P(\lambda)$.*

Minimal indices: Let $0 \leq \eta_1 \leq \eta_2 \leq \cdots \leq \eta_p$ and $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_p$ be, respectively, the left and right minimal indices of $P(\lambda)$. Then, the left and right minimal indices of $SRL_{\tau_0}(\lambda)$ are, respectively,

$$\eta_1 + \frac{k-1}{2} \leq \eta_2 + \frac{k-1}{2} \leq \cdots \leq \eta_p + \frac{k-1}{2} \quad \text{and}$$
$$\varepsilon_1 + \frac{k-1}{2} \leq \varepsilon_2 + \frac{k-1}{2} \leq \cdots \leq \varepsilon_p + \frac{k-1}{2}\,.$$

REMARK 4.6. *It was proved in [8, Section 3]: (a) that left minimal indices are equal to right minimal indices for symmetric and palindromic matrix polynomials; (b) that left minimal bases are transposes of right minimal bases for symmetric matrix polynomials; and, (c) for palindromic matrix polynomials, if the vectors of a right minimal basis are reversed and transposed, then a left minimal basis is obtained. Therefore, for symmetric and palindromic polynomials only right minimal bases and indices need to be recovered.*

5. Conclusions. We have developed easy recovery procedures for the eigenvectors of regular matrix polynomials and for the minimal indices and bases of square singular matrix polynomials from the corresponding ones of any generalized Fiedler linearization. Except for a few particular linearizations, these recovery procedures consist simply in extracting adequate blocks from the eigenvectors or minimal bases of the linearization, and in shifting left and right minimal indices by certain quantities that can be easily determined. Therefore, the recovery methods we propose do not represent any computational cost. This is, at a first glance, surprising because the class of generalized Fiedler pencils is a wide set containing many pencils with widely varying structures and properties. The results in this work allow us to use generalized Fiedler pencils to solve numerically polynomial eigenvalue problems, which can be useful, for instance, to solve symmetric and palindromic polynomial eigenvalue problems with odd degree arising in control and in algebraic-differential ordinary equations.

REFERENCES

[1] A. AMIRASLANI, R. M. CORLESS AND P. LANCASTER, *Linearization of matrix polynomials expressed in polynomial bases*, IMA J. Numer. Anal., 29 (2009), pp. 141–157.

[2] E. N. ANTONIOU AND S. VOLOGIANNIDIS, *A new family of companion forms of polynomial matrices*, Electron. J. Linear Algebra, 11 (2004), pp. 78–87.

[3] R. BYERS, V. MEHRMANN, AND H. XU, *Trimmed linearizations for structured matrix polynomials*, Linear Algebra Appl., 429 (2008), pp. 2373–2400.

[4] R. BYERS, D. S. MACKEY, V. MEHRMANN, AND H. XU, *Symplectic, BVD, and palindromic approaches to discrete-time control problems*, To appear in Jubilee Collection of Papers dedicated to the 60th Anniversary of Mihail Konstantinov, P. Petkov and N. Christov, eds, Publishing House of the Bulgarian Academy

of Sciences, Sofia. Also available as MIMS EPrint 2008.35 (2008), The University of Manchester, at http://www.manchester.ac.uk/mims/eprints

[5] J. W. DEMMEL AND B. KÅGSTRÖM, *The generalized Schur decomposition of an arbitrary pencil $A - \lambda B$: Robust software with error bounds and applications. Part I: Theory and Algorithms*, ACM T. Math. Software, 19 (1993), pp. 160–174.

[6] J. W. DEMMEL AND B. KÅGSTRÖM, *The generalized Schur decomposition of an arbitrary pencil $A - \lambda B$: Robust software with error bounds and applications. Part II: Software and Applications*, ACM T. Math. Software, 19 (1993), pp. 175–201.

[7] F. DE TERÁN AND F. M. DOPICO, *Sharp lower bounds for the dimension of linearizations of matrix polynomials*, Electron. J. Linear Algebra, 17 (2008), pp. 518–531.

[8] F. DE TERÁN, F. M. DOPICO, AND D. S. MACKEY, *Linearizations of singular matrix polynomials and the recovery of minimal indices*, Electron. J. Linear Algebra, 18 (2009), pp. 371–402.

[9] F. DE TERÁN, F. M. DOPICO, AND D. S. MACKEY, *Fiedler companion linearizations and the recovery of minimal indices*, SIAM J. Matrix Anal. Appl., 31 (2010), pp. 2181–2204.

[10] F. DE TERÁN, F. M. DOPICO, AND D. S. MACKEY, *Structured linearizations for palindromic matrix polynomials of odd degree*, submitted. Also available as MIMS EPrint 2010.33 (2010), The University of Manchester, at http://www.manchester.ac.uk/mims/eprints

[11] A. EDELMAN, E. ELMROTH, AND B. KÅGSTRÖM, *A geometric approach to perturbation theory of matrices and matrix pencils.* Part II. *A stratification-enhanced staircase algorithm*, SIAM J. Matrix Anal. Appl., 20 (1999), pp. 667–699.

[12] M. FIEDLER, *A note on companion matrices*, Linear Algebra Appl., 372 (2003), pp. 325–331.

[13] G. D. FORNEY, *Minimal bases of rational vector spaces, with applications to multivariable linear systems*, SIAM J. Control, 13 (1975), pp. 493–520.

[14] F. R. GANTMACHER, *The Theory of Matrices*, Chelsea, New York, 1959.

[15] I. GOHBERG, M. A. KAASHOEK, AND P. LANCASTER, *General theory of regular matrix polynomials and band Toeplitz operators*, Int. Equat. Oper. Th., 11 (1988), pp. 776–882.

[16] I. GOHBERG, P. LANCASTER, AND L. RODMAN, *Matrix Polynomials*, Academic Press, New York, 1982.

[17] G. GOLUB AND C. VAN LOAN, *Matrix Computations*, Third Ed., Johns Hopkins University Press, Baltimore, Maryland, 1996.

[18] N. J. HIGHAM, R-C. LI, AND F. TISSEUR, *Backward error of polynomial eigenproblems solved by linearization*, SIAM J. Matrix Anal. Appl., 29 (2007), pp. 1218–1241.

[19] N. J. HIGHAM, D. S. MACKEY, AND F. TISSEUR, *The conditioning of linearizations of matrix polynomials*, SIAM J. Matrix Anal. Appl., 28 (2006), pp. 1005–1028.

[20] N. J. HIGHAM, D. S. MACKEY, N. MACKEY, AND F. TISSEUR, *Symmetric linearizations for matrix polynomials*, SIAM J. Matrix Anal. Appl., 29 (2006), pp. 143–159.

[21] N. J. HIGHAM, D. S. MACKEY, F. TISSEUR AND S. D. GARVEY, *Scaling, sensitivity and stability in the numerical solution of quadratic eigenvalue problems*, Int. J. Numerical Methods in Engineering, 73 (2008), pp. 344–360.

[22] A. HILLIGES, C. MEHL, AND V. MEHRMANN, *On the solution of palindromic eigenvalue problems*, in Proceedings of the 4th European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS), Jyväskylä, Finland, 2004. CD-ROM.

[23] M. HOFER, N. FINGER, J. SCHÖBERL, S. ZAGLMAYR, U. LANGER, AND R. LERCH, *Finite element simulation of wave propagation in periodic piezoelectric SAW structures*, IEEE Transactions on UFFC, 53 (2006), pp. 1192-1201.

[24] T-M. HUANG, W-W. LIN, AND J. QIANG, *Structure preserving algorithms for palindromic quadratic eigenvalue problems arising from vibration of fast trains*, SIAM J. Matrix Anal. Appl., 30 (2009), pp. 1566-1592.

[25] T. KAILATH, *Linear Systems*, Prentice Hall, Englewood Cliffs, NJ, 1980.

[26] P. LANCASTER AND P. PSARRAKOS, *A note on weak and strong linearizations of regular matrix polynomials*. Available as MIMS-EPrint 2006.72, Manchester Institute for Mathematical Sciences.

[27] D. S. MACKEY, N. MACKEY, C. MEHL, AND V. MEHRMANN, *Vector spaces of linearizations for matrix polynomials*, SIAM J. Matrix Anal. Appl., 28 (2006), pp. 971–1004.

[28] D. S. MACKEY, N. MACKEY, C. MEHL, AND V. MEHRMANN, *Structured polynomial eigenvalue problems: Good vibrations from good linearizations*, SIAM J. Matrix Anal. Appl., 28 (2006), pp. 1029–1051.

[29] F. TISSEUR AND K. MEERBERGEN, *The quadratic eigenvalue problem*, SIAM Review, 43 (2001), pp. 235–286.

[30] P. VAN DOOREN, *The computation of Kronecker's canonical form of a singular pencil*, Linear Algebra Appl., 27 (1979), pp. 103–140.

[31] P. VAN DOOREN AND P. DEWILDE, *The Eigenstructure of an Arbitrary Polynomial Matrix: Computational Aspects*, Linear Algebra Appl., 50 (1983), pp. 545–579.

[32] S. ZAGLMAYR, *Eigenvalue problems in SAW-filter simulations*, Diplomarbeit, Inst. of Comp. Mathematics, J. Kepler Univ. Linz, Austria, 2002.